On Dirichlet series similar to Hadamard compositions in half-plane

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Let \(F(s) = \sum_{n=1}^{\infty} a_n \exp\{s\lambda_n\}\) and \(F_j(s) = \sum_{n=1}^{\infty} a_{n,j} \exp\{s\lambda_n\}\), \(j = 1, p\), be Dirichlet series with exponents \(0 \leq \lambda_n \uparrow +\infty, n \to \infty\), and the abscissas of absolutely convergence equal to 0. The function \(F\) is called Hadamard composition of the genus \(m \geq 1\) of the functions \(F_j\) if \(a_n = P(a_{n,1}, \ldots, a_{n,p})\), where \(P(x_1, \ldots, x_p) = \sum_{k_1 + \cdots + k_p = m} c_{k_1} x_1^{k_1} \cdots x_p^{k_p}\) is a homogeneous polynomial of degree \(m\). In terms of generalized orders and convergence classes the connection between the growth of the functions \(F_j\) and the growth of the Hadamard composition \(F\) of the genus \(m \geq 1\) of \(F_j\) is investigated. The pseudostarlikeness and pseudoconvexity of the Hadamard composition of the genus \(m \geq 1\) are studied.

Key words and phrases: Dirichlet series, Hadamard composition, generalized order, generalized type, generalized convergence class, pseudostarlikeness, pseudoconvexity.

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Introduction

Let \(f_j(z) = \sum_{n=0}^{\infty} a_{n,j} z^n, 1 \leq j \leq p\), be entire transcendental functions. As in [12], we say that the function \(f(z) = \sum_{n=0}^{\infty} a_n z^n\) is similar to Hadamard composition of the functions \(f_j\) if \(a_n = w(a_{n,1}, \ldots, a_{n,p})\) for all \(n\), where \(w : \mathbb{C}^p \to \mathbb{C}\) is some function. Clearly, if \(p = 2\) and \(w(a_{n,1}, a_{n,2}) = a_{n,1} a_{n,2}\), then \(f = (f_1 * f_2)\) is the Hadamard composition (product) of the functions \(f_1\) and \(f_2\) (see [5]). Obtained by J. Hadamard properties of this composition find the applications in the theory of the analytic continuation of the functions represented by power series [1, 6]. E.G. Calys [2] investigated the functions similar to Hadamard compositions with \(|w(x, y)| = \sqrt{|xy|}\) and proved, in particular, the following theorem.

Theorem A ([2]). Let entire functions \(f_j, j = 1, 2\), have the same order \(\sigma[f_j] = \varphi \in (0, +\infty)\) and types \(\sigma[f_j] = \sigma_j\). Suppose that \(a_{n,1} \neq 0\) and \(|a_{n,2}| \geq |a_{n,1}|/l(1/|a_{n,1}|)\) for all \(n \geq n_0\), where \(l\) is slowly varying function. If \(|a_{n}| = (1 + o(1)) \sqrt{|a_{n,1}| |a_{n,2}|}\) as \(n \to \infty\), then function \(f\) has the order \(\varphi[f] = \varphi\) and the type \(\sigma[f] \leq \sqrt{\sigma_1 \sigma_2}\).

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In article [8], Theorem A is generalized to the case of entire Dirichlet series of finite generalized order, and instead of two entire functions $f_1$ and $f_2$ it is considered $n \geq 2$ entire Dirichlet series. Analogues of the results from [8] for Dirichlet series absolutely convergent in the half-plane are obtained in [9].

To formulate them, we denote by $L$ the class of positive continuous functions $\alpha$ on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0)$ for $x \leq x_0$ and $0 < \alpha(x) \uparrow +\infty$ as $x_0 \leq x \uparrow +\infty$. We say that $\alpha \in L^0$ if $\alpha \in L$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_{si}$ if $\alpha \in L$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x \to +\infty$ for each $c \in (0, +\infty)$, i.e. $\alpha$ is slowly increasing function. Clearly, $L_{si} \subset L^0$.

Let $\Lambda = (\lambda_n)$ be an increasing to $+\infty$ sequence of nonnegative number, $S(\Lambda, 0)$ be the class of Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_n \exp\{s\lambda_n\}, \quad s = \sigma + it,$$

(1)

with the sequence $(\lambda_n)$ of exponents and the abscissa of absolutely convergence $\sigma_a[F] = 0$ and $M(\sigma, F) = \sup \{|F(\sigma + it)| : t \in \mathbb{R}\}$ for $\sigma \in (-\infty, 0)$.

If $\alpha \in L$, $\beta \in L$ and $F \in S(\Lambda, 0)$ then the value

$$\phi_{a, \beta}^0[F] = \lim_{\sigma \to 0} \frac{\alpha(\ln M(\sigma, F))}{\beta(1/|\sigma|)},$$

is called [3] generalized order of $F$. If $\phi_{a, \beta}^0[F] \in (0, +\infty)$, the generalized type is defined [9] as

$$T_{a, \beta}^{01}[F] = \lim_{\sigma \to 0} \frac{\ln M(\sigma, F)}{\alpha^{-1}(1/|\beta|)}, \quad q = \phi_{a, \beta}^0[F].$$

**Theorem B ([9]).** Let $\beta \in L_{si}$, $\alpha(e^x) \in L^0$, $\alpha^{-1}(c\beta(x)) \in L_{si}$,

$$\frac{x}{\beta^{-1}(ca(x))} \uparrow +\infty, \quad \alpha \left(\frac{x}{\beta^{-1}(ca(x))}\right) \sim \alpha(x) \quad \text{as} \quad x_0 \leq x \to +\infty \quad (2)$$

and $n = o\left(\alpha^{-1}(c\beta(\lambda_n))\right)$ as $n \to \infty$ for every $c \in (0, +\infty)$. Suppose that Dirichlet series $F_j \in S(\Lambda, 0)$ of the form

$$F_j(s) = \sum_{n=1}^{\infty} a_{n, j} \exp\{s\lambda_n\}, \quad 1 \leq j \leq p,$$

(3)

have the same generalized order $\phi_{a, \beta}^0[F_j] = q \in (0, +\infty)$, the types $T_{a, \beta}^{01}[F_j] \in (0, +\infty)$, $|a_{n, 1}| \geq n \geq n_0$ and $\ln \ln |a_{n, j}| \geq \left(1 + o(1)\right) \ln \ln |a_{n, 1}|$ as $n \to \infty$ for $2 \leq j \leq p$. If $\omega_j > 0$ with $\sum_{j=1}^{p} \omega_j = 1$ and $\ln |a_n| = (1 + o(1)) \prod_{j=1}^{p} \ln^{\omega_j} |a_{n, j}|$, then function $F$ of the form (1) has the generalized order $\phi_{a, \beta}^0[F] = q$ and the type $T_{a, \beta}^{01}[F] \leq \prod_{j=1}^{p} T_{a, \beta}^{01}[F_j]^{\omega_j}$.

If $T_{a, \beta}^{01}[F] = 0$, then for the characteristic of the growth of the functions $F \in S(\Lambda, 0)$ we define a generalized convergence class by the condition

$$\int_0^c \frac{\ln M(\sigma, F)}{|\sigma|^2 \alpha^{-1}(1/|\sigma|)} \ d\sigma < +\infty, \quad q = \phi_{a, \beta}^0[F].$$

(4)
Theorem C ([12]). Let \( \alpha \in L^0 \), \( \beta \in L^0 \) and \( \alpha^{-1}(\beta(x)) \in L^0 \) for every \( c \in (0, +\infty) \). If all functions (3) belong to the generalized convergence class, \( \lim_{n \to \infty} \frac{\ln n}{\ln |n_{\omega_j}|} \leq \ln \omega \) for all 

\[ 1 \leq j \leq p, \omega_j > 0, \quad \sum_{j=1}^{p} \omega_j = 1 \quad \text{and} \quad |a_n| < \prod_{j=1}^{p} |a_{n,j}|^{\omega_j} \quad \text{as} \quad n \to \infty, \]

then the function (1) also belongs to the same convergence class. If, in addition, \( |a_{n,1}| > 0 \) for all \( n \geq 1 \) and \( |a_{n,j}| \leq |a_{n,1}| \) as \( n \to \infty \) for all \( 1 \leq j \leq p \), then the belonging of \( F \) to the generalized convergence class implies the belonging of all \( F_j \) to the same convergence class.

Here we consider the case when \( w \) is a homogeneous polynomial.

1 Definition and convergence of the Hadamard composition of the genus \( m \)

Recall that a polynomial is named homogeneous if all monomials with nonzero coefficients have the identical degree. A polynomial \( P(x_1, \ldots, x_p) \) is homogeneous to the degree \( m \) if and only if \( P(tx_1, \ldots, tx_p) = t^m P(x_1, \ldots, x_p) \) for all \( t \) from the domain of the polynomial. Dirichlet series (1) is called a Hadamard composition of genus \( m \) of Dirichlet series (3) if \( a_n = P(a_{n,1}, \ldots, a_{n,p}) \), where

\[ P(x_1, \ldots, x_p) = \sum_{k_1 + \cdots + k_p = m} c_{k_1, \ldots, k_p} x_1^{k_1} \cdots x_p^{k_p}. \]

is a homogeneous polynomial of degree \( m \geq 1 \). We remark that the usual Hadamard composition is a special case of the Hadamard composition of the genus \( m = 2 \).

Therefore, if the function \( F \) is Hadamard composition of genus \( m \geq 1 \) of the functions \( F_j \)

\[ |a_n| \leq \sum_{k_1 + \cdots + k_p = m} |c_{k_1, \ldots, k_p}| |a_{n,1}|^{k_1} \cdots |a_{n,p}|^{k_p}. \]  \hspace{1cm} (5)

It is known [10, 14] that if \( \ln n = o(\lambda_n) \) as \( n \to \infty \) then \( \sigma_a[F] = \alpha[F] := \lim_{n \to \infty} \frac{1}{\lambda_n} \ln \frac{1}{|a_n|} \).

Hence, it follows that if all \( F_j \in S(\Lambda, 0) \), i.e. \( \sigma_a[F_j] = 0 \), then \( |a_{n,j}| \leq \exp\{\epsilon \lambda_n\} \) for every \( \epsilon > 0 \), all \( n \geq n_0(\epsilon) \) and all \( 1 \leq j \leq p \). Therefore, (5) implies \( |a_n| \leq C \exp\{\epsilon m \lambda_n\} \), where

\[ C = \sum_{k_1 + \cdots + k_p = m} |c_{k_1, \ldots, k_p}|. \]  \hspace{1cm} Hence \( \alpha[F] = \lim_{n \to \infty} \frac{1}{\lambda_n} \ln \frac{1}{|a_n|} \geq -m \epsilon, \)

i.e. in view of the arbitrariness of \( \epsilon \) we get \( \sigma_a[F] = \alpha[F] \geq 0 \).

In order for \( \sigma_a[F] = 0 \) additional conditions on \( a_{n,j} \) are required. For example, suppose that \( |c_{m_0,0}| |a_{n,1}|^m \to +\infty \) and \( |a_{n,j}| = o\big(|a_{n,1}|^m\big) \) as \( n \to \infty \). We put

\[ \Sigma_n = \sum_{k_1 + \cdots + k_p = m, k_i \neq m} c_{k_1, \ldots, k_p} (a_{n,1})^{k_1} \cdots (a_{n,p})^{k_p} = \sum_{k_1 + \cdots + k_p = m} c_{k_1, \ldots, k_p} (a_{n,1})^{k_1} \cdots (a_{n,p})^{k_p} - c_{m_0,0} (a_{n,1})^m. \]

Since for each monomial of the polynomial \( \Sigma'_n \) the sum of the exponents is equal to \( m \), we have

\[ \frac{|a_{n,1}|^{k_1} \cdots |a_{n,p}|^{k_p}}{|a_{n,1}|^m} = \frac{|a_{n,2}|^{k_2} \cdots |a_{n,p}|^{k_p}}{|a_{n,1}|^{m - k_1}} \to 0, \quad n \to \infty, \]

and thus \( \Sigma'_n = o\big(|a_{n,1}|^m\big) \) as \( n \to \infty \). Therefore,

\[ |a_n| \geq |c_{m_0,0}| |a_{n,1}|^m - |\Sigma'_n| = |c_{m_0,0}| |a_{n,1}|^m - o\big(|a_{n,1}|^m\big) \geq \frac{|c_{m_0,0}|}{2} |a_{n,1}|^m, \]

\[ n \geq n_0, \]
whence $\sigma_n[F] = \alpha[F] \leq m \lim_{n \to \infty} \frac{1}{\ln |a_{n,1}|} = 0$, i.e. $\sigma_n[F] = 0$.

**Growth of Hadamard compositions of the genus $m$**

Since the polynomial $P(x_1, \ldots, x_p)$ is homogeneous of the degree $m \geq 1$, we have

$$a_n e^{\mu \lambda_n} = \sum_{k_1 + \cdots + k_p = m} c_{k_1 \cdots k_p} (a_{n,1} e^{\sigma \lambda_n})^{k_1} \cdots (a_{n,p} e^{\sigma \lambda_n})^{k_p}. \quad (6)$$

Let $\mu(\sigma, F) = \max \{|a_n| \exp \{\sigma \lambda_n\} : n \geq 0\}$ be the maximal term of series (1). We remark that $\mu(\sigma, F) \uparrow +\infty$ as $\sigma \uparrow 0$ if and only if $\lim_{n \to \infty} |a_n| = +\infty$. In what follows, we will assume that this condition is satisfied.

Since (6) implies

$$|a_n| e^{m \sigma \lambda_n} \leq \sum_{k_1 + \cdots + k_p = m} |c_{k_1 \cdots k_p}| (a_{n,1} e^{\sigma \lambda_n})^{k_1} \cdots (a_{n,p} e^{\sigma \lambda_n})^{k_p},$$

we have

$$\mu(m \sigma, F) \leq \sum_{k_1 + \cdots + k_p = m} |c_{k_1 \cdots k_p}| \mu(\sigma, F_1)^{k_1} \cdots \mu(\sigma, F_p)^{k_p},$$

whence for all $\sigma < 0$ enough large we get

$$\ln \mu(m \sigma, F) \leq \sum_{k_1 + \cdots + k_p = m} \ln \left(\sum_{k_1 + \cdots + k_p = m} |c_{k_1 \cdots k_p}| \mu(\sigma, F_1)^{k_1} \cdots \mu(\sigma, F_p)^{k_p}\right) + \ln (m + 1)$$

$$= \sum_{k_1 + \cdots + k_p = m} \left(\ln |c_{k_1 \cdots k_p}| + k_1 \ln \mu(\sigma, F_1) + \cdots + k_p \ln \mu(\sigma, F_p)\right) + \ln (m + 1) \quad (7)$$

$$= \sum_{k_1 + \cdots + k_p = m} (k_1 \ln \mu(\sigma, F_1) + \cdots + k_p \ln \mu(\sigma, F_p)) + C_1,$$

where $C_1 = \sum_{k_1 + \cdots + k_p = m} \ln^+ |c_{k_1 \cdots k_p}| + \ln (m + 1)$.

We remark also that if $|c_{m0\cdots0}| |a_{n,1}|^m \to +\infty$ and $|a_{n,j}| = o(|a_{n,1}|)$ as $n \to \infty$, then, as above, $|a_n| \geq |c_{m0\cdots0}| |a_{n,1}|^m / 2$ for $n \geq n_0$ and therefore for $1 \leq j \leq p$ and all $\sigma < 0$ enough large

$$\ln \mu(\sigma, F_j) \leq \ln \mu(\sigma, F_1) \leq \ln \mu(\sigma, F) + C_2. \quad (8)$$

From (7) and (8) it follows that we need a connection between the growth of $M(\sigma, F)$ and the growth of $\mu(\sigma, F)$. By Cauchy inequality $\mu(\sigma, F) \leq M(\sigma, F)$, the following lemmas contain estimates of $M(\sigma, F)$ from above. Such estimates depend on the behavior of the coefficients, the relative growth of the functions $\alpha$ and $\beta$, and the growth of the sequence $\Lambda$.

**Lemma 1 ([4]).** If $F \in S(\Lambda, 0)$ and

$$\lim_{n \to \infty} \frac{\ln n}{\ln |a_n|} = h < +\infty, \quad (9)$$

then for every $\varepsilon > 0$ and all $\sigma_0(\varepsilon) \leq \sigma < 0$

$$M(\sigma, F) \leq K(\varepsilon) \mu \left(\frac{1 - \varepsilon}{1 + h} \sigma, F\right)^{1+h+\varepsilon}, \quad K(\varepsilon) = \text{const} > 0.$$
Lemma 2 ([4]). If \( F \in S(\Lambda, 0), \alpha \in L_{si}, \beta \in L_{si} \) and \( c \in (0, +\infty) \)
\[
\alpha(\lambda_n) = o\left( \beta \left( \frac{\lambda_n}{\ln n} \right) \right), \quad n \to \infty,
\]
then for every \( \varepsilon \in (0, 1) \) and all \( \sigma_0(\varepsilon) \leq \sigma < 0 \)
\[
M(\sigma, F) \leq \mu \left( (1 - \varepsilon) \sigma, F \right) \exp \left\{ \alpha^{-1}(\varepsilon \beta(1/|\sigma|)) \right\}.
\]

In the following lemma there are no conditions on the functions \( \alpha \) and \( \beta \).

Lemma 3 ([13]). If \( F \in S(\Lambda, 0) \) and
\[
\lim_{n \to \infty} \frac{\ln n}{\lambda_n} = h_0 < \infty,
\]
where \( \gamma \) is a positive continuous decreasing to 0 function on \([0, +\infty)\) such that \( t \gamma(t) \uparrow +\infty \) as \( t \to +\infty \), then for every \( \varepsilon > 0 \) there exists \( K(\varepsilon) > 0 \) such that for all \( \sigma_0(\varepsilon) \leq \sigma < 0 \)
\[
M(\sigma, F) \leq \mu \left( \frac{\sigma}{1 + \varepsilon}, F \right) \left( \exp \left\{ \frac{\varepsilon|\sigma|}{1 + \varepsilon} \gamma^{-1} \left( \frac{\varepsilon|\sigma|}{(1 + \varepsilon)^2(h_0 + \varepsilon^2)} \right) \right\} + K(\varepsilon) \right).
\]

These lemmas imply the following statements.

Lemma 4. Let \( F \in S(\Lambda, 0), \phi_{a,b}^0[\mu(F)] = \lim_{\sigma \to 0} \frac{\alpha(\ln \mu(\sigma, F))}{\beta(1/|\sigma|)} \) and one of the following conditions be satisfied:

a) condition (9) holds and either \( \alpha \in L_{si}, \beta \in L_{si} \) and \( h < +\infty \) or \( \alpha \in L^0, \beta \in L^0 \) and \( h = 0 \);

b) \( \alpha \in L_{si}, \beta \in L^0 \) and condition (10) holds;

c) \( \alpha \in L_{si}, \beta \in L^0, \) condition (11) holds and \( \lim_{x \to +\infty} \frac{\alpha(\gamma^{-1}(1/x))}{\beta(x)} = 0. \)

Then \( \phi_{a,b}^0[F] = \phi_{a,b}^0[\mu(F)]. \)

Proof. If condition (9) with \( h < +\infty \) holds, then by Lemma 1 we get
\[
\ln \mu(\sigma, F) \leq \ln M(\sigma, F) \leq (1 + h + \varepsilon) \ln \mu \left( \frac{1 - \varepsilon}{1 + h}, \sigma, F \right) + o(1), \quad \sigma \uparrow 0,
\]
and if \( \alpha \in L_{si} \) and \( \beta \in L_{si} \), then
\[
\phi_{a,b}^0[\mu(F)] \leq \phi_{a,b}^0[F] \leq \lim_{\sigma \to 0} \frac{\alpha(\ln \mu \left( \frac{1 - \varepsilon}{1 + h}, \sigma, F \right))}{\beta(1/|\sigma|)} \leq \lim_{\sigma \to 0} \frac{\alpha(\ln \mu \left( \frac{1 - \varepsilon}{1 + h}, \sigma, F \right))}{\beta(1/|\sigma|)} \lim_{x \to +\infty} \frac{\beta \left( \frac{1 + h}{1 - \varepsilon} x \right)}{\beta(x)} = \phi_{a,b}^0[\mu(F)].
\]

In [15] it is proved that if \( \beta \in L^0, \) then \( \lim_{x \to +\infty} \beta((1 + \varepsilon)x)/\beta(x) = A(\varepsilon) \downarrow 1 \) as \( \varepsilon \downarrow 0 \) and
\[
\lim_{x \to +\infty} \beta(Kx)/\beta(x) = B(K) < +\infty \quad \text{for} \quad K = \text{const} > 0.
\]
Therefore, if $h = 0$, $\alpha \in L^0$ and $\beta \in L^0$, then

$$
\phi_{a,b}^o[\mu(F)] \leq \phi_{a,b}^o[F] \leq \lim_{\sigma \to 0} \frac{\alpha \left( (1 + \varepsilon) \ln \mu \left((1 - \varepsilon)\sigma, F\right) \right)}{\beta(1/|\sigma|)}
$$

$$
= \lim_{\sigma \to 0} \frac{\alpha \left( \ln \mu \left((1 - \varepsilon)\sigma, F\right) \right)}{\beta(1/(1 - \varepsilon)|\sigma|)} = \lim_{\sigma \to 0} \frac{\alpha \left( (1 + \varepsilon) \ln \mu \left((1 - \varepsilon)\sigma, F\right) \right)}{\beta(1/(1 - \varepsilon)|\sigma|)}
$$

$$
= \phi_{a,b}^o[\mu(F)] A_1(\varepsilon) A_2(\varepsilon),
$$

where $A_j(\varepsilon) \downarrow 1$ as $\varepsilon \downarrow 0$. Hence in view of the arbitrariness of $\varepsilon$ we obtain the equality

$$
\phi_{a,b}^o[F] = \phi_{a,b}^o[\mu(F)].
$$

In the case $b)$, by Lemma 2 we have

$$
\ln M(\sigma, F) \leq \ln \mu((1 - \varepsilon)\sigma, F) + \alpha^{-1}(\varepsilon\beta(1/|\sigma|))
$$

and since $\alpha \in L_{si}$, we get

$$
\alpha \left( \ln M(\sigma, F) \right) \leq \alpha \left( 2 \max \left\{ \ln \mu((1 - \varepsilon)\sigma, F), \alpha^{-1}(\varepsilon\beta(1/|\sigma|)) \right\} \right)
$$

$$
= (1 + o(1)) \alpha \left( \max \left\{ \ln \mu((1 - \varepsilon)\sigma, F), \alpha^{-1}(\varepsilon\beta(1/|\sigma|)) \right\} \right)
$$

$$
= (1 + o(1)) \max \left\{ \alpha \left( \ln \mu((1 - \varepsilon)\sigma, F) \right), \varepsilon\beta(1/|\sigma|) \right\}
$$

$$
\leq (1 + o(1)) \left( \alpha \left( \ln \mu((1 - \varepsilon)\sigma, F) \right) + \varepsilon\beta(1/|\sigma|) \right)
$$

as $\sigma \uparrow 0$, whence, as above, in view of the condition $\beta \in L^0$ we get

$$
\phi_{a,b}^o[\mu(F)] \leq \phi_{a,b}^o[F] \leq \phi_{a,b}^o[\mu(F)] A(\varepsilon) + \varepsilon,
$$

where $A(\varepsilon) \downarrow 1$ as $\varepsilon \downarrow 0$, and thus, in view of the arbitrariness of $\varepsilon$ we obtain the equality

$$
\phi_{a,b}^o[F] = \phi_{a,b}^o[\mu(F)].
$$

Finally, as above, from (12) we have

$$
\alpha \left( \ln \mu(\sigma, F) \right) \leq \alpha \left( \ln M(\sigma, F) \right)
$$

$$
\leq (1 + o(1)) \max \left\{ \alpha \left( \ln \mu \left(\frac{\sigma}{1 + \varepsilon}, F\right) \right), \alpha \left( \gamma^{-1} \left( \frac{\varepsilon|\sigma|}{(1 + \varepsilon)^2(h_0 + \varepsilon^2)} \right) \right) \right\}
$$

(13)

as $\sigma \uparrow 0$. Clearly,

$$
\lim_{\sigma \to 0} \frac{\alpha \left( \ln \mu \left(\frac{\sigma}{1 + \varepsilon}, F\right) \right)}{\beta(1/|\sigma|)} \leq \lim_{\sigma \to 0} \frac{\alpha \left( \ln \mu \left(\frac{\sigma}{1 + \varepsilon}, F\right) \right)}{\beta((1 + \varepsilon)/|\sigma|)} = \phi_{a,b}^o[\mu(F)] A(\varepsilon)
$$

and, similarly,

$$
\lim_{\sigma \to 0} \frac{\alpha \left( \gamma^{-1} \left( \frac{\varepsilon|\sigma|}{(1 + \varepsilon)^2(h_0 + \varepsilon^2)} \right) \right)}{\beta(1/|\sigma|)} \leq \lim_{\sigma \to 0} \frac{\alpha \left( \gamma^{-1}(|\sigma|) \right)}{\beta(1/|\sigma|)} = B \left( \frac{\varepsilon}{(1 + \varepsilon)^2(h_0 + \varepsilon^2)} \right).
$$

Since $A(\varepsilon) \downarrow 1$ as $\varepsilon \downarrow 0$, $B \left( \frac{\varepsilon}{(1 + \varepsilon)^2(h_0 + \varepsilon^2)} \right) < +\infty$ and

$$
\lim_{\sigma \to 0} \frac{\alpha \left( \gamma^{-1}(|\sigma|) \right)}{\beta(1/|\sigma|)} = 0,
$$

from (13) we obtain the equality

$$
\phi_{a,b}^o[F] = \phi_{a,b}^o[\mu(F)].
$$

Lemma 4 is proved.

□
Lemma 5. Let $F \in S(A, 0)$, $T_{a, \beta}^{01}[\mu(F)] = \lim_{\sigma \to 0} \frac{\ln \mu(\sigma, F)}{\alpha^{-1}(\sigma)}$, $\varrho = \sigma_T^{01}[F]$, $\alpha^{-1}(\sigma(x)) \in L^0$ for each $c \in (0, +\infty)$. If one of the conditions a) with $h = 0$, b), c) of Lemma 4 is satisfied, then $T_{a, \beta}^{01}[F] = T_{a, \beta}^{01}[\mu(F)]$.

Proof. If condition (9) holds with $h = 0$, then by Lemma 1 for every $\epsilon \in (0, 1)$ and all $\sigma \in (\sigma_0(\epsilon), 0)$ in view of the condition $\alpha^{-1}(\sigma(x)) \in L^0$ for each $c \in (0, +\infty)$ we have

$$T_{a, \beta}^{01}[\mu(F)] \leq T_{a, \beta}^{01}[F] \leq (1 + \epsilon) \lim_{\sigma \to 0} \frac{\ln \mu((1 - \epsilon)\sigma, F)}{\alpha^{-1}(\sigma)} + \frac{\alpha^{-1}(\varrho(1/(|\sigma|)))}{\alpha^{-1}(\sigma)} = (1 + \epsilon)T_{a, \beta}^{01}[\mu(F)]A(\epsilon),$$

where $A(\epsilon) \downarrow 1$ as $\epsilon \downarrow 0$. Thus, $T_{a, \beta}^{01}[F] = T_{a, \beta}^{01}[\mu(F)]$.

If the condition b) holds, then by Lemma 2 as above we get

$$T_{a, \beta}^{01}[\mu(F)] \leq T_{a, \beta}^{01}[F] \leq \lim_{\sigma \to 0} \frac{\ln \mu((1 - \epsilon)\sigma, F)}{\alpha^{-1}(\sigma)} + \frac{\alpha^{-1}(\varrho(1/(|\sigma|)))}{\alpha^{-1}(\sigma)} \leq T_{a, \beta}^{01}[\mu(F)]A(\epsilon) + A^*(\epsilon),$$

where $A(\epsilon) \downarrow as \epsilon \downarrow 0$ and the condition $a \in L^1$ implies $A^*(\epsilon) = \lim_{\epsilon \to +\infty} \frac{\alpha^{-1}(\epsilon x)}{\alpha^{-1}(\sigma)} = 0$ for $\epsilon < \sigma$. Therefore, in view of the arbitrariness of $\epsilon$ the equality $T_{a, \beta}^{01}[F] = T_{a, \beta}^{01}[\mu(F)]$ is correct. Finally, as above, from (12) we have

$$\ln M(\sigma, F) \leq \ln \mu \left( \frac{\sigma}{1 + \epsilon \sigma} \right) + \gamma^{-1} \left( \frac{\epsilon \sigma}{(1 + \epsilon)^2(h_0 + \epsilon^2)} \right) + K_1(\epsilon),$$

whence, as above, $T_{a, \beta}^{01}[\mu(F)] \leq T_{a, \beta}^{01}[F] \leq T_{a, \beta}^{01}[\mu(F)]A(\epsilon) + C(\epsilon)$, where $A(\epsilon) \to 1$ as $\epsilon \downarrow 0$ and

$$C(\epsilon) = \lim_{\sigma \to 0} \frac{\gamma^{-1} \left( \frac{\epsilon \sigma}{(1 + \epsilon)^2(h_0 + \epsilon^2)} \right)}{\alpha^{-1}(\sigma)} \leq \lim_{\sigma \to 0} \frac{\gamma^{-1}(1/|\sigma|)}{\alpha^{-1}(\sigma)} \frac{\varrho(1/|\sigma|)}{\epsilon|\sigma|} \leq C(\epsilon) \leq \lim_{\sigma \to 0} \frac{\gamma^{-1}(1/|\sigma|)}{\alpha^{-1}(\sigma)} \frac{\varrho(1/|\sigma|)}{\epsilon|\sigma|} \leq \frac{\alpha^{-1}(\varrho(1/|\sigma|))}{\alpha^{-1}(\sigma)} \frac{\gamma^{-1}(1/|\sigma|)}{\varrho(1/|\sigma|)}.$$
Theorem 1. Let \( \alpha^{-1}(c\beta(x)) \in L^0 \) for each \( c \in (0, +\infty) \) and one of the conditions a) with \( h = 0, b), c) \) of Lemma 4 is satisfied.

Suppose that for \( 1 \leq j \leq p \) the functions \( F_j \in S(\Lambda, 0) \) have the same generalized order \( \varrho_{\alpha, \beta}[F_j] = \varrho \in (0, +\infty) \) and the types \( T_{a, \beta}^{01}[F_j] = T_j^{01} \in [0, +\infty) \) such that \( |a_{n,1}|^m \to +\infty \) and \( |a_{n,j}| = o(|a_{n,1}|) \) as \( n \to \infty \) for \( 2 \leq j \leq p \).

If the function \( F \in S(\Lambda, 0) \) is the Hadamard composition of the genus \( m \geq 1 \) of the functions \( F_j \) and either \( m = 1 \) or \( \beta \in L_{sir} \) then \( \varrho_{a, \beta}[F] = \varrho \) and

\[
T_{a, \beta}^{01}[F] = Q(m) \sum_{k_1 + \cdots + k_p = m} \left( k_1 T_1^{01} + \cdots + k_p T_p^{01} \right),
\]

where \( Q(m) := \lim_{x \to +\infty} \frac{\alpha^{-1}(q\beta(mx))}{\alpha^{-1}(q\beta(x))} = 1 \) if either \( m = 1 \) or \( \alpha^{-1}(q\beta(x)) \in L_{sir} \).

Proof. Since the functions \( F_j \in S(\Lambda, 0) \) have the same generalized order \( \varrho_{a, \beta}[F_j] = \varrho \in (0, +\infty) \), for every \( \varrho_1 > \varrho \) and all \( \sigma \in [\sigma_0(\varrho_1), 0) \) we have \( \ln \mu(\sigma, F_j) \leq \alpha^{-1}(\varrho_1 \beta(1/|\sigma|)) \). Therefore, from (7) we get

\[
\ln \mu(m\sigma, F) \leq \sum_{k_1 + \cdots + k_p = m} (k_1 + \cdots + k_p) \alpha^{-1}(\varrho_1 \beta(1/|\sigma|)) + C_1 = C_2 \alpha^{-1}(\varrho_1 \beta(1/|\sigma|)) + C_1,
\]

where \( C_1 = \sum_{k_1 + \cdots + k_p = m} \ln |c_{k_1 \cdots k_p}| + \ln (m + 1) \) and \( C_2 = \sum_{k_1 + \cdots + k_p = m} (k_1 + \cdots + k_p) \). Since \( \alpha \in L_{sir} \), it follows that

\[
\varrho_{a, \beta}[\mu(F)] = \lim_{\sigma \to 0^+} \frac{\alpha(\ln \mu(m\sigma, F))}{\beta(1/(m|\sigma|))} \leq \lim_{\sigma \to 0^+} \frac{\alpha(\varrho_1 \beta(1/|\sigma|))}{\beta(1/(m|\sigma|))} = \varrho_1 \lim_{x \to +\infty} \frac{\beta(mx)}{\beta(x)} = \varrho_1,
\]

provided either \( m = 1 \) or \( \beta \in L_{sir} \). In view of the arbitrariness of \( \varrho \) we get \( \varrho_{a, \beta}[\mu(F)] \leq \varrho \).

If \( |c_{m0...0}| |a_{n,1}|^m \to +\infty \) and \( |a_{n,j}| = o(|a_{n,1}|) \) as \( n \to \infty \) then from (8) we get \( \varrho_{a, \beta}[\mu(F_j)] \leq \varrho_{a, \beta}[\mu(F)] \). Thus, \( \varrho_{a, \beta}[\mu(F)] = \varrho_{a, \beta}[\mu(F_j)] \) and by Lemma 4 \( \varrho_{a, \beta}[F] = \varrho \).

Also, from (7) we obtain

\[
T_{a, \beta}^{01}[\mu(F)] = \lim_{\sigma \to 0^+} \frac{\ln \mu(m\sigma, F)}{\alpha^{-1}(\varrho_1 \beta(1/|\sigma|))} \leq \sum_{k_1 + \cdots + k_p = m} \left( k_1 \lim_{\sigma \to 0^+} \frac{\ln \mu(\sigma, F_1)}{\alpha^{-1}(\varrho_1 \beta(1/|\sigma|))} + \cdots + k_p \lim_{\sigma \to 0^+} \frac{\ln \mu(\sigma, F_p)}{\alpha^{-1}(\varrho_1 \beta(1/|\sigma|))} \right) \lim_{\sigma \to 0^+} \frac{\alpha^{-1}(\varrho_1 \beta(1/|\sigma|))}{\alpha^{-1}(\varrho_1 \beta(1/(m|\sigma|)))}
\]

\[
= Q(m) \sum_{k_1 + \cdots + k_p = m} \left( k_1 T_{a, \beta}^{02}[\mu(F_1)] + \cdots + k_p T_{a, \beta}^{01}[\mu(F_p)] \right).
\]

Since by Lemma 5 we have \( T_{a, \beta}^{01}[\mu(F)] = T_{a, \beta}^{01}[F] \) and \( T_{a, \beta}^{01}[\mu(F_j)] = T_{a, \beta}^{01}[F_j] \), the proof of Theorem 1 is complete.

Since \( \varrho_{a, \beta}[F] = \lim_{\sigma \to 0^+} \frac{\ln \exp \left\{ \frac{\alpha(\ln M(\sigma, F))}{\beta(1/|\sigma|)} \right\}}{\ln \exp \left\{ \frac{\alpha(\ln M(\sigma, F))}{\beta(1/|\sigma|)} \right\}} \), the generalized type can be determined [9] by the formula

\[
T_{a, \beta}^{02}[F] = \lim_{\sigma \to 0^+} \frac{\exp \left\{ \frac{\alpha(\ln M(\sigma, F))}{\beta(1/|\sigma|)} \right\}}{\exp \left\{ \frac{\alpha(\ln M(\sigma, F))}{\beta(1/|\sigma|)} \right\}}, \quad \varrho = \varrho_{a, \beta}[F].
If we put $\alpha_1(x) = \exp \{ \alpha(x) \}$ and $\beta_1(x) = \exp \{ \epsilon \beta(x) \}$, then $T^{02}_{\alpha_1, \beta_1}[F] = \epsilon^{02}_{\alpha_1, \beta_1}[F]$. Therefore, Lemma 3 implies the following statement.

**Lemma 6.** Let $F \in S(\Lambda, 0), 0 < \epsilon < +\infty$, $T^{02}_{\alpha, \beta}[\mu(F)] = \lim_{\sigma \to 0} \exp \left\{ \alpha \left( \ln \mu(\sigma, F) \right) \right\}$ and one of the following conditions be satisfied:

**d)** condition (9) holds and either $\epsilon^a \in L_{si}, \epsilon^b \in L_{si}$ and $h < +\infty$ or $\epsilon^a \in L^0, \epsilon^b \in L^0$ and $h = 0$;

**e)** $\epsilon^a \in L_{si}, \epsilon^b \in L^0$ and

$$\exp \{ \alpha(\lambda_n) \} = o \left( \exp \left\{ \epsilon \beta \left( \frac{\lambda_n}{\ln n} \right) \right\} \right), \quad n \to \infty;$$

(14)

**f)** $e^a \in L_{si}, e^b \in L^0$, condition (12) holds and $\lim_{x \to +\infty} \exp \left\{ \alpha \left( \gamma^{-1}(1/x) \right) \right\}$ = 0.

Then $T^{02}_{\alpha, \beta}[F] = T^{02}_{\alpha, \beta}[\mu(F)]$.

Using Lemmas 4 and 6 we prove the following theorem.

**Theorem 2.** Let one of the conditions **d** of Lemma 6 and **b), c** of Lemma 4 is satisfied.

Suppose that for $1 \leq j \leq p$ the functions $F_j \in S(\Lambda, 0)$ have the same generalized order $\epsilon^{0}_{\alpha, \beta}[F_j] = \epsilon \in (0, +\infty)$ and the types $T^{02}_{\alpha, \beta}[F_j] = T^{02}_{j} \in [0, +\infty), |c_{m_0...0}| a_{n,j} |m \to +\infty$ and $|a_{n,j}| = o |a_{n,1}|$ as $n \to \infty$ for $2 \leq j \leq p$.

If the function $F \in S(\Lambda, 0)$ is the Hadamard composition of the genus $m \geq 1$ of the functions $F_j$ and either $m = 1$ or $\beta \in L_{si}$ then $\epsilon^{0}_{\alpha, \beta}[F] = \epsilon$ and

$$T^{02}_{\alpha, \beta}[F] \leq Q(m) \sum_{k_1 + \cdots + k_p = m} \left( k_1 \epsilon^{01}_{1} + \cdots + k_p \epsilon^{01}_{p} \right),$$

where $Q_2(m) = \lim_{\sigma \to 0} \exp \left\{ \epsilon \beta (1/|\sigma|) \right\}$ = 1 if either $m = 1$ or $\exp \{ \epsilon \beta(x) \} \in L_{si}$.

**Proof.** Clearly, if $\epsilon^a \in L_{si} \quad (\epsilon^a \in L^0)$, then $\alpha \in L_{si} \quad (\alpha \in L^0)$ and, therefore, the condition **d** of Lemma 6 implies the condition **a** of Lemma 4. Then by Lemma 4 for each function $F \in S(\Lambda, 0)$ we have $\epsilon^{0}_{\alpha, \beta}[F] = \epsilon^{0}_{\alpha, \beta}[\mu(F)]$. If condition (10) holds, then for every $\epsilon \in (0, \epsilon)$ and all $n \geq n_0(\epsilon)$ we have $\alpha(\lambda_n) \leq \epsilon \beta(\lambda_n / \ln n)$ and, thus,

$$\exp \{ \alpha(\lambda_n) \} \leq \exp \{ \epsilon \beta(\lambda_n / \ln n) \}$$

$$= \exp \{ \epsilon^{0}_{\beta}(\lambda_n / \ln n) \} \exp \{ -(\epsilon - \epsilon) \beta(\lambda_n / \ln n) \} = o \left( \exp \{ \epsilon \beta(\lambda_n / \ln n) \} \right)$$

as $n \to \infty$, i.e. (14) holds. Similarly, the condition $\lim_{x \to +\infty} \frac{\alpha \left( \gamma^{-1}(1/x) \right)}{\beta(x)} = 0$ implies the condition

$$\lim_{x \to +\infty} \frac{\exp \left\{ \alpha \left( \gamma^{-1}(1/x) \right) \right\}}{\exp \{ \epsilon \beta(x) \}} = 0.$$
Theorem 3. Let \( \alpha^{-1}(c\beta(x)) \in L^0 \) for each \( c \in (0, +\infty) \) and one of the conditions a), b), c) of Lemma 4 is satisfied. If all functions \( F_j \in S(\Lambda, 0) \) have the same generalized order \( \varphi_{a,\beta}^0[F_j] = \varphi \in (0, +\infty) \) and belong to \( C_0^0 \), then the Hadamard composition \( F \in S(\Lambda, 0) \) of the genus \( m \geq 1 \) of the functions \( F_j \) has the generalized order \( \varphi_{a,\beta}^0[F] = \varphi \in (0, +\infty) \) and belongs to \( C_0^0 \). If, in addition, \( |c_{m0...0}|a_{n1}|^m \to +\infty \) and \( |a_{nj}| = o(|a_{n1}|) \) as \( n \to \infty \) for \( 2 \leq j \leq p \) then the condition \( F \in C_0^0 \) implies \( F_j \in C_0^0 \).

Proof. By Lemma 4 we have \( \varphi_{a,\beta}^0[F] = \varphi_{a,\beta}^0[\mu(F)] \) and \( \varphi_{a,\beta}^0[F_j] = \varphi_{a,\beta}^0[\mu(F_j)] \). Let us show that condition (4) is satisfied if and only if

\[
\int_{c_0}^{0} \frac{\ln \mu(\sigma, F)}{|\sigma|^{2\alpha-1}(\varphi(1/|\sigma|))} d\sigma < +\infty, \quad \varphi = \varphi_{a,\beta}^0[F]. \tag{15}
\]

Indeed, by Cauchy inequality condition (4) implies (15). On the other hand, if (9) holds then by Lemma 1

\[
\int_{c_0}^{0} \frac{\ln \mu(\sigma, F)}{|\sigma|^{2\alpha-1}(\varphi(1/|\sigma|))} d\sigma \leq (1 + h + \epsilon) \int_{c_0}^{0} \frac{\ln \mu \left( \frac{1-h}{\epsilon} \sigma, F \right)}{|\sigma|^{2\alpha-1}(\varphi(1/|\sigma|))} d\sigma + \int_{c_0}^{0} \frac{\ln K(\epsilon)}{|\sigma|^{2\alpha-1}(\varphi(1/|\sigma|))} d\sigma,
\]

whence in view of the condition \( h^{-1}(c\beta(x)) \in L^0 \) for each \( c \in (0, +\infty) \) it follows that (15) implies (4).

If (19) holds, then by Lemma 2 for \( \epsilon < \varphi \)

\[
\int_{c_0}^{0} \frac{\ln M(\sigma, F)}{|\sigma|^{2\alpha-1}(\varphi(1/|\sigma|))} d\sigma \leq \int_{c_0}^{0} \frac{\ln \mu((1-\epsilon)\sigma, F)}{|\sigma|^{2\alpha-1}(\varphi(1/|\sigma|))} d\sigma + \int_{c_0}^{0} \frac{d\sigma}{|\sigma|^{2\alpha-1}((\varphi - \epsilon)\beta(1/|\sigma|))},
\]

whence as above it follows that (15) implies (4).
Finally, if (11) holds, then by Lemma 3
\[
\int_{c_0}^{0} \frac{\ln M(\sigma, F)}{|\sigma|^{2\alpha^{-1}(q\sigma(1/|\sigma|))}} \, d\sigma \leq \int_{c_0}^{0} \frac{\ln \mu(\sigma/(1+\epsilon), F)}{|\sigma|^{2\alpha^{-1}(q\sigma(1/|\sigma|))}} \, d\sigma + \int_{c_0}^{0} \frac{\gamma^{-1}(\delta \sigma)}{|\sigma|^{2\alpha^{-1}(q\sigma(1/|\sigma|))}} \, d\sigma,
\]
where \(\delta = \frac{e(1+x)^{-2}}{(\ln x + x^2)}\). The condition \(\lim_{x \to +\infty} a(\gamma^{-1}(1/x)) = 0\) implies \(\gamma^{-1}(1/x) \leq a^{-1}(e\beta(x))\) for every \(\epsilon > 0\) and all \(x \geq x_0(\epsilon)\). Therefore, as above, in view of the condition \(a^{-1}(c\beta(x)) \in L^0\) we get \(\int_{c_0}^{0} \frac{\gamma^{-1}(\delta \sigma)}{|\sigma|^{2\alpha^{-1}(q\sigma(1/|\sigma|))}} \, d\sigma \leq +\infty\) and, thus, (15) implies (4). The equivalence of the conditions (4) and (15) is proven.

If all \(F_j \in C_{\alpha,\beta}^0\) then, from (7) we obtain
\[
\int_{c_0}^{0} \frac{\ln \mu(m\sigma, F)}{|\sigma|^{2\alpha^{-1}(q\sigma(1/|\sigma|))}} \, d\sigma \leq \sum_{k_1+\ldots+k_p=m} \left( k_1 \int_{c_0}^{0} \frac{\ln \mu(\sigma, F_1)}{|\sigma|^{2\alpha^{-1}(q\sigma(1/|\sigma|))}} \, d\sigma + \ldots + k_p \int_{c_0}^{0} \frac{\ln \mu(\sigma, F_p)}{|\sigma|^{2\alpha^{-1}(q\sigma(1/|\sigma|))}} \, d\sigma \right) + C < +\infty,
\]
where \(C\) is some constant. Since \(a^{-1}(c\beta(x)) \in L^0\) for each \(c \in (0, +\infty)\), hence it follows that \(\int_{c_0}^{0} \frac{\ln \mu(m\sigma, F)}{|\sigma|^{2\alpha^{-1}(q\sigma(1/|\sigma|))}} \, d\sigma < +\infty\) and, thus, \(F \in C_{\alpha,\beta}^0\).

On the contrary, if \(|c_m, 0| \cdot |a_{n,1}|^m \to +\infty\) and \(|a_{n,j}| = o(|a_{n,1}|)\) as \(n \to \infty\) for \(2 \leq j \leq p\) and \(F \in C_{\alpha,\beta}^0\), then (8) implies
\[
\int_{c_0}^{0} \frac{\ln \mu(\sigma, F)}{|\sigma|^{2\alpha^{-1}(q\sigma(1/|\sigma|))}} \, d\sigma \leq \int_{c_0}^{0} \frac{\ln \mu(\sigma, F)}{|\sigma|^{2\alpha^{-1}(q\sigma(1/|\sigma|))}} \, d\sigma \leq \int_{c_0}^{0} \frac{\ln \mu(\sigma, F)}{|\sigma|^{2\alpha^{-1}(q\sigma(1/|\sigma|))}} \, d\sigma + C < +\infty,
\]
where \(C\) is some constant, i.e. all \(F_j \in C_{\alpha,\beta}^0\). The proof of Theorem 3 is complete.

In addition to class \(C_{\alpha,\beta}^0\) to study the growth of functions \(F \in S(\Lambda, 0)\) one can use the generalized convergence class \(C_{\alpha,\beta}^0\) defined in [11] by the condition
\[
\int_{c_0}^{0} \frac{\alpha(\ln M(\sigma, F))}{|\sigma|^{2\beta(1/|\sigma|)}} \, d\sigma < +\infty, \quad \alpha \in L, \beta \in L.
\] (16)

In order for (16) to hold, it is necessary that \(\int_{c_0}^{0} \frac{d\sigma}{|\sigma|^{2\beta(1/|\sigma|)}} < +\infty\), that is \(\int_{x_0}^{\infty} \frac{dx}{\beta(x)} < +\infty\). Now we prove the following theorem.

**Theorem 4.** Let \(\alpha \in L^0, \beta \in L^0\), one of the following conditions be satisfied: a) (10) holds; b) (12) holds and \(\int_{x_0}^{\infty} \frac{d\gamma^{-1}(1/x)}{\beta(x)} < +\infty\). Suppose that the function \(F \in S(\Lambda, 0)\) is Hadamard composition of the genus \(m \geq 1\) of the functions \(F_j \in S(\Lambda, 0)\). If \(F_j \in C_{\alpha,\beta}^0\) for all \(1 \leq j \leq p\), then \(F \in C_{\alpha,\beta}^0\). If \(F \in C_{\alpha,\beta}^0\) and, in addition, \(|c_m, 0| \cdot |a_{n,1}|^m \to +\infty\) and \(|a_{n,j}| = o(|a_{n,1}|)\) as \(n \to \infty\) for \(2 \leq j \leq p\), then \(F_j \in C_{\alpha,\beta}^0\) for all \(1 \leq j \leq p\).

**Proof.** At first, we show that condition (16) holds if and only if
\[
\int_{c_0}^{0} \frac{\alpha(\ln \mu(\sigma, F))}{|\sigma|^{2\beta(1/|\sigma|)}} \, d\sigma < +\infty.
\] (17)
Indeed, by Cauchy inequality condition (16) implies (17). On the other hand, if (9) holds, then since $\alpha \in L^0$ and $\beta \in L^0$, by Lemma 1 we have

$$
\int_{c_0}^{0} \frac{\alpha}{|\sigma|^2\beta(1/|\sigma|)} d\sigma \leq \int_{c_0}^{0} \frac{\alpha}{|\sigma|^2\beta(1/|\sigma|)} d\sigma \\
\leq K_1 \int_{c_0}^{0} \frac{\alpha}{|\sigma|^2\beta(1/|\sigma|)} d\sigma \leq K_1 K_2 \int_{c_0}^{0} \frac{\alpha}{|\sigma|^2\beta(1/|\sigma|)} d\sigma < +\infty,
$$
i.e. relations (16) and (17) are equivalent. The equivalence of relations (16) and (17) in the case of condition b) is proved in [12].

If all $F_j \in C^0_{\alpha,\beta}$, then from (7) we obtain

$$
\int_{c_0}^{0} \frac{\alpha}{|\sigma|^2\beta(1/|\sigma|)} d\sigma \\
\leq \sum_{k_1 + \cdots + k_p = m} \left( k_1 \int_{c_0}^{0} \frac{\alpha}{|\sigma|^2\beta(1/|\sigma|)} d\sigma + \cdots + k_p \int_{c_0}^{0} \frac{\alpha}{|\sigma|^2\beta(1/|\sigma|)} d\sigma \right) + \text{const} < +\infty.
$$

Since $\beta \in L^0$, hence (17) follows and, thus, $F \in C^0_{\alpha,\beta}$.

On the contrary, if $|c_{m0..0}|/|a_{n,j}| \to +\infty$ and $|a_{n,j}| = o(|a_{n,1}|)$ as $n \to \infty$ for $2 \leq j \leq p$ and $F \in C^0_{\alpha,\beta}$, then (8) implies

$$
\int_{c_0}^{0} \frac{\alpha}{|\sigma|^2\beta(1/|\sigma|)} d\sigma \leq \int_{c_0}^{0} \frac{\alpha}{|\sigma|^2\beta(1/|\sigma|)} d\sigma \leq \int_{c_0}^{0} \frac{\alpha}{|\sigma|^2\beta(1/|\sigma|)} d\sigma + \text{const} < +\infty,
$$
i.e. all $F_j \in C^0_{\alpha,\beta}$. The proof of Theorem 4 is complete.

3 Pseudostarlikeness and pseudoconvexity

Here we consider Dirichlet series of the form

$$
F(s) = e^{\alpha s} - \sum_{n=1}^{\infty} a_n \exp\{s\lambda_n\}, \quad a_n > 0, \quad \lambda_1 > h > 0,
$$
absolutely convergent in half-plane $\Pi_0 = \{ s : \Re s < 0 \}$.

A conformal function (18) in $\Pi_0$ is said [16] to be pseudostarlike of the order $\alpha \in [0, h)$ and the type $\beta > 0$ if

$$
\left| \frac{F'(s)}{F(s)} - h \right| < \beta \left| \frac{F'(s)}{F(s)} - (2\alpha - h) \right|, \quad s \in \Pi_0,
$$
and is said to be pseudoconvex of the order $\alpha \in [0, h)$ and the type $\beta > 0$ if

$$
\left| \frac{F''(s)}{F'(s)} - h \right| < \beta \left| \frac{F''(s)}{F'(s)} - (2\alpha - h) \right|, \quad s \in \Pi_0.
$$

If $\beta = 1$ we obtain the definition of the pseudostarlikeness and the pseudoconvexity of the order $\alpha \in [0, h)$ and if $\beta = 1$ and $\alpha = 0$ we obtain the definition [7,17] of the pseudostarlikeness and the pseudoconvexity of function (18).

We need the following lemma.
Lemma 7 ([16]). Function (18) is pseudostarlike of the order \( \alpha \in [0, h) \) and the type \( \beta > 0 \) if and only if
\[
\sum_{n=1}^{\infty} \{ (1 + \beta) \lambda_n - 2 \beta \alpha - h(1 - \beta) \} a_n \leq 2 \beta (h - \alpha),
\]
and is pseudoconvex of the order \( \alpha \in [0, h) \) and the type \( \beta > 0 \) if and only if
\[
\sum_{n=1}^{\infty} \lambda_n \{ (1 + \beta) \lambda_n - 2 \beta \alpha - h(1 - \beta) \} a_n \leq 2 \beta h (h - \alpha).
\]

Let Dirichlet series
\[
F_j(s) = e^{\lambda} - \sum_{n=1}^{\infty} a_{n,j} \exp \{ s \lambda_n \}, \quad a_{n,j} > 0 (1 \leq j \leq p), \quad \lambda_1 > h > 0,
\]
be absolutely convergent in \( \Pi_0 = \{ s : \Re s < 0 \} \). We say that function (18) is a positive Hadamard composition of genus \( m \) of Dirichlet series (22) if \( a_n = P(a_{n,1}, \ldots, a_{n,p}) \), where \( P(x_1, \ldots, x_p) \) is a homogeneous polynomial of degree \( m \geq 1 \) with non-negative coefficients.

Using Lemma 7, we can find conditions under which the pseudostarlikeness (pseudoconvexity) of all functions \( F_j \) implies the pseudostarlikeness (pseudoconvexity) of the function \( F \). We will have to distinguish between cases \( m = 1 \) and \( m \geq 2 \). Remark that a homogeneous polynomial of the degree \( m = 1 \) has the form \( P(x_1, \ldots, x_p) = c_1 x_1 + \cdots + c_p x_p \).

Theorem 5. Let functions (22) are pseudostarlike of the order \( \alpha_j \in [0, h) \) for all \( 1 \leq j \leq p \) and function (18) is a positive Hadamard composition of genus \( m \) of functions (22). If either \( m = 1 \) and \( 0 < c_1 + \cdots + c_p \leq 1 \) or \( m \geq 2 \) and \( \sum_{n=1}^{\infty} (h/\lambda_n)^{m-1} \sum_{k_1+\cdots+k_p=m} c_{k_1 \cdots k_p} \leq 1 \), then function (18) is pseudostarlike of the order \( \alpha = \min_{1 \leq j \leq p} \alpha_j \).

Proof. If the function \( F_j \) is pseudostarlike of the order \( \alpha_j \in [0, h) \) then by Lemma 7 we have \( \sum_{n=1}^{\infty} (\lambda_n - \alpha_j) a_{n,j} \leq h - \alpha_j \) and, thus,
\[
a_{n,j} \leq \frac{h - \alpha_j}{\lambda_n - \alpha_j} < 1, \quad j = 1, 2.
\]

Therefore, if \( F \) is the positive Hadamard composition of genus \( m = 1 \) of the functions \( F_j \), that is \( a_n = c_1 a_{n,1} + \cdots + c_p a_{n,p} \), then
\[
\sum_{n=1}^{\infty} (\lambda_n - \alpha) a_n = c_1 \sum_{n=1}^{\infty} (\lambda_n - \alpha) a_{n,1} + \cdots + c_p \sum_{n=1}^{\infty} (\lambda_n - \alpha) a_{n,p} = c_1 \sum_{n=1}^{\infty} \frac{\lambda_n - \alpha}{\lambda_n - \alpha_1} (\lambda_n - \alpha_1) a_{n,1} + \cdots + c_p \sum_{n=1}^{\infty} \frac{\lambda_n - \alpha}{\lambda_n - \alpha_p} (\lambda_n - \alpha_p) a_{n,p}
\]
\leq c_1 \sum_{n=1}^{\infty} \frac{h - \alpha}{h - \alpha_1} (\lambda_n - \alpha_1) a_{n,1} + \cdots + c_p \sum_{n=1}^{\infty} \frac{h - \alpha}{h - \alpha_p} (\lambda_n - \alpha_p) a_{n,p}
\leq c_1 (h - \alpha) + \cdots + c_p (h - \alpha) = (c_1 + \cdots + c_p)(h - \alpha) \leq h - \alpha,
\]
i.e. the function \( F \) is pseudostarlike of the order \( \alpha = \min_{1 \leq j \leq p} \alpha_j \).
Now let $m \geq 2$. Then $a_n = \sum_{k_1 + \cdots + k_p = m} c_{k_1 \cdots k_p} a_{n_1} \cdots a_{n_p}$ and in view of (23) we have
\[
\sum_{n=1}^{\infty} \{ \lambda_n - \alpha \} a_n \leq \sum_{n=1}^{\infty} \{ \lambda_n - \alpha \} \sum_{k_1 + \cdots + k_p = m} c_{k_1 \cdots k_p} \frac{(h - \alpha_1)}{\lambda_n - \alpha_1} \cdots \frac{(h - \alpha_p)}{\lambda_n - \alpha_p}
\leq \sum_{n=1}^{\infty} \{ \lambda_n - \alpha \} \sum_{k_1 + \cdots + k_p = m} c_{k_1 \cdots k_p} \frac{(h - \alpha)}{\lambda_n - \alpha}
= \sum_{n=1}^{\infty} \{ \lambda_n - \alpha \} \sum_{k_1 + \cdots + k_p = m} c_{k_1 \cdots k_p} \frac{(h - \alpha)^m}{\lambda_n - \alpha}
= \sum_{n=1}^{\infty} \left( \frac{h - \alpha}{\lambda_n - \alpha} \right)^{m-1} (h - \alpha) \sum_{k_1 + \cdots + k_p = m} c_{k_1 \cdots k_p}
\leq \sum_{n=1}^{\infty} \left( \frac{h}{\lambda_n} \right)^{m-1} (h - \alpha) \sum_{k_1 + \cdots + k_p = m} c_{k_1 \cdots k_p} \leq h - \alpha,
\]
i.e. the function $F$ is pseudostarlike of the order $\alpha = \min_{1 \leq j \leq p} \alpha_j$. Theorem 5 is proved. \qed

Using (21) and repeating the proof of Theorem 5, it is easy to prove the following assertion.

**Proposition 1.** Let functions (22) are pseudoconvex of the order $\alpha_j \in [0, h)$ for all $1 \leq j \leq p$, and function (18) is a positive Hadamard composition of genus $m$ of functions (22). If either $m = 1$ and $0 < c_1 + \cdots + c_p \leq 1$ or $m \geq 2$ and $\sum_{n=1}^{\infty} (h/\lambda_n)^{2(m-1)} \sum_{k_1 + \cdots + k_p = m} c_{k_1 \cdots k_p} \leq 1$, then function (18) is pseudoconvex of the order $\alpha = \min_{1 \leq j \leq p} \alpha_j$.

For pseudostarlike functions of the order $\alpha \in [0, h)$ and the type $\beta \geq 0$ the following theorem is true.

**Theorem 6.** Let functions (22) are pseudostarlike of the order $\alpha \in [0, h)$ and the type $\beta_j \geq 0$ for all $1 \leq j \leq p$, and function (18) is a positive Hadamard composition of genus $m$ of functions (22). If either $m = 1$ and $0 < c_1 + \cdots + c_p \leq 1$ or $m \geq 2$ and
\[
\left( \sum_{k_1 + \cdots + k_p = m} c_{k_1 \cdots k_p} \right)^\infty \left( \frac{2h}{\lambda_n - h} \right)^{m-1} \leq 1
\]
then function (18) is pseudostarlike of the order $\alpha$ and the type $\beta = \max_{1 \leq j \leq p} \beta_j$.

**Proof.** Let the functions $F_j$ are pseudostarlike of the order $\alpha_j \in [0, h)$ and the type $\beta_j \geq 0$. If $F$ is a positive Hadamard composition of genus $m = 1$ of the functions $F_j$, then in view of (20) we get
\[
\sum_{n=1}^{\infty} \{ (1 + \beta) \lambda_n - 2 \beta \alpha - h(1 - \beta) \} a_n
= c_1 \sum_{n=1}^{\infty} \{ (1 + \beta) \lambda_n - 2 \beta \alpha - h(1 - \beta) \} a_{n_1} + \cdots + c_p \sum_{n=1}^{\infty} \{ (1 + \beta) \lambda_n - 2 \beta \alpha - h(1 - \beta) \} a_{n_p}
= c_1 \sum_{n=1}^{\infty} \frac{(1 + \beta_1) \lambda_n - 2 \beta_1 \alpha - h(1 - \beta_1)}{(1 + \beta_1) \lambda_n - 2 \beta_1 \alpha - h(1 - \beta_1)} \{ (1 + \beta_1) \lambda_n - 2 \beta_1 \alpha - h(1 - \beta_1) \} a_{n_1} + \cdots
+ c_p \sum_{n=1}^{\infty} \frac{(1 + \beta_p) \lambda_n - 2 \beta_p \alpha - h(1 - \beta_p)}{(1 + \beta_p) \lambda_n - 2 \beta_p \alpha - h(1 - \beta_p)} \{ (1 + \beta_p) \lambda_n - 2 \beta_p \alpha - h(1 - \beta_p) \} a_{n_p}
\]
\[
\leq c_1 \sum_{n=1}^{\infty} \frac{\beta}{\beta_1} \{(1 + \beta_1)\lambda_n - 2\beta_1 \alpha - h(1 - \beta_1)\} a_{n,1} + \ldots
+ c_p \sum_{n=1}^{\infty} \frac{\beta}{\beta_p} \{(1 + \beta_p)\lambda_n - 2\beta_p \alpha - h(1 - \beta_p)\} a_{n,p}
\leq c_1 \frac{\beta}{\beta_1} 2\beta_1 (h - \alpha) + \ldots + c_p \frac{\beta}{\beta_p} 2\beta_p (h - \alpha) = 2(c_1 + \ldots + c_p) \beta(h - \alpha) \leq 2\beta(h - \alpha),
\]
i.e. the function \( F \) is pseudostarlike of the order \( \alpha \) and the type \( \beta = \max_{1 \leq j \leq p} \beta_j \).

If \( F_j \) is the positive Hadamard composition of genus \( m \geq 2 \) of the functions \( F_j \), then in view of (20) we get
\[
a_n \leq \sum_{k_1 + \ldots + k_p = m} c_{k_1} \ldots \frac{2\beta_1 (h - \alpha)}{(1 + \beta_1)\lambda_n - 2\beta_1 \alpha - h(1 - \beta_1)} k_1 \ldots \frac{2\beta_p (h - \alpha)}{(1 + \beta_p)\lambda_n - 2\beta_p \alpha - h(1 - \beta_p)} k_p
\leq \sum_{k_1 + \ldots + k_p = m} c_{k_1} \ldots \frac{2\beta(h - \alpha)}{(1 + \beta)\lambda_n - 2\beta \alpha - h(1 - \beta)} k_1 \ldots \frac{2\beta(h - \alpha)}{(1 + \beta)\lambda_n - 2\beta \alpha - h(1 - \beta)} k_p
= \left( \frac{2\beta(h - \alpha)}{(1 + \beta)\lambda_n - 2\beta \alpha - h(1 - \beta)} \right)^m \sum_{k_1 + \ldots + k_p = m} c_{k_1} \ldots c_{k_p}.
\]
Hence, it follows that
\[
\sum_{n=1}^{\infty} \{(1 + \beta)\lambda_n - 2\beta \alpha - h(1 - \beta)\} a_n
\leq 2\beta(h - \alpha) \sum_{k_1 + \ldots + k_p = m} c_{k_1} \ldots c_{k_p} \sum_{n=1}^{\infty} \left( \frac{2\beta(h - \alpha)}{(1 + \beta)\lambda_n - 2\beta \alpha - h(1 - \beta)} \right)^{m-1}
\leq 2\beta(h - \alpha) \left( \sum_{k_1 + \ldots + k_p = m} c_{k_1} \ldots c_{k_p} \right) \sum_{n=1}^{\infty} \left( \frac{2h}{\lambda_n - h} \right)^{m-1} \leq 2\beta(h - \alpha),
\]
i.e. the function \( F \) is pseudostarlike of the order \( \alpha \) and the type \( \beta = \max_{1 \leq j \leq p} \beta_j \). Theorem 6 is proved.

Using (21) and repeating the proof of Theorem 6, it is easy to prove the following assertion.

**Proposition 2.** Let functions (22) are pseudoconvex of the order \( \alpha \in [0, h) \) and the type \( \beta_j \geq 0 \) for all \( 1 \leq j \leq p \) and function (18) is a positive Hadamard composition of genus \( m \) of functions (22). If either \( m = 1 \) and \( 0 < c_1 + \ldots + c_p \leq 1 \) or \( m \geq 2 \) and
\[
\left( \sum_{k_1 + \ldots + k_p = m} c_{k_1} \ldots c_{k_p} \right) \sum_{n=1}^{\infty} \left( \frac{2h}{\lambda_n - h} \right)^{2(m-1)} \leq 1
\]
then function (18) is pseudoconvex of the order \( \alpha \) and the type \( \beta = \max_{1 \leq j \leq p} \beta_j \).
References

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