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Factorization of the matrices of discrete wavelet transform on the Galois functions base

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The paper deals with the factorization of the matrices of discrete wavelet transform based on the Galois functions of different orders. It is used the known method of factorization of the matrices of the discrete Haar transform. Factorized matrices of transforms are presented in the form of a product of sparse matrices. This representation is the basis for building fast transforms algorithms.

Key words and phrases: factorization of the matrices, discrete wavelet transform, wavelet system on the Galois functions base.

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Introduction

Discrete orthogonal and wavelet transforms are widely used in practice, for example, in digital signal processing. In fact, all discrete orthogonal transforms and discrete wavelet transforms in matrix form are performed on the basis of the matrix multiplication algorithm, that is, multiplication of the input signal model vector by the transformation matrix.

Significant number of calculations when performing this operation results in low processing speed and limited application of transforms. Matrix factorization, i.e. the representation of a matrix in the form of a product of sparse matrices [10], is the basis of fast transforms algorithms and reduction of the number of computational operations when performing transforms in discrete bases. Sparse matrix is a matrix in which most of the elements are zero.

Matrix factorization is a tool for designing different fast algorithms [2, 3, 10, 11]. Factorization have been used in classical algorithms of signal processing (such as Fourier transform, Hadamard transform, Haar transform, discrete cosine transform [1, 3, 10, 11]) and, more recently, in wavelet transforms [3, 4, 7].

Article of I.J. Good [2] was the first work in which matrix factorization was the basis for reducing of the number of calculations, when performing discrete Fourier transform. The concept of factorization and decomposition of transformation matrices to simpler ones is explained in [10]. Methods of factorization of matrices of widely used Fourier, Walsh, Haar transforms are described in [10, 11]. Haar transform is also the wavelet transform. Wavelet transform is the time-frequency representation of a signal, it is done in the basis, built from one mother wavelet function using scale changes and parallel shifts.

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Over the years we have seen a number of works on ways of factorization of matrices [1,3,8] and their application in different systems [3,8], in different transforms and wavelet transforms [1,3,4,9]. The automatically generated sparse factorizations and fast algorithms for matrices corresponding to digital signal processing transforms including the discrete Fourier, cosine, Hartley and Haar transforms are described in [1].

The problem of developing matrix factorization algorithms is actual, since new efficient algorithms for fast discrete transforms are being built on their base [1,3,4,8,9].

The works most closely related to some of the ideas in this paper are that on defining algorithm for computing the Walsh-Hadamard transform, which consists entirely of Haar wavelet transforms and method of matrix factorization [9]. This paper partly is based on the known factorization methods of L.P. Yaroslavskyi [10], which is used for Haar matrices and other matrices with a certain internal structure and dimensions equal to the powers of 2 [10,11].

In [7], a family of discrete wavelet systems on the Galois function base is constructed. At the same time, the possibility of factorization of the matrices of constructed systems for the development of fast transforms algorithms and their application is not known.

The goal of this article is the factorization of matrices of discrete wavelet systems on the Galois function base, which are used in discrete wavelet transforms, that is, the representation of matrices in the form of a product of sparse matrices.

1 Matrices of wavelet systems on the Galois functions base

For constructing of a system of wavelets on the Galois functions base for discrete transforms of signals modeled by functions $f \in L_2\big([0,1)\big)$ as a mother wavelet it is used the first function of a recursively ordered Galois system [6].

Mother wavelet $Gal_p(\theta)$ is defined in [7] as the the first in order function $Gal_{p,0}(\theta)$ of a recursively ordered Galois system (see [6])

$$\{Gal_{p,i}(\theta)\}, \quad p = 1,2,3,\ldots, \quad i = 0,1,\ldots,2^p-1.$$

Values of the first Galois function in the recursively ordered system are calculated according to the elements of recursive sequence [6, p.36]. The rule of formation of sequence elements is determined by a vector of coefficients of irreducible polynomial in Galois field $GF(2^p)$ [5]. A sequence with elements $\{0,1\}$ is generated based on each irreducible polynomial. Examples of creating recursive sequences are shown in [7].

Example 1. *Vector of coefficients* $(p_0,p_1,p_2) = (1,1,1)$ *corresponds to irreducible polynomial* x^2+x+1*, which generates Galois field* $GF\left(2^2\right)$*. Non-zero elements of vector determine the rule* $p_{i+2} = p_i \oplus_2 p_{i+1}$ *for the formation of a recursive sequence, where* $\oplus_2$ *is sign of addition modulo two. Initial vector with unitary elements* $(v_0,v_1) = (1,1)$ *is chosen as a primary vector. From the primary vector according to this rule* $v_{i+2} = v_i \oplus_2 v_{i+1}$ *there are defined the elements of a recursive sequence, which are repeated with period* 2^n-1*. Fragment of* $n-1$ *zero elements of the sequence is supplemented by one zero. Elements of supplemented sequence are denoted as*

$$g_i : \{0, v_{i+2}, v_i, v_{i+1}\} = \{g_0, g_1, g_2, g_3\} = \{0,0,1,1\}.$$

Example 2. *Vector of coefficients* $(p_0, p_1, p_2, p_3) = (1, 1, 0, 1)$ *corresponds to irreducible polynomial* $x^3 + x^2 + 1$ *which generates Galois field* $GF\left(2^3\right)$. *This vector determines the next rule* $p_{i+3} = p_i \oplus_2 p_{i+1}$ *for the formation of a recursive sequence. From the initial vector* $(v_0, v_1, v_2) = (1, 1, 1)$ *according to the rule* $v_{i+3} = v_i \oplus_2 v_{i+1}$ *there are defined the elements of a recursive sequence, supplemented by zero and submitted the following fragment*

$$\{0, v_{i+3}, v_{i+4}, v_{i+5}, v_{i+6}, v_i, v_{i+1}, v_{i+2}\} = \{g_0, g_1, g_2, \dots, g_7\} = \{0, 0, 0, 1, 0, 1, 1, 1\}.$$

Elements of a recursive sequence supplemented by one zero in fragment of consecutive zeros are denoted as $\{g_0, g_1, g_2, \dots, g_{2^p-1}\}$ (see [7]).

Values of mother wavelet at the points $\theta = \theta_j = j$ in the interval $\theta \in [0, M)$, $M = 2^p$, $j = 0, 1, \dots, 2^p - 1$ are obtained from the elements of a recursive sequences $g_j \in \{0, 1\}$ via transform

$$Gal_p\left(\theta_j\right) = 1 - 2g_j, \tag{1}$$

where g_j are elements of a recursive sequence.

Functions $Gal_p\left(\theta\right)$ are continuous constants in the intervals $\theta \in [j, j+1)$ and take values

$$Gal_p\left(\theta\right) = Gal_p\left(\theta_j\right). \tag{2}$$

According to (1) and (2), these functions are $Gal_p\left(\theta\right) = \pm 1$.

System of wavelet function is formed on the basis of mother function $Gal_p\left(\theta\right)$ with the help of scale and parallel shift and it is defined as

$$Gal_{p,m,k}(t) = 2^{\frac{m-1}{2}} Gal_p\left(2^{m-1}t - Nk\right), \tag{3}$$

where $t = \frac{N}{M}\theta$, $\theta \in [0, N)$, $N = 2^n$ is the quantity of functions in the system, $n = 1, 2, 3, \dots$, $m = 0, 1, \dots, \log_2 N + 1$, $k = 0, 1, \dots, N \cdot 2^{-m}$. Non-normalized functions $Gal_{p,m,k}(t) = \pm 1$ [7] and normalized functions $Gal_{p,m,k}(t) = \pm\sqrt{\frac{2^{m-1}}{N}}$ are piecewise-constants in intervals $t \in \left[\frac{q}{l}, \frac{q+1}{l}\right)$, where $q = 0, 1, \dots, lN - 1$, $l = 2^{p-1}$.

The set $\left\{Gal_{p,m,k}(t)\right\}$ of systems, based on mother wavelets for different $p = 1, 2, 3, \dots$, forms a family of wavelet functions on the Galois functions basis [7]. Values of piecewise-constant functions $Gal_{1,m,k}(t)$ and $Gal_{2,m,k}(t)$ at equidistant points $t = j\frac{N}{2^p}$ are written as a square matrices. Values of functions $Gal_{p,m,k}(t)$ for $p \geq 3$ at points $t = \frac{q}{l}$ are written as a rectangular matrix. Each row of this matrix corresponds to a discrete function. This matrix is a wavelet transform matrix.

These materials offer the factorization algorithm for matrices with values of discrete wavelet functions $Gal_{p,m,k}(t)$ with mother wavelet functions Galois.

An overview of works on the same type of matrix representation of various fast discrete orthogonal transformations is made in [11]. In this representation, the transformed sequence of samples of the input signal is represented in the form of a column vector, and the transform is represented by a matrix. When multiplying the transformation matrix by the specified column vector, we get the column vector that is the result of the transform.

Paper [10] describes a single approach to the derivation of fast algorithms, which is based on the representation of transformation matrices in the form of sums of Kronecker matrices. It is also proved that transformation matrices of dimension 2^n can be factorized, that is, represented in the form of a product of sparse matrices. Factorized representations of some matrices are given in [10, 11].

In the proposed materials, the described method of factorization [10] is used, which requires specifying the main definitions and properties.

Definition 1. *The direct sum of any pair of matrices A of size $m \times n$ and B of size $p \times q$ is a matrix of size $(m+p) \times (n+q)$ defined as*

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

The direct sum of matrices is a special type of block matrix and it is denoted by $\oplus$. The direct sum of matrices has the following properties

$$A_1 A_2 A_3 \ldots A_n \oplus B_1 B_2 \ldots B_n = (A_1 \oplus B_1)(A_2 \oplus B_2) \ldots (A_n \oplus B_n), \tag{4}$$
$$(A_1 \oplus A_2 \oplus \ldots \oplus A_n)^T = \left(A_1^T \oplus A_2^T \oplus \ldots \oplus A_n^T\right).$$

Definition 2. *The right direct (Kronecker) product of the matrices A of size $m \times n$ and B of size $p \times q$ is called the matrix $C = A \otimes B$, composed of $m \times n$ submatrices, each of which is the product of the corresponding element of the matrix A by the matrix B.*

The Kronecker product is denoted by $\otimes$.

2 Factorization of wavelet matrices based on Galois functions

It is established that the matrices of discrete values of wavelet functions $Gal_{p,m,k}(t)$ are factorized and presented in the form of the product of sparse matrices of smaller dimension.

It is known [7] from the construction that the mother wavelets $Gal_1(\theta)$ and $Gal_2(\theta)$ of systems by orders $n = 1$ and $n = 2$ are Haar wavelets. The systems of wavelet functions built on their basis are the orthogonal Haar systems. That is why this paper contains two cases of factorization of matrices for wavelet functions of orders $p = 1, p = 2$ and $p \geq 3$.

2.1 Factorization of wavelet matrices based on Galois functions of order $p = 2$

Let $WGAL_{p,N}$ denote the matrix of size $N \times M$ of the discrete wavelet transform, built on the of Galois functions base of the pth order, N is the number of functions in the system. In the case of $p = 2$, the matrix $WGAL_{2,N}$ is a square matrix.

Example 3. *The matrix of discrete values of four wavelet functions $\{Gal_{2,m,k}(t)\}$, constructed according to the formula* (3) *from the mother Galois wavelet $Gal_2(\theta)$ of the order $n = 2$, has the form*

$$WGAL_{2,4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}.$$

The known method of factorization of the matrices of the discrete Haar transform and other orthogonal transforms [10, 11] is reduced to executing operations with the following matrices

$$V_2^0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; V_2^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; G_2^1 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \; G_2^2 = \begin{bmatrix} 1 & -1 \end{bmatrix}.$$

The next formula is a recurrent expression of the Haar matrix [10]

$$\overline{HAR_{2^n}} = \left[\overline{HAR_{2^{n-1}}} \oplus I_{2^{n-1}}\right] \left[\begin{bmatrix} I_{2^{n-1}} \otimes G_2^1 \\ I_{2^{n-1}} \otimes G_2^2 \end{bmatrix} \right], \tag{5}$$

where $\overline{HAR_{2^n}}$ is Haar matrix with elements $\{\pm 1, 0\}$ and without scale factor used for the simplification of notation and transformation [10]; I_q is identity matrix of size q.

Proposition 1. *Square matrix of size $2^n \times 2^n$ of the wavelet system $Gal_{2,m,k}(t)$, defined by the mother wavelet functions Galois of the second order, is factorized*

$$WGAL_{2,2^n} = \frac{1}{2^{\frac{n}{2}}} \left[1 \oplus \bigoplus_{i=0}^{n-1} 2^{\frac{i}{2}} I_{2^i}\right] \times \prod_{i=0}^{n-1} \left[\begin{bmatrix} I_{2^i} \otimes G_2^1 \\ I_{2^i} \otimes G_2^2 \end{bmatrix} \oplus I_{2^n - 2^{i+1}} \right].$$

Proof. We apply recurrent expression of the Haar matrix (5) for the square Galois wavelet matrix defined by the mother wavelet functions Galois of the second order

$$\overline{WGAL_{2,2^n}} = \left[\overline{WGAL_{2,2^{n-1}}} \oplus I_{2^{n-1}}\right] \left[\begin{bmatrix} I_{2^{n-1}} \otimes G_2^1 \\ I_{2^{n-1}} \otimes G_2^2 \end{bmatrix} \right],$$

where $\overline{WGAL_{2,2^n}}$ is the matrix with elements $\{\pm 1, 0\}$ and without scale factor, we use it for the simplification of notation and transformation. The expression of the Galois matrix $\overline{WGAL_{2,2^{n-1}}}$ is written in terms of the matrix $\overline{WGAL_{2,2^{n-2}}}$

$$\begin{aligned} \overline{WGAL_{2,2^n}} &= \left[\left[\overline{WGAL_{2,2^{n-2}}} \oplus I_{2^{n-2}}\right] \begin{bmatrix} I_{2^{n-2}} \otimes G_2^1 \\ I_{2^{n-2}} \otimes G_2^2 \end{bmatrix} \oplus I_{2^{n-1}} \right] \begin{bmatrix} I_{2^{n-1}} \otimes G_2^1 \\ I_{2^{n-1}} \otimes G_2^2 \end{bmatrix} \\ &= \left[\left[\overline{WGAL_{2,2^{n-2}}} \oplus I_{2^{n-2}}\right] \begin{bmatrix} I_{2^{n-2}} \otimes G_2^1 \\ I_{2^{n-2}} \otimes G_2^2 \end{bmatrix} \oplus I_{2^{n-1}} \times I_{2^{n-1}} \right] \begin{bmatrix} I_{2^{n-1}} \otimes G_2^1 \\ I_{2^{n-1}} \otimes G_2^2 \end{bmatrix}. \end{aligned}$$

For further transformations, it is used the property (4) of the direct sum of the matrices. This implies

$$\overline{WGAL_{2,2^n}} = \left[\overline{WGAL_{2,2^{n-2}}} \oplus I_{2^{n-2}} \oplus I_{2^{n-1}}\right] \left[\begin{bmatrix} I_{2^{n-2}} \otimes G_2^1 \\ I_{2^{n-2}} \otimes G_2^2 \end{bmatrix} \oplus I_{2^{n-1}} \right] \begin{bmatrix} I_{2^{n-1}} \otimes G_2^1 \\ I_{2^{n-1}} \otimes G_2^2 \end{bmatrix}.$$

Since $I_{2^{n-2}} \oplus I_{2^{n-1}} = I_{3 \cdot 2^{n-2}}$, then

$$\overline{WGAL_{2,2^n}} = \left[\overline{WGAL_{2,2^{n-2}}} \oplus I_{3 \cdot 2^{n-2}}\right] \left[\begin{bmatrix} I_{2^{n-2}} \otimes G_2^1 \\ I_{2^{n-2}} \otimes G_2^2 \end{bmatrix} \oplus I_{2^{n-1}} \right] \begin{bmatrix} I_{2^{n-1}} \otimes G_2^1 \\ I_{2^{n-1}} \otimes G_2^2 \end{bmatrix}. \tag{6}$$

Continuing the reasoning about the other size of the matrix 2^n, we obtain an expression for the factorization of the wavelet transform matrix based on Galois functions of order $n = 2$, namely

$$WGAL_{2,2^n} = \frac{1}{2^{\frac{n}{2}}} \left[1 \oplus \bigoplus_{i=0}^{n-1} 2^{\frac{i}{2}} I_{2^i}\right] \times \prod_{i=0}^{n-1} \left[\begin{bmatrix} I_{2^i} \otimes G_2^1 \\ I_{2^i} \otimes G_2^2 \end{bmatrix} \oplus I_{2^n - 2^{i+1}} \right] \tag{7}$$

$$= C_{2^n}^{GAL} \times \prod_{i=0}^{n-1} \left[\begin{bmatrix} I_{2^i} \otimes G_2^1 \\ I_{2^i} \otimes G_2^2 \end{bmatrix} \oplus I_{2^n - 2^{i+1}} \right], \tag{8}$$

where $C_{2^n}^{GAL} = \left\{ c_{k,l} = 2^{\frac{k_c}{2}} \delta_{k-l} \right\}$ is the diagonal matrix; k_c is the number of the oldest non-zero digit of row number in binary system, δ_k is Kronecker delta function

$$\delta_{k,l} = \begin{cases} 1, & k = 0, \\ 0, & k \neq 0. \end{cases}$$

Thus, the wavelet transform matrix based on Galois functions is represented as a product (8) of n sparse matrices. Similarly it is done factorization of wavelet matrices based on Galois functions of order $p = 1$. It is possible to construct a matrix of any size $2^n \times 2^n$ based on Galois functions of order 2 and to execute its factorization by (7), (8). □

Example 4. *Let us give an example of factorization of the matrix $WGAL_{2,8}$. Wavelet transform matrix based on Galois functions of order $p = 2$*

$$WGAL_{2,8} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}.$$

The formula (6) *is used to represent the product*

$$WGAL_{2,8} = C_{2^3}^{GAL} \left[\overline{WGAL_{2,2}} \oplus I_{3\cdot 2} \right] \left[\begin{bmatrix} I_2 \otimes G_2^1 \\ I_2 \otimes G_2^2 \end{bmatrix} \oplus I_4 \right] \begin{bmatrix} I_4 \otimes G_2^1 \\ I_4 \otimes G_2^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

2.2 Factorization of wavelet matrices based on Galois functions of order $p \geq 3$

The factorization of the matrices of wavelet functions on the Galois functions basis by orders $p \geq 3$ is performed using the method and factorization formula for the case $p = 2$.

Since wavelet systems of order $p \geq 3$ are built on the basis of other mother wavelets, it is proposed to form matrices of values of mother wavelets of orders $p \geq 3$ instead of matrices G_2^1 and G_2^2 and substitute in the second factor of the formula (7).

A feature of systems $\{Gal_{p,m,k}(t)\}$ of order $p \geq 3$ is their representation by rectangular matrices instead of square ones in case $p = 2$. The resulting matrices $WGAL_{p,N}$ have the size $N \times Nl$. Thus, in these cases, it is necessary to factorize rectangular matrices.

Instead of the matrices G_2^1 and G_2^2 in subsection 2.1 it is proposed to define the matrix of ones $G_{2^n}^1 = [h_0 \quad h_1 \quad \ldots \quad h_{2^n-1}]$ with elements $\{h_i, h_i = 1\}$ and the matrix with values g_i of mother wavelet function $Gal_{p,1,0}(t)$ respectively

$$G_{2^n}^2 = [g_0 \quad g_1 \quad \ldots \quad g_{2^n-1}], \quad i = 0, 1, \ldots, 2^n - 1, \quad p \geq 3.$$

Example 5. *Elements g_i of matrix $G_{2^n}^2$ correspond to the values of mother wavelet function $Gal_{3,1,0}(\frac{q}{l})$ and submitted by the set $G_8^2 = [1, 1, 1, -1, 1, -1, -1, -1]$. In this case, the matrix $WGAL_{3,8}$ has the size* 8×32. *To form the first row of the* 8×32 *matrix $\overline{WGAL_{3,8}}$, each element of G_8^2 must be duplicated* 4 *times:*

$$[1,1,1,1,1,1,1,1,1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1].$$

Example 6. *Elements g_i of matrix $G_{2^n}^2$ correspond to the values of mother wavelet function $Gal_{4,1,0}\left(\frac{q}{l}\right)$ and submitted by the following set*

$$G_{16}^2 = [1,1,1,1,-1,-1,-1,-1,1,-1,1,-1,-1,1,1,-1].$$

In this case, the matrix $WGAL_{4,16}$ has the size 16×80. *To form the first row of the* 16×80 *matrix $\overline{WGAL_{4,16}}$, each element of G_{16}^2 must be duplicated* 5 *times.*

Proposition 2. *Matrix of the wavelet transform on the Galois functions $Gal_{p,m,k}(t)$ base, defined by the mother wavelet functions Galois of the order $p \geq 3$, is factorized*

$$WGAL_{n,N} = \frac{1}{2^{\frac{\log_2 N}{2}+n}} \left[1 \oplus \bigoplus_{j=0}^{n-1} 2^{j/2} I_N\right] \times \left[\begin{bmatrix} G_{2^n}^1 \\ G_{2^n}^2 \end{bmatrix} \oplus I_{N-2^{n+1}} \otimes G_{2^n-1}^1\right]$$
$$\times \prod_{r=1}^{\log_2 N-1} \begin{bmatrix} I_{2^r} \otimes \bigoplus_{i=0}^{2^n-1} [h_{2i} \quad h_{2i+1}] \\ I_{2^r} \otimes \bigoplus_{i=0}^{2^n-1} [g_{2i} \quad g_{2i+1}] \end{bmatrix} \oplus I_{2^{n-1}N-2^{r+n}}.$$

Proof. We modify the factorization method, given in subsection 2.1, using the expression (5), to enable the factorization of rectangular matrices $WGAL_{n,N}$ of size $N \times Nl$ of the different structure. Multipliers in the product are denoted $WG_1, WG_2, \ldots, WG_{\log_2 N}$. It is established that sparse matrices WG_1 have the following structure

$$WG_1 = \begin{bmatrix} G_{2^n}^1 \\ G_{2^n}^2 \end{bmatrix} \oplus I_{N-2^{r+1}} \otimes G_{2^n-1}^1. \tag{9}$$

At the first step, as a result of multiplication by WG_1, the rectangular matrix is transformed into a square one. The structure of the matrices WG_{r+1} is similar to the factors in expression (5), but instead of G_2^1 and G_2^2, matrices are formed from two adjacent elements of $G_{2^n}^1$ and $G_{2^n}^2$, namely

$$WG_{r+1} = \begin{bmatrix} I_{2^r} \otimes \bigoplus_{i=0}^{2^{n-1}} [h_{2i} \quad h_{2i+1}] \\ I_{2^r} \otimes \bigoplus_{i=0}^{2^{n-1}} [g_{2i} \quad g_{2i+1}] \end{bmatrix} \oplus I_{2^{n-1}N-2^{r+1}}, \tag{10}$$

where $r = 1, \ldots, \log_2 N - 1$, $p \geq 3$.

We form matrix of discrete wavelet transform with mother wavelet functions Galois as a product of sparse matrices

$$WGAL_{n,N} = C \times WG_1 \times WG_2 \times \cdots \times WG_{log_2 N}, \tag{11}$$

where $C = \frac{1}{2^{\frac{log_2 N}{2}+n}} \left[1 \oplus \bigoplus_{j=0}^{n-1} 2^{j/2} I_N\right]$ is the matrix of coefficients. Substitution of expressions (9) and (10) into (11) gives

$$WGAL_{n,N} = \frac{1}{2^{\frac{log_2 N}{2}+n}} \left[1 \oplus \bigoplus_{j=0}^{n-1} 2^{j/2} I_N\right] \times \left[\begin{bmatrix} G_{2^n}^1 \\ G_{2^n}^2 \end{bmatrix} \oplus I_{N-2^{n+1}} \otimes G_{2^{n-1}}^1\right]$$
$$\times \prod_{r=1}^{log_2 N-1} \begin{bmatrix} I_{2^r} \otimes \bigoplus_{i=0}^{2^{n-1}} [h_{2i} \quad h_{2i+1}] \\ I_{2^r} \otimes \bigoplus_{i=0}^{2^{n-1}} [g_{2i} \quad g_{2i+1}] \end{bmatrix} \oplus I_{2^{n-1}N-2^{r+n}}.$$

□

Thus, this materials present a simple factorization of the matrices of the wavelet functions on the Galois function base, which are used as matrices of the wavelet transforms. The possibility of factorization of matrices of discrete wavelet systems on Galois functions base is proved. A way of decomposing rectangular wavelet transform matrices on Galois functions base has been developed. The matrix decomposition is in the form of the Kronecker products of identity matrices and lower order matrices. The result of decomposition is the representation of matrices in the form of a product of sparse matrices. It is will be used to develop a fast algorithm for the wavelet transform.

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Здійснено факторизацію матриць дискретного вейвлет-пеетворення на основі функцій Галуа різних порядків. Використано відомий метод факторизації матриць дискретного перетворення Хаара. Факторизовані матриці перетворень представляються у формі добутку розріджених матриць. Це подання є основою побудови швидких алгоритмів перетворень.

Ключові слова і фрази: факторизація матриць, дискретне вейвлет-перетворення, система вейвлет-функцій на основі функцій Галуа.