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On Dirichlet series similar to Hadamard compositions in half-plane

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Let $F(s)=\sum\limits_{n=1}^{\infty}a_n\exp\{s\lambda_n\}$ and $F_j(s)=\sum\limits_{n=1}^{\infty}a_{n,j}\exp\{s\lambda_n\}$, $j=\overline{1,p}$, be Dirichlet series with exponents $0\le\lambda_n\uparrow+\infty$, $n\to\infty$, and the abscissas of absolutely convergence equal to 0. The function F is called Hadamard composition of the genus $m\ge 1$ of the functions F_j if $a_n=P(a_{n,1},\dots,a_{n,p})$, where $P(x_1,\dots,x_p)=\sum\limits_{k_1+\dots+k_p=m}c_{k_1\dots k_p}x_1^{k_1}\cdots x_p^{k_p}$ is a homogeneous polynomial of degree m. In terms of generalized orders and convergence classes the connection between the growth of the functions F_j and the growth of the Hadamard composition F of the genus $m\ge 1$ of F_j is investigated. The pseudostarlikeness and pseudoconvexity of the Hadamard composition of the genus $m\ge 1$ are studied.

Key words and phrases: Dirichlet series, Hadamard composition, generalized order, generalized type, generalized convergence class, pseudostarlikeness, pseudoconvexity.

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Introduction

Let $f_j(z)=\sum\limits_{n=0}^{\infty}a_{n,j}z^n$, $1\le j\le p$, be entire transcedental functions. As in [12], we say that the function $f(z)=\sum\limits_{n=0}^{\infty}a_nz^n$ is similar to the Hadamard composition of the functions f_j if $a_n=w(a_{n,1},\dots,a_{n,p})$ for all n, where $w:\mathbb{C}^p\to\mathbb{C}$ is some function. Clearly, if $p=2$ and $w(a_{n,1},a_{n,2})=a_{n,1}a_{n,2}$, then $f=(f_1*f_2)$ is the Hadamard composition (product) of the functions f_1 and f_2 (see [5]). Obtained by J. Hadamard properties of this composition find the applications in the theory of the analytic continuation of the functions represented by power series [1,6]. E.G. Calys [2] investigated the functions similar to Hadamard compositions with $|w(x,y)|=\sqrt{|xy|}$ and proved, in particular, the following theorem.

Theorem A ([2]). *Let entire functions f_j, $j=1,2$, have the same order $\varrho[f_j]=\varrho\in(0,+\infty)$ and types $\sigma[f_j]=\sigma_j$. Suppose that $a_{n,1}\ne 0$ and $|a_{n,2}|\ge|a_{n,1}|/l\big(1/|a_{n,1}|\big)$ for all $n\ge n_0$, where l is slowly varying function. If $|a_n|=\big(1+o(1)\big)\sqrt{|a_{n,1}||a_{n,2}|}$ as $n\to\infty$, then function f has the order $\varrho[f]=\varrho$ and the type $\sigma[f]\le\sqrt{\sigma_1\sigma_2}$.*

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In article [8], Theorem A is generalized to the case of entire Dirichlet series of finite generalized order, and instead of two entire functions f_1 and f_2 it is considered $n \geq 2$ entire Dirichlet series. Analogues of the results from [8] for Dirichlet series absolutely convergent in the half-plane are obtained in [9].

To formulate them, we denote by L the class of positive continuous functions α on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0)$ for $x \leq x_0$ and $0 < \alpha(x) \uparrow +\infty$ as $x_0 \leq x \uparrow +\infty$. We say that $\alpha \in L^0$ if $\alpha \in L$ and $\alpha\big((1+o(1))x\big) = \big(1+o(1)\big)\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_{si}$ if $\alpha \in L$ and $\alpha(cx) = \big(1+o(1)\big)\alpha(x)$ as $x \to +\infty$ for each $c \in (0, +\infty)$, i.e. α is slowly increasing function. Clearly, $L_{si} \subset L^0$.

Let $\Lambda = (\lambda_n)$ be an increasing to $+\infty$ sequence of nonnegative number, $S(\Lambda, 0)$ be the class of Diriclet series

$$F(s) = \sum_{n=1}^{\infty} a_n \exp\{s\lambda_n\}, \qquad s = \sigma + it, \tag{1}$$

with the sequence (λ_n) of exponents and the abscissa of absolutely convergence $\sigma_a[F] = 0$ and $M(\sigma, F) = \sup\big\{|F(\sigma + it)| : t \in \mathbb{R}\big\}$ for $\sigma \in (-\infty, 0)$.

If $\alpha \in L$, $\beta \in L$ and $F \in S(\Lambda, 0)$ then the value

$$\varrho^0_{\alpha,\beta}[F] = \varlimsup_{\sigma\uparrow 0} \frac{\alpha\big(\ln M(\sigma, F)\big)}{\beta\big(1/|\sigma|\big)}$$

is called [3] generalized order of F. If $\varrho^0_{\alpha,\beta}[F] \in (0, +\infty)$, the generalized type is defined [9] as

$$T^{01}_{\alpha,\beta}[F] = \varlimsup_{\sigma\uparrow 0} \frac{\ln M(\sigma, F)}{\alpha^{-1}\big(\varrho\beta(1/|\sigma|)\big)}, \qquad \varrho = \varrho^0_{\alpha,\beta}[F].$$

Theorem B ([9]). *Let $\beta \in L_{si}$, $\alpha(e^x) \in L^0$, $\alpha^{-1}\big(c\beta(x)\big) \in L_{si}$,*

$$\frac{x}{\beta^{-1}\big(c\alpha(x)\big)} \uparrow +\infty, \qquad \alpha\left(\frac{x}{\beta^{-1}\big(c\alpha(x)\big)}\right) \sim \alpha(x) \quad \text{as } \ x_0 \leq x \to +\infty \tag{2}$$

and $\ln n = o\Big(\alpha^{-1}\big(c\beta(\lambda_n)\big)\Big)$ as $n \to \infty$ for every $c \in (0, +\infty)$. Suppose that Dirichlet series $F_j \in S(\Lambda, 0)$ of the form

$$F_j(s) = \sum_{n=1}^{\infty} a_{n,j} \exp\{s\lambda_n\}, \qquad 1 \leq j \leq p, \tag{3}$$

have the same generalized order $\varrho^0_{\alpha,\beta}[F_j] = \varrho \in (0, +\infty)$, the types $T^{01}_{\alpha,\beta}[F_j] \in (0, +\infty)$, $|a_{n,1}| > 0$ for $n \geq n_0$ and $\ln\ln|a_{n,j}| \geq \big(1+o(1)\big)\ln\ln|a_{n,1}|$ as $n \to \infty$ for $2 \leq j \leq p$. If $\omega_j > 0$ with $\sum_{j=1}^{p} \omega_j = 1$ and $\ln|a_n| = \big(1+o(1)\big)\prod_{j=1}^{p} \ln^{\omega_j}|a_{n,j}|$, then function F of the form (1) *has the generalized order $\varrho^0_{\alpha,\beta}[F] = \varrho$ and the type $T^{01}_{\alpha,\beta}[F] \leq \prod_{j=1}^{p} T^{01}_{\alpha,\beta}[F_j]^{\omega_j}$.*

If $T^{01}_{\alpha,\beta}[F] = 0$, then for the characteristic of the growth of the functions $F \in S(\Lambda, 0)$ we define a generalized convergence class by the condition

$$\int_{\sigma_0}^{0} \frac{\ln M(\sigma, F)}{|\sigma|^2\alpha^{-1}\Big(\varrho\beta\big(1/|\sigma|\big)\Big)}\, d\sigma < +\infty, \qquad \varrho = \varrho^0_{\alpha,\beta}[F]. \tag{4}$$

Theorem C ([12]). *Let $\alpha \in L^0$, $\beta \in L^0$ and $\alpha^{-1}(c\beta(x)) \in L^0$ for every $c \in (0, +\infty)$. If all functions* (3) *belong to the generalized convergence class, $\overline{\lim\limits_{n\to\infty}} \dfrac{\ln n}{\ln |a_{n,j}|} \le h_0 < +\infty$ for all $1 \le j \le p$, $\omega_j > 0$, $\sum\limits_{j=1}^{p} \omega_j = 1$ and $|a_n| \asymp \prod\limits_{j=1}^{p} |a_{n,j}|^{\omega_j}$ as $n \to \infty$, then the function* (1) *also belongs the same convergence class. If, in addition, $|a_{n,1}| > 0$ for all $n \ge 1$ and $|a_{n,j}| \asymp |a_{n,1}|$ as $n \to \infty$ for all $1 \le j \le p$, then the belonging of F to the generalized convergence class implies the belonging of all F_j to the same convergence class.*

Here we consider the case when w is a homogeneous polynomial.

1 Definition and convergence of the Hadamard composition of the genus *m*

Recall that a polynomial is named homogeneous if all monomials with nonzero coefficients have the identical degree. A polynomial $P(x_1, \dots, x_p)$ is homogeneous to the degree m if and only if $P(tx_1, \dots, tx_p) = t^m P(x_1, \dots, x_p)$ for all t from the domain of the polynomial. Dirichlet series (1) is called a Hadamard composition of genus m of Dirichlet series (3) if $a_n = P(a_{n,1}, \dots, a_{n,p})$, where

$$P(x_1, \dots, x_p) = \sum_{k_1+\dots+k_p=m} c_{k_1\dots k_p} x_1^{k_1} \cdots x_p^{k_p}.$$

is a homogeneous polynomial of degree $m \ge 1$. We remark that the usual Hadamard composition is a special case of the Hadamard composition of the genus $m = 2$.

Therefore, if the function F is Hadamard composition of genus $m \ge 1$ of the functions F_j then

$$|a_n| \le \sum_{k_1+\dots+k_p=m} |c_{k_1\dots k_p}| |a_{n,1}|^{k_1} \cdots |a_{n,p}|^{k_p}. \tag{5}$$

It is known [10, 14] that if $\ln n = o(\lambda_n)$ as $n \to \infty$ then $\sigma_a[F] = \alpha[F] := \varliminf\limits_{n\to\infty} \dfrac{1}{\lambda_n} \ln \dfrac{1}{|a_n|}$. Hence, it follows that if all $F_j \in S(\Lambda, 0)$, i.e. $\sigma_a[F_j] = 0$, then $|a_{n,j}| \le \exp\{\varepsilon\lambda_n\}$ for every $\varepsilon > 0$, all $n \ge n_0(\varepsilon)$ and all $1 \le j \le p$. Therefore, (5) implies $|a_n| \le C\exp\{m\varepsilon\lambda_n\}$, where $C = \sum\limits_{k_1+\dots+k_p=m} |c_{k_1\dots k_p}|$. Hence $\alpha[F] = \varliminf\limits_{n\to\infty} \dfrac{1}{\lambda_n} \ln \dfrac{1}{|a_n|} \ge -m\varepsilon$, i.e. in view of the arbitrariness of ε we get $\sigma_a[F] = \alpha[F] \ge 0$.

In order for $\sigma_a[F] = 0$ additional conditions on $a_{n,j}$ are required. For example, suppose that $|c_{m0\dots0}||a_{n,1}|^m \to +\infty$ and $|a_{n,j}| = o(|a_{n,1}|)$ as $n \to \infty$. We put

$$\begin{aligned}\Sigma_n' &= \sum_{k_1+\dots+k_p=m,\, k_1\ne m} c_{k_1\dots k_p}(a_{n,1})^{k_1} \cdots (a_{n,p})^{k_p} \\ &= \sum_{k_1+\dots+k_p=m} c_{k_1\dots k_p}(a_{n,1})^{k_1} \cdots (a_{n,p})^{k_p} - c_{m0\dots0}(a_{n,1})^m.\end{aligned}$$

Since for each monomial of the polynomial Σ_n' the sum of the exponents is equal to m, we have

$$\frac{|a_{n,1}|^{k_1} \cdots |a_{n,p}|^{k_p}}{|a_{n,1}|^m} = \frac{|a_{n,2}|^{k_2} \cdots |a_{n,p}|^{k_p}}{|a_{n,1}|^{m-k_1}} \to 0, \qquad n \to \infty,$$

and thus $\Sigma_n' = o(|a_{n,1}|^m)$ as $n \to \infty$. Therefore,

$$|a_n| \ge |c_{m0\dots0}||a_{n,1}|^m - |\Sigma_n'| = |c_{m0\dots0}||a_{n,1}|^m - o(|a_{n,1}|^m) \ge \frac{|c_{m0\dots0}|}{2}|a_{n,1}|^m, \qquad n \ge n_0^*,$$

whence $\sigma_a[F]=\alpha[F]\le m\varliminf\limits_{n\to\infty}\dfrac{1}{\lambda_n}\ln\dfrac{1}{|a_{n,1}|}=0$, i.e. $\sigma_a[F]=0$.

Growth of Hadamard compositions of the genus m

Since the polynomial $P(x_1,\dots,x_p)$ is homogeneous of the degree $m\ge 1$, we have

$$a_ne^{ms\lambda_n}=\sum_{k_1+\dots+k_p=m}c_{k_1\dots k_p}\left(a_{n,1}e^{s\lambda_n}\right)^{k_1}\cdots\left(a_{n,p}e^{s\lambda_n}\right)^{k_p}. \tag{6}$$

Let $\mu(\sigma,F)=\max\left\{|a_n|\exp\{\sigma\lambda_n\}:\ n\ge 0\right\}$ be the maximal term of series (1). We remark that $\mu(\sigma,F)\uparrow+\infty$ as $\sigma\uparrow 0$ if and only if $\varlimsup\limits_{n\to\infty}|a_n|=+\infty$. In what follows, we will assume that this condition is satisfied.

Since (6) implies

$$|a_n|e^{m\sigma\lambda_n}\le\sum_{k_1+\dots+k_p=m}|c_{k_1\dots k_p}|\left(|a_{n,1}|e^{\sigma\lambda_n}\right)^{k_1}\cdots\left(|a_{n,p}|e^{\sigma\lambda_n}\right)^{k_p},$$

we have

$$\mu(m\sigma,F)\le\sum_{k_1+\dots+k_p=m}|c_{k_1\dots k_p}|\mu(\sigma,F_1)^{k_1}\cdots\mu(\sigma,F_p)^{k_p},$$

whence for all $\sigma<0$ enough large we get

$$\begin{aligned}\ln\mu(m\sigma,F)&\le\sum_{k_1+\dots+k_p=m}\ln\left(|c_{k_1\dots k_p}|\mu(\sigma,F_1)^{k_1}\cdots\mu(\sigma,F_p)^{k_p}\right)+\ln(m+1)\\&=\sum_{k_1+\dots+k_p=m}\Big(\ln(|c_{k_1\dots k_p}|+k_1\ln\mu(\sigma,F_1)+\dots+k_p\ln\mu(\sigma,F_p)\Big)+\ln(m+1)\\&=\sum_{k_1+\dots+k_p=m}(k_1\ln\mu(\sigma,F_1)+\dots+k_p\ln\mu(\sigma,F_p))+C_1,\end{aligned} \tag{7}$$

where $C_1=\sum\limits_{k_1+\dots+k_p=m}\ln^+|c_{k_1\dots k_p}|+\ln(m+1)$.

We remark also that if $|c_{m0\dots0}||a_{n,1}|^m\to+\infty$ and $|a_{n,j}|=o\big(|a_{n,1}|\big)$ as $n\to\infty$, then, as above, $|a_n|\ge|c_{m0\dots0}||a_{n,1}|^m/2$ for $n\ge n_0^*$ and therefore for $1\le j\le p$ and all $\sigma<0$ enough large

$$\ln\mu(\sigma,F_j)\le\ln\mu(\sigma,F_1)\le\ln\mu(\sigma,F)+C_2. \tag{8}$$

From (7) and (8) it follows that we need a connection between the growth of $M(\sigma,F)$ and the growth of $\mu(\sigma,F)$. By Cauchy inequality $\mu(\sigma,F)\le M(\sigma,F)$, the following lemmas contain estimates of $M(\sigma,F)$ from above. Such estimates depend on the behavior of the coefficients, the relative growth of the functions α and β, and the growth of the sequence Λ.

Lemma 1 ([4])**.** *If $F\in S(\Lambda,0)$ and*

$$\varlimsup_{n\to\infty}\frac{\ln n}{\ln|a_n|}=h<+\infty, \tag{9}$$

then for every $\varepsilon>0$ and all $\sigma_0(\varepsilon)\le\sigma<0$

$$M(\sigma,F)\le K(\varepsilon)\mu\left(\frac{1-\varepsilon}{1+h}\sigma,F\right)^{1+h+\varepsilon},\qquad K(\varepsilon)=const>0.$$

To replace condition (10) by a condition on the growth of the sequence Λ, we have the following lemma.

Lemma 2 ([4]). *If $F \in S(\Lambda, 0)$, $\alpha \in L_{si}$, $\beta \in L_{si}$ and $c \in (0, +\infty)$*

$$\alpha(\lambda_n) = o\left(\beta\left(\frac{\lambda_n}{\ln n}\right)\right), \qquad n \to \infty, \tag{10}$$

then for every $\varepsilon \in (0,1)$ and all $\sigma_0(\varepsilon) \le \sigma < 0$

$$M(\sigma, F) \le \mu\big((1-\varepsilon)\sigma, F\big) \exp\left\{\alpha^{-1}\big(\varepsilon\beta(1/|\sigma|)\big)\right\}.$$

In the following lemma there are no conditions on the functions α and β.

Lemma 3 ([13]). *If $F \in S(\Lambda, 0)$ and*

$$\varlimsup_{n\to\infty} \frac{\ln n}{\lambda_n \gamma(\lambda_n)} = h_0 < \infty, \tag{11}$$

where γ is a positive continuous decreasing to 0 function on $[0, +\infty)$ such that $t\gamma(t) \uparrow +\infty$ as $t \to +\infty$, then for every $\varepsilon > 0$ there exists $K(\varepsilon) > 0$ such that for all $\sigma_0(\varepsilon) \le \sigma < 0$

$$M(\sigma, F) \le \mu\left(\frac{\sigma}{1+\varepsilon}, F\right)\left(\exp\left\{\frac{\varepsilon|\sigma|}{1+\varepsilon}\gamma^{-1}\left(\frac{\varepsilon|\sigma|}{(1+\varepsilon)^2(h_0+\varepsilon^2)}\right)\right\} + K(\varepsilon)\right). \tag{12}$$

These lemmas imply the following statements.

Lemma 4. *Let $F \in S(\Lambda, 0)$, $\varrho^0_{\alpha,\beta}[\mu(F)] = \varlimsup\limits_{\sigma\uparrow 0} \dfrac{\alpha\big(\ln \mu(\sigma, F)\big)}{\beta(1/|\sigma|)}$ and one of the following conditions be satisfied:*

a) *condition (9) holds and either $\alpha \in L_{si}$, $\beta \in L_{si}$ and $h < +\infty$ or $\alpha \in L^0$, $\beta \in L^0$ and $h = 0$;*

b) *$\alpha \in L_{si}$, $\beta \in L^0$ and condition (10) holds;*

c) *$\alpha \in L_{si}$, $\beta \in L^0$, condition (11) holds and $\varlimsup\limits_{x\to+\infty} \dfrac{\alpha\big(\gamma^{-1}(1/x)\big)}{\beta(x)} = 0$.*

Then $\varrho^0_{\alpha,\beta}[F] = \varrho^0_{\alpha,\beta}[\mu(F)]$.

Proof. If condition (9) with $h < +\infty$ holds, then by Lemma 1 we get

$$\ln \mu(\sigma, F) \le \ln M(\sigma, F) \le (1+h+\varepsilon) \ln \mu\left(\frac{1-\varepsilon}{1+h}\sigma, F\right) + o(1), \qquad \sigma \uparrow 0,$$

and if $\alpha \in L_{si}$ and $\beta \in L_{si}$, then

$$\varrho^0_{\alpha,\beta}[\mu(F)] \le \varrho^0_{\alpha,\beta}[F] \le \varlimsup_{\sigma\uparrow 0} \frac{\alpha \ln \mu\left(\frac{1-\varepsilon}{1+h}\sigma, F\right)}{\beta(1/|\sigma|)}$$

$$\le \varlimsup_{\sigma\uparrow 0} \frac{\alpha \ln \mu\left(\frac{1-\varepsilon}{1+h}\sigma, F\right)}{\beta\left(\frac{1+h}{(1-\varepsilon)|\sigma|}\right)} \varlimsup_{x\to+\infty} \frac{\beta\left(\frac{1+h}{1-\varepsilon}x\right)}{\beta(x)} = \varrho^0_{\alpha,\beta}[\mu(F)].$$

In [15] it is proved that if $\beta \in L^0$, then $\varlimsup\limits_{x\to+\infty} \beta\big((1+\varepsilon)x\big)/\beta(x) = A(\varepsilon) \downarrow 1$ as $\varepsilon \downarrow 0$ and $\varlimsup\limits_{x\to+\infty} \beta(Kx)/\beta(x) = B(K) < +\infty$ for $K = \text{const} > 0$.

Therefore, if $h=0$, $\alpha\in L^0$ and $\beta\in L^0$, then

$$\begin{aligned}\varrho^0_{\alpha,\beta}[\mu(F)] &\le \varrho^0_{\alpha,\beta}[F] \le \varlimsup_{\sigma\uparrow 0}\frac{\alpha\Big((1+\varepsilon)\ln\mu\big((1-\varepsilon)\sigma,F\big)\Big)}{\beta(1/|\sigma|)}\\ &\le \varlimsup_{\sigma\uparrow 0}\frac{\alpha\Big(\ln\mu\big((1-\varepsilon)\sigma,F\big)\Big)}{\beta\big(1/(1-\varepsilon)|\sigma|\big)}\varlimsup_{\sigma\uparrow 0}\frac{\alpha\Big((1+\varepsilon)\ln\mu\big((1-\varepsilon)\sigma,F\big)\Big)}{\alpha\Big(\ln\mu\big((1-\varepsilon)\sigma,F\big)\Big)}\varlimsup_{\sigma\uparrow 0}\frac{\beta\big(1/(1-\varepsilon)|\sigma|\big)}{\beta\big(1/|\sigma|\big)}\\ &= \varrho^0_{\alpha,\beta}[\mu(F)]A_1(\varepsilon)A_2(\varepsilon),\end{aligned}$$

where $A_j(\varepsilon)\downarrow 1$ as $\varepsilon\downarrow 0$. Hence in view of the arbitrariness of ε we obtain the equality $\varrho^0_{\alpha,\beta}[F]=\varrho^0_{\alpha,\beta}[\mu(F)]$.

In the case **b)**, by Lemma 2 we have $\ln M(\sigma,F)\le\ln\mu\big((1-\varepsilon)\sigma,F\big)+\alpha^{-1}\big(\varepsilon\beta(1/|\sigma)\big)$ and since $\alpha\in L_{si}$, we get

$$\begin{aligned}\alpha\big(\ln M(\sigma,F)\big) &\le \alpha\Big(2\max\Big\{\ln\mu\big((1-\varepsilon)\sigma,F\big),\alpha^{-1}\big(\varepsilon\beta(1/|\sigma)\big)\Big\}\Big)\\ &= (1+o(1))\alpha\Big(\max\Big\{\ln\mu\big((1-\varepsilon)\sigma,F\big),\alpha^{-1}\big(\varepsilon\beta(1/|\sigma)\big)\Big\}\Big)\\ &= (1+o(1))\max\Big\{\alpha\Big(\ln\mu\big((1-\varepsilon)\sigma,F\big)\Big),\varepsilon\beta(1/|\sigma)\Big\}\\ &\le (1+o(1))\Big(\alpha\Big(\ln\mu\big((1-\varepsilon)\sigma,F\big)\Big)+\varepsilon\beta(1/|\sigma)\Big)\end{aligned}$$

as $\sigma\uparrow 0$, whence, as above, in view of the condition $\beta\in L^0$ we get

$$\varrho^0_{\alpha,\beta}[\mu(F)]\le\varrho^0_{\alpha,\beta}[F]\le\varrho^0_{\alpha,\beta}[\mu(F)]A(\varepsilon)+\varepsilon,$$

where $A(\varepsilon)\downarrow 1$ as $\varepsilon\downarrow 0$, and thus, in view of the arbitrariness of ε we obtain the equality $\varrho^0_{\alpha,\beta}[F]=\varrho^0_{\alpha,\beta}[\mu(F)]$. Finally, as above, from (12) we have

$$\begin{aligned}\alpha\big(\ln\mu(\sigma,F)\big) &\le \alpha\big(\ln M(\sigma,F)\big)\\ &\le (1+o(1))\max\left\{\alpha\left(\ln\mu\left(\frac{\sigma}{1+\varepsilon},F\right)\right),\alpha\left(\gamma^{-1}\left(\frac{\varepsilon|\sigma|}{(1+\varepsilon)^2(h_0+\varepsilon^2)}\right)\right)\right\}\end{aligned}\tag{13}$$

as $\sigma\uparrow 0$. Clearly,

$$\varlimsup_{\sigma\uparrow 0}\frac{\alpha\left(\ln\mu\left(\frac{\sigma}{1+\varepsilon},F\right)\right)}{\beta\big(1/|\sigma|\big)}\le\varlimsup_{\sigma\uparrow 0}\frac{\alpha\left(\ln\mu\left(\frac{\sigma}{1+\varepsilon},F\right)\right)}{\beta\big((1+\varepsilon)/|\sigma|\big)}\varlimsup_{\sigma\uparrow 0}\frac{\beta\big((1+\varepsilon)/|\sigma|\big)}{\beta\big(1/|\sigma|\big)}=\varrho^0_{\alpha,\beta}[\mu(F)]A(\varepsilon)$$

and, similarly,

$$\varlimsup_{\sigma\uparrow 0}\frac{\alpha\left(\gamma^{-1}\left(\frac{\varepsilon|\sigma|}{(1+\varepsilon)^2(h_0+\varepsilon^2)}\right)\right)}{\beta\big(1/|\sigma|\big)}\le\varlimsup_{\sigma\uparrow 0}\frac{\alpha\big(\gamma^{-1}(|\sigma|)\big)}{\beta\big(1/|\sigma|\big)}B\left(\frac{\varepsilon}{(1+\varepsilon)^2(h_0+\varepsilon^2)}\right).$$

Since $A(\varepsilon)\downarrow 1$ as $\varepsilon\downarrow 0$, $B\left(\frac{\varepsilon}{(1+\varepsilon)^2(h_0+\varepsilon^2)}\right)<+\infty$ and $\varlimsup\limits_{\sigma\uparrow 0}\frac{\alpha\big(\gamma^{-1}(|\sigma|)\big)}{\beta\big(1/|\sigma|\big)}=0$, from (13) we obtain the equality $\varrho^0_{\alpha,\beta}[F]=\varrho^0_{\alpha,\beta}[\mu(F)]$. Lemma 4 is proved. ☐

Lemma 5. *Let $F\in S(\Lambda,0)$, $T^{01}_{\alpha,\beta}[\mu(F)]=\overline{\lim\limits_{\sigma\uparrow 0}}\,\dfrac{\ln\mu(\sigma,F)}{\alpha^{-1}(\varrho\beta(1/|\sigma|))}$, $\varrho=\varrho^0_{\alpha,\beta}[F]$, $\alpha^{-1}(c\beta(x))\in L^0$ for each $c\in(0,+\infty)$. If one of the conditions* ***a)*** *with $h=0$,* ***b), c)*** *of Lemma 4 is satisfied, then $T^{01}_{\alpha,\beta}[F]=T^{01}_{\alpha,\beta}[\mu(F)]$.*

Proof. If condition (9) holds with $h=0$, then by Lemma 1 for every $\varepsilon\in(0,1)$ and all $\sigma\in(\sigma_0(\varepsilon),0)$ in view of the condition $\alpha^{-1}(c\beta(x))\in L^0$ for each $c\in(0,+\infty)$ we have

$$T^{01}_{\alpha,\beta}[\mu(F)]\le T^{01}_{\alpha,\beta}[F]\le(1+\varepsilon)\,\overline{\lim_{\sigma\uparrow 0}}\,\frac{\ln\mu\big((1-\varepsilon)\sigma,F\big)}{\alpha^{-1}\Big(\varrho\beta\big(1/((1-\varepsilon)|\sigma|)\big)\Big)}\,\overline{\lim_{\sigma\uparrow 0}}\,\frac{\alpha^{-1}\Big(\varrho\beta\big(1/((1-\varepsilon)|\sigma|)\big)\Big)}{\alpha^{-1}\Big(\varrho\beta\big(1/|\sigma|\big)\Big)}$$
$$=(1+\varepsilon)T^{01}_{\alpha,\beta}[\mu(F)]\,A(\varepsilon),$$

where $A(\varepsilon)\downarrow 1$ as $\varepsilon\downarrow 0$. Thus, $T^{01}_{\alpha,\beta}[F]=T^{01}_{\alpha,\beta}[\mu(F)]$.

If the condition **b)** holds, then by Lemma 2 as above we get

$$T^{01}_{\alpha,\beta}[\mu(F)]\le T^{01}_{\alpha,\beta}[F]\le\overline{\lim_{\sigma\uparrow 0}}\,\frac{\ln\mu\big((1-\varepsilon)\sigma,F\big)}{\alpha^{-1}\Big(\varrho\beta\big(1/|\sigma|\big)\Big)}+\overline{\lim_{\sigma\uparrow 0}}\,\frac{\alpha^{-1}\Big(\varepsilon\beta\big(1/|\sigma|\big)\Big)}{\alpha^{-1}\Big(\varrho\beta\big(1/|\sigma|\big)\Big)}\le T^{01}_{\alpha,\beta}[\mu(F)]\,A(\varepsilon)+A^*(\varepsilon),$$

where $A(\varepsilon)\downarrow$ as $\varepsilon\downarrow 0$ and the condition $\alpha\in L_{si}$ implies $A^*(\varepsilon)=\overline{\lim\limits_{x\to+\infty}}\,\dfrac{\alpha^{-1}(\varepsilon x)}{\alpha^{-1}(\varrho x)}=0$ for $\varepsilon<\varrho$. Therefore, in view of the arbitrariness of ε the equality $T^{01}_{\alpha,\beta}[F]=T^{01}_{\alpha,\beta}[\mu(F)]$ is correct. Finally, as above, from (12) we have

$$\ln M(\sigma,F)\le\ln\mu\left(\frac{\sigma}{1+\varepsilon},F\right)+\gamma^{-1}\left(\frac{\varepsilon|\sigma|}{(1+\varepsilon)^2(h_0+\varepsilon^2)}\right)+K_1(\varepsilon),$$

whence, as above, $T^{01}_{\alpha,\beta}[\mu(F)]\le T^{01}_{\alpha,\beta}[F]\le T^{01}_{\alpha,\beta}[\mu(F)]\,A(\varepsilon)+C(\varepsilon)$, where $A(\varepsilon)\to 1$ as $\varepsilon\downarrow 0$ and

$$C(\varepsilon)=\overline{\lim_{\sigma\uparrow 0}}\,\frac{\gamma^{-1}\left(\dfrac{\varepsilon|\sigma|}{(1+\varepsilon)^2(h_0+\varepsilon^2)}\right)}{\alpha^{-1}\Big(\varrho\beta\big(1/|\sigma|\big)\Big)}\le\overline{\lim_{\sigma\uparrow 0}}\,\frac{\gamma^{-1}(1/|\sigma|)}{\alpha^{-1}\left(\varrho\beta\left(\dfrac{(1+\varepsilon)^2(h_0+\varepsilon^2)}{\varepsilon|\sigma|}\right)\right)}$$
$$\le\overline{\lim_{\sigma\uparrow 0}}\,\frac{\gamma^{-1}(1/|\sigma|)}{\alpha^{-1}\Big(\varrho\beta\big(1/|\sigma|\big)\Big)}\,\overline{\lim_{\sigma\uparrow 0}}\,\frac{\alpha^{-1}\Big(\varrho\beta\big(1/|\sigma|\big)\Big)}{\alpha^{-1}\left(\varrho^0\beta\left(\dfrac{(1+\varepsilon)^2(h_0+\varepsilon^2)}{\varepsilon|\sigma|}\right)\right)}.$$

Since the condition $\overline{\lim\limits_{x\to+\infty}}\,\dfrac{\alpha\big(\gamma^{-1}(1/x)\big)}{\beta(x)}=0$ implies $\overline{\lim\limits_{x\to+\infty}}\,\dfrac{\gamma^{-1}(1/x)}{\alpha^{-1}(c\beta(x))}=0$ for each $c\in(0,+\infty)$, and $\overline{\lim\limits_{x\to+\infty}}\,\dfrac{\alpha^{-1}(\varrho\beta(Kx))}{\alpha^{-1}(\varrho\beta(x))}=B(K)<+\infty$, we have $C(\varepsilon)=0$. Therefore, $T^{01}_{\alpha,\beta}[F]=T^{01}_{\alpha,\beta}[\mu(F)]$, and Lemma 5 is proved. □

Using Lemmas 4 and 5 we prove the following theorem.

Theorem 1. *Let $\alpha^{-1}\big(c\beta(x)\big) \in L^0$ for each $c \in (0,+\infty)$ and one of the conditions* ***a)*** *with $h=0$,* ***b), c)*** *of Lemma 4 is satisfied.*

Suppose that for $1 \le j \le p$ the functions $F_j \in S(\Lambda, 0)$ have the same generalized order $\varrho^0_{\alpha,\beta}[F_j] = \varrho \in (0,+\infty)$ and the types $T^{01}_{\alpha,\beta}[F_j] = T^{01}_j \in [0,+\infty)$, $|c_{m0\dots0}||a_{n,1}|^m \to +\infty$ and $|a_{n,j}| = o\big(|a_{n,1}|\big)$ as $n\to\infty$ for $2 \le j \le p$.

If the function $F \in S(\Lambda, 0)$ is the Hadamard composition of the genus $m \ge 1$ of the functions F_j and either $m=1$ or $\beta \in L_{si}$, then $\varrho^0_{\alpha,\beta}[F] = \varrho$ and

$$T^{01}_{\alpha,\beta}[F] \le Q(m) \sum_{k_1+\dots+k_p=m} (k_1 T^{01}_1 + \dots + k_p T^{01}_p),$$

where $Q_1(m) := \varlimsup\limits_{x\to+\infty} \dfrac{\alpha^{-1}\big(\varrho\beta(mx)\big)}{\alpha^{-1}\big(\varrho\beta(x)\big)} = 1$ *if either $m=1$ or $\alpha^{-1}(\varrho\beta(x)) \in L_{si}$.*

Proof. Since the functions $F_j \in S(\Lambda,0)$ have the same generalized order $\varrho^0_{\alpha,\beta}[F_j] = \varrho \in (0,+\infty)$, for every $\varrho_1 > \varrho$ and all $\sigma \in \big[\sigma_0(\varrho_1), 0\big)$ we have $\ln\mu(\sigma, F_j) \le \alpha^{-1}\Big(\varrho_1\beta\big(1/|\sigma|\big)\Big)$. Therefore, from (7) we get

$$\ln\mu(m\sigma, F) \le \sum_{k_1+\dots+k_p=m} (k_1+\dots+k_p)\alpha^{-1}\Big(\varrho_1\beta\big(1/|\sigma|\big)\Big) + C_1 = C_2\alpha^{-1}\Big(\varrho_1\beta\big(1/|\sigma|\big)\Big) + C_1,$$

where $C_1 = \sum\limits_{k_1+\dots+k_p=m} \ln^+|c_{k_1\dots k_p}| + \ln(m+1)$ and $C_2 = \sum\limits_{k_1+\dots+k_p=m}(k_1+\dots+k_p)$. Since $\alpha \in L_{si}$, it follows that

$$\varrho^0_{\alpha,\beta}\big[\mu(F)\big] = \varlimsup_{\sigma\uparrow 0} \frac{\alpha\big(\ln\mu(m\sigma,F)\big)}{\beta\Big(1/\big(m|\sigma|\big)\Big)} \le \varlimsup_{\sigma\uparrow 0}\frac{\varrho_1\beta\big(1/|\sigma|\big)}{\beta\Big(1/\big(m|\sigma|\big)\Big)} = \varrho_1 \varlimsup_{x\to} \frac{\beta(mx)}{\beta(x)} = \varrho_1,$$

provided either $m=1$ or $\beta \in L_{si}$. In view of the arbitrariness of ϱ we get $\varrho^0_{\alpha,\beta}\big[\mu(F)\big] \le \varrho$.

If $|c_{m0\dots0}||a_{n,1}|^m \to +\infty$ and $|a_{n,j}| = o\big(|a_{n,1}|\big)$ as $n\to\infty$ then from (8) we get $\varrho^0_{\alpha,\beta}\big[\mu(F_j)\big] \le \varrho^0_{\alpha,\beta}\big[\mu(F_1)\big] \le \varrho^0_{\alpha,\beta}\big[\mu(F)\big]$. Thus, $\varrho^0_{\alpha,\beta}\big[\mu(F)\big] = \varrho^0_{\alpha,\beta}\big[\mu(F_j)\big]$ and by Lemma 4 $\varrho^0_{\alpha,\beta}[F] = \varrho$.

Also, from (7) we obtain

$$T^{01}_{\alpha,\beta}\big[\mu(F)\big] = \varlimsup_{\sigma\uparrow 0}\frac{\ln\,\mu(m\sigma,F)}{\alpha^{-1}\Big(\varrho\beta\big(1/\big(m|\sigma|\big)\big)\Big)}$$

$$\le \sum_{k_1+\dots+k_p=m}\left(k_1 \varlimsup_{\sigma\uparrow 0}\frac{\ln\,\mu(\sigma,F_1)}{\alpha^{-1}\Big(\varrho\beta\big(1/|\sigma|\big)\Big)} + \dots + k_p \varlimsup_{\sigma\uparrow 0}\frac{\ln\,\mu(\sigma,F_p)}{\alpha^{-1}\Big(\varrho\beta\big(1/|\sigma|\big)\Big)}\right)\varlimsup_{\sigma\uparrow 0}\frac{\alpha^{-1}\Big(\varrho\beta\big(1/|\sigma|\big)\Big)}{\alpha^{-1}\Big(\varrho\beta\big(1/\big(m|\sigma|\big)\big)\Big)}$$

$$= Q_1(m)\sum_{k_1+\dots+k_p=m}\Big(k_1 T^{02}_{\alpha,\beta}\big[\mu(F_1)\big] + \dots + k_p T^{01}_{\alpha,\beta}\big[\mu(F_p)\big]\Big).$$

Since by Lemma 5 we have $T^{01}_{\alpha,\beta}\big[\mu(F)\big] = T^{01}_{\alpha,\beta}[F]$ and $T^{01}_{\alpha,\beta}\big[\mu(F_j)\big] = T^{01}_{\alpha,\beta}[F_j]$, the proof of Theorem 1 is complete. □

Since $\varrho^0_{\alpha,\beta}[F] = \varlimsup\limits_{\sigma\uparrow 0}\dfrac{\ln\exp\Big\{\alpha\big(\ln M(\sigma,F)\big)\Big\}}{\ln\exp\Big\{\beta\big(1/|\sigma|\big)\Big\}}$, the generalized type can be determined [9] by the formula

$$T^{02}_{\alpha,\beta}[F] = \varlimsup_{\sigma\uparrow 0}\frac{\exp\Big\{\alpha\big(\ln M(\sigma,F)\big)\Big\}}{\exp\Big\{\varrho\beta\big(1/|\sigma|\big)\Big\}}, \quad \varrho = \varrho^0_{\alpha,\beta}[F].$$

If we put $\alpha_1(x) = \exp\{\alpha(x)\}$ and $\beta_1(x) = \exp\{\varrho\beta(x)\}$, then $T^{02}_{\alpha,\beta}[F] = \varrho^0_{\alpha_1,\beta_1}[F]$. Therefore, Lemma 3 implies the following statement.

Lemma 6. *Let $F \in S(\Lambda, 0)$, $0 < \varrho < +\infty$, $T^{02}_{\alpha,\beta}[\mu(F)] = \overline{\lim\limits_{\sigma\uparrow 0}} \dfrac{\exp\{\alpha(\ln\mu(\sigma, F))\}}{\exp\{\varrho\beta(1/|\sigma|)\}}$ and one of the following conditions be satisfied:*

d) *condition (9) holds and either $e^\alpha \in L_{si}$, $e^\beta \in L_{si}$ and $h < +\infty$ or $e^\alpha \in L^0$, $e^\beta \in L^0$ and $h = 0$;*

e) *$e^\alpha \in L_{si}$, $e^\beta \in L^0$ and*

$$\exp\{\alpha(\lambda_n)\} = o\Big(\exp\Big\{\varrho\beta\Big(\frac{\lambda_n}{\ln n}\Big)\Big\}\Big), \qquad n \to \infty; \tag{14}$$

f) *$e^\alpha \in L_{si}$, $e^\beta \in L^0$, condition (12) holds and $\overline{\lim\limits_{x\to+\infty}} \dfrac{\exp\{\alpha(\gamma^{-1}(1/x))\}}{\exp\{\varrho\beta(x)\}} = 0.$*

Then $T^{02}_{\alpha,\beta}[F] = T^{02}_{\alpha,\beta}[\mu(F)]$.

Using Lemmas 4 and 6 we prove the following theorem.

Theorem 2. *Let one of the conditions **d)** of Lemma 6 and **b), c)** of Lemma 4 is satisfied.*

Suppose that for $1 \le j \le p$ the functions $F_j \in S(\Lambda, 0)$ have the same generalized order $\varrho^0_{\alpha,\beta}[F_j] = \varrho \in (0, +\infty)$ and the types $T^{02}_{\alpha,\beta}[F_j] = T^{02}_j \in [0, +\infty)$, $|c_{m0\ldots0}||a_{n,1}|^m \to +\infty$ and $|a_{n,j}| = o(|a_{n,1}|)$ as $n \to \infty$ for $2 \le j \le p$.

If the function $F \in S(\Lambda, 0)$ is the Hadamard composition of the genus $m \ge 1$ of the functions F_j and either $m = 1$ or $\beta \in L_{si}$, then $\varrho^0_{\alpha,\beta}[F] = \varrho$ and

$$T^{02}_{\alpha,\beta}[F] \le Q(m) \sum_{k_1+\cdots+k_p=m} \Big(k_1 T^{01}_1 + \cdots + k_p T^{01}_p\Big),$$

where $Q_2(m) := \overline{\lim\limits_{\sigma\uparrow 0}} \dfrac{\exp\{\varrho\beta(1/|\sigma|)\}}{\exp\{\varrho\beta(1/(m|\sigma|)\}} = 1$ if either $m = 1$ or $\exp\{\varrho\beta(x)\} \in L_{si}$.

Proof. Clearly, if $e^\alpha \in L_{si}$ $(e^\alpha \in L^0)$, then $\alpha \in L_{si}$ $(\alpha \in L^0)$ and, therefore, the condition **d)** of Lemma 6 implies the condition **a)** of Lemma 4. Then by Lemma 4 for each function $F \in S(\Lambda, 0)$ we have $\varrho^0_{\alpha,\beta}[F] = \varrho^0_{\alpha,\beta}[\mu(F)]$. If condition (10) holds, then for every $\varepsilon \in (0, \varrho)$ and all $n \ge n_0(\varepsilon)$ we have $\alpha(\lambda_n) \le \varepsilon\beta(\lambda_n/\ln n)$ and, thus,

$$\begin{aligned}\exp\{\alpha(\lambda_n)\} &\le \exp\{\varepsilon\beta(\lambda_n/\ln n)\}\\ &= \exp\{\varrho^0\beta(\lambda_n/\ln n)\}\exp\{-(\varrho-\varepsilon)\beta(\lambda_n/\ln n)\} = o\Big(\exp\{\varrho\beta(\lambda_n/\ln n)\}\Big)\end{aligned}$$

as $n \to \infty$, i.e. (14) holds. Similarly, the condition $\overline{\lim\limits_{x\to+\infty}} \dfrac{\alpha(\gamma^{-1}(1/x))}{\beta(x)} = 0$ implies the condition $\overline{\lim\limits_{x\to+\infty}} \dfrac{\exp\{\alpha(\gamma^{-1}(1/x))\}}{\exp\{\varrho\beta(x)\}} = 0$. Therefore, by Lemma 6 for each function $F \in S(\Lambda, 0)$ we have $T^{02}_{\alpha,\beta}[F] = T^{02}_{\alpha,\beta}[\mu(F)]$.

From (7) we obtain

$$T^{02}_{\alpha,\beta}[\mu(F)] = \overline{\lim_{\sigma\uparrow 0}} \frac{\exp\left\{\alpha\big(\ln\mu(m\sigma, F)\big)\right\}}{\exp\left\{\varrho\beta\big(1/(m|\sigma|\big)\right\}}$$

$$\leq \sum_{k_1+\dots+k_p=m} \left(k_1 \overline{\lim_{\sigma\uparrow 0}} \frac{\exp\left\{\alpha\big(\ln\mu(\sigma, F_1)\big)\right\}}{\exp\left\{\varrho\beta\big(1/|\sigma|\big)\right\}} + \dots + k_p \overline{\lim_{\sigma\uparrow 0}} \frac{\exp\left\{\alpha\big(\ln\mu(\sigma, F_1)\big)\right\}}{\exp\left\{\varrho\beta\big(1/|\sigma|\big)\right\}} \right)$$

$$\times \overline{\lim_{\sigma\uparrow 0}} \frac{\exp\left\{\varrho^0\beta\big(1/|\sigma|\big)\right\}}{\exp\left\{\varrho\beta\big(1/(m|\sigma|)\big)\right\}} = Q_2(m) \sum_{k_1+\dots+k_p=m} \Big(k_1 T^{01}_{\alpha,\beta}[\mu(F_1)] + \dots + k_p T^{01}_{\alpha,\beta}[\mu(F_p)] \Big).$$

Theorem 2 is proved. □

2 Convergence classes of Hadamard composition of the genus m

Here by C^0_ϱ we denote a generalized convergence class of the functions $F \in S(\Lambda, 0)$ defined by condition (4). In order for (4) to hold, it is necessary that $\int_{\sigma_0}^0 \frac{d\sigma}{|\sigma|^2\alpha^{-1}(\varrho\beta(1/|\sigma|))} < +\infty$. Since ϱ can be arbitrary, we will assume in what follows that $\int_{x_0}^\infty \frac{dx}{\alpha^{-1}(c\beta(x))} < +\infty$ for any $c \in (0, +\infty)$. Now we prove the following theorem.

Theorem 3. *Let $\alpha^{-1}\big(c\beta(x)\big) \in L^0$ for each $c \in (0, +\infty)$ and one of the conditions* ***a), b), c)*** *of Lemma 4 is satisfied. If all functions $F_j \in S(\Lambda, 0)$ have the same generalized order $\varrho^0_{\alpha,\beta}[F_j] = \varrho \in (0, +\infty)$ and belong to C^0_ϱ, then the Hadamard composition $F \in S(\Lambda, 0)$ of the genus $m \geq 1$ of the functions F_j has the generalized order $\varrho^0_{\alpha,\beta}[F_j] = \varrho \in (0, +\infty)$ and belongs to C^0_ϱ. If, in addition, $|c_{m0\dots0}||a_{n,1}|^m \to +\infty$ and $|a_{n,j}| = o\big(|a_{n,1}|\big)$ as $n \to \infty$ for $2 \leq j \leq p$ then the condition $F \in C^0_\varrho$ implies $F_j \in C^0_\varrho$.*

Proof. By Lemma 4 we have $\varrho^0_{\alpha,\beta}[F] = \varrho^0_{\alpha,\beta}\big[\mu(F)\big]$ and $\varrho^0_{\alpha,\beta}[F_j] = \varrho^0_{\alpha,\beta}\big[\mu(F_j)\big]$. Let us show that condition (4) is satisfied if and only if

$$\int_{\sigma_0}^0 \frac{\ln\mu(\sigma, F)}{|\sigma|^2\alpha^{-1}\Big(\varrho\beta\big(1/|\sigma|\big)\Big)} d\sigma < +\infty, \qquad \varrho = \varrho^0_{\alpha,\beta}[F]. \tag{15}$$

Indeed, by Cauchy inequality condition (4) implies (15). On the other hand, if (9) holds then by Lemma 1

$$\int_{\sigma_0}^0 \frac{\ln M(\sigma, F)}{|\sigma|^2\alpha^{-1}\big(\varrho\beta(1/|\sigma|)\big)} d\sigma \leq (1+h+\varepsilon) \int_{\sigma_0}^0 \frac{\ln\mu\left(\frac{1-\varepsilon}{1+h}\sigma, F\right)}{|\sigma|^2\alpha^{-1}\big(\varrho\beta(1/|\sigma|)\big)} d\sigma + \int_{\sigma_0}^0 \frac{\ln K(\varepsilon)}{|\sigma|^2\alpha^{-1}\big(\varrho\beta(1/|\sigma|)\big)} d\sigma,$$

whence in view of the condition $\alpha^{-1}\big(c\beta(x)\big) \in L^0$ for each $c \in (0, +\infty)$ it follows that (15) implies (4).

If (19) holds, then by Lemma 2 for $\varepsilon < \varrho$

$$\int_{\sigma_0}^0 \frac{\ln\, M(\sigma, F)}{|\sigma|^2\alpha^{-1}\big(\varrho\beta(1/|\sigma|)\big)} d\sigma \leq \int_{\sigma_0}^0 \frac{\ln\mu\big((1-\varepsilon)\sigma, F\big)}{|\sigma|^2\alpha^{-1}\big(\varrho\beta(1/|\sigma|)\big)} d\sigma + \int_{\sigma_0}^0 \frac{d\sigma}{|\sigma|^2\alpha^{-1}\big((\varrho-\varepsilon)\beta(1/|\sigma|)\big)},$$

whence as above it follows that (15) implies (4).

Finally, if (11) holds, then by Lemma 3

$$\int_{\sigma_0}^{0} \frac{\ln M(\sigma,F)}{|\sigma|^2\alpha^{-1}\big(\varrho\beta(1/|\sigma|)\big)}d\sigma \le \int_{\sigma_0}^{0} \frac{\ln \mu\big(\sigma/(1+\varepsilon),F\big)}{|\sigma|^2\alpha^{-1}\big(\varrho\beta(1/|\sigma|)\big)}d\sigma + \int_{\sigma_0}^{0} \frac{\gamma^{-1}(\delta\sigma)}{|\sigma|^2\alpha^{-1}\big(\varrho\beta(1/|\sigma|)\big)}d\sigma,$$

where $\delta = \frac{\varepsilon(1+\varepsilon)^{-2}}{(h_0+\varepsilon^2)}$. The condition $\overline{\lim\limits_{x\to+\infty}} \frac{\alpha\left(\gamma^{-1}(1/x)\right)}{\beta(x)} = 0$ implies $\gamma^{-1}(1/x) \le \alpha^{-1}\big(\varepsilon\beta(x)\big)$ for every $\varepsilon > 0$ and all $x \ge x_0(\varepsilon)$. Therefore, as above, in view of the condition $\alpha^{-1}\big(c\beta(x)\big) \in L^0$ we get $\int_{\sigma_0}^{0} \frac{\gamma^{-1}(\delta\sigma)}{|\sigma|^2\alpha^{-1}(\varrho\beta(1/|\sigma|))}d\sigma < +\infty$ and, thus, (15) implies (4). The equivalence of the conditions (4) and (15) is proven.

If all $F_j \in C^0_\varrho$ then, from (7) we obtain

$$\begin{aligned}&\int_{\sigma_0}^{0} \frac{\ln \mu(m\sigma,F)}{|\sigma|^2\alpha^{-1}\big(\varrho\beta(1/|\sigma|)\big)}d\sigma\\ &\le \sum_{k_1+\dots+k_p=m}\left(k_1\int_{\sigma_0}^{0} \frac{\ln \mu(\sigma,F_1)}{|\sigma|^2\alpha^{-1}\big(\varrho\beta(1/|\sigma|)\big)}d\sigma + \dots + k_p\int_{\sigma_0}^{0} \frac{\ln \mu(\sigma,F_p)}{|\sigma|^2\alpha^{-1}\big(\varrho\beta(1/|\sigma|)\big)}d\sigma\right) + C < +\infty,\end{aligned}$$

where C is some constant. Since $\alpha^{-1}\big(c\beta(x)\big) \in L^0$ for each $c \in (0,+\infty)$, hence it follows that $\int_{\sigma_0}^{0} \frac{\ln \mu(\sigma,F)}{|\sigma|^2\alpha^{-1}(\varrho\beta(1/|\sigma|))}d\sigma < +\infty$ and, thus, $F \in C^0_\varrho$.

On the contrary, if $|c_{m0\ldots0}||a_{n,1}|^m \to +\infty$ and $|a_{n,j}| = o\big(|a_{n,1}|\big)$ as $n \to \infty$ for $2 \le j \le p$ and $F \in C^0_\varrho$, then (8) implies

$$\int_{\sigma_0}^{0} \frac{\ln \mu(\sigma,F_j)}{|\sigma|^2\alpha^{-1}\big(\varrho\beta(1/|\sigma|)\big)}d\sigma \le \int_{\sigma_0}^{0} \frac{\ln \mu(\sigma,F_1)}{|\sigma|^2\alpha^{-1}\big(\varrho\beta(1/|\sigma|)\big)}d\sigma \le \int_{\sigma_0}^{0} \frac{\ln \mu(\sigma,F)}{|\sigma|^2\alpha^{-1}\big(\varrho\beta(1/|\sigma|)\big)}d\sigma + C < +\infty,$$

where C is some constant, i.e. all $F_j \in C^0_\varrho$. The proof of Theorem 3 is complete. □

In addition to class C^0_ϱ, to study the growth of functions $F \in S(\Lambda,0)$ one can use the generalized convergence class $C^0_{\alpha,\beta}$ defined in [11] by the condition

$$\int_{\sigma_0}^{0} \frac{\alpha\big(\ln M(\sigma,F)\big)}{|\sigma|^2\beta\big(1/|\sigma|\big)}d\sigma < +\infty, \qquad \alpha \in L,\ \beta \in L. \tag{16}$$

In order for (16) to hold, it is necessary that $\int_{\sigma_0}^{0} \frac{d\sigma}{|\sigma|^2\beta(1/|\sigma|)} < +\infty$, that is $\int_{x_0}^{\infty} \frac{dx}{\beta(x)} < +\infty$. Now we prove the following theorem.

Theorem 4. *Let $\alpha \in L^0$, $\beta \in L^0$, one of the following conditions be satisfied: a)* (10) *holds; b)* (12) *holds and $\int_{x_0}^{\infty} \frac{\alpha(\gamma^{-1}(1/x))}{\beta(x)}dx < +\infty$. Suppose that the function $F \in S(\Lambda,0)$ is Hadamard composition of the genus $m \ge 1$ of the functions $F_j \in S(\Lambda,0)$. If $F_j \in C^0_{\alpha,\beta}$ for all $1 \le j \le p$, then $F \in C^0_{\alpha,\beta}$. If $F \in C^0_{\alpha,\beta}$ and, in addition, $|c_{m0\ldots0}||a_{n,1}|^m \to +\infty$ and $|a_{n,j}| = o\big(|a_{n,1}|\big)$ as $n \to \infty$ for $2 \le j \le p$, then $F_j \in C^0_{\alpha,\beta}$ for all $1 \le j \le p$.*

Proof. At first, we show that condition (16) holds if and only if

$$\int_{\sigma_0}^{0} \frac{\alpha\big(\ln \mu(\sigma,F)\big)}{|\sigma|^2\beta\big(1/|\sigma|\big)}d\sigma < +\infty. \tag{17}$$

Indeed, by Cauchy inequality condition (16) implies (17). On the other hand, if (9) holds, then since $\alpha \in L^0$ and $\beta \in L^0$, by Lemma 1 we have

$$\int_{\sigma_0}^{0} \frac{\alpha\big(\ln M(\sigma,F)\big)}{|\sigma|^2\beta\big(1/|\sigma|\big)}d\sigma \leq \int_{\sigma_0}^{0} \frac{\alpha\Big((1+h+\varepsilon)\ln\mu\left(\frac{1-\varepsilon}{1+h}\sigma,F\right)+\ln K(\varepsilon)\Big)}{|\sigma|^2\beta\big(1/|\sigma|\big)}d\sigma$$
$$\leq K_1\int_{\sigma_0}^{0} \frac{\alpha\Big(\ln\mu\left(\frac{1-\varepsilon}{1+h}\sigma,F\right)\Big)}{|\sigma|^2\beta\big(1/|\sigma|\big)}d\sigma \leq K_1K_2\int_{\sigma_0}^{0} \frac{\alpha\big(\ln\mu(\sigma,F)\big)}{|\sigma|^2\beta\big(1/|\sigma|\big)}d\sigma < +\infty,$$

i.e. relations (16) and (17) are equivalent. The equivalence of relations (16) and (17) in the case of condition **b)** is proved in [12].

If all $F_j \in C^0_{\alpha,\beta}$, then from (7) we obtain

$$\int_{\sigma_0}^{0} \frac{\alpha\big(\ln\mu(m\sigma,F)\big)}{|\sigma|^2\beta\big(1/|\sigma|\big)}d\sigma$$
$$\leq \sum_{k_1+\cdots+k_p=m}\left(k_1\int_{\sigma_0}^{0} \frac{\alpha\big(\ln\mu(\sigma,F_1)\big)}{|\sigma|^2\beta\big(1/|\sigma|\big)}d\sigma+\cdots+k_p\int_{\sigma_0}^{0} \frac{\alpha\big(\ln\mu(\sigma,F_1)\big)}{|\sigma|^2\beta\big(1/|\sigma|\big)}d\sigma\right)+\text{const}<+\infty.$$

Since $\beta \in L^0$, hence (17) follows and, thus, $F \in C^0_{\alpha,\beta}$.

On the contrary, if $|c_{m0\ldots0}||a_{n,1}|^m \to +\infty$ and $|a_{n,j}| = o\big(|a_{n,1}|\big)$ as $n \to \infty$ for $2 \leq j \leq p$ and $F \in C^0_{\alpha,\beta}$, then (8) implies

$$\int_{\sigma_0}^{0} \frac{\alpha\big(\ln\mu(\sigma,F_j)\big)}{|\sigma|^2\beta\big(1/|\sigma|\big)}d\sigma \leq \int_{\sigma_0}^{0} \frac{\alpha\big(\ln\mu(\sigma,F_1)\big)}{|\sigma|^2\beta\big(1/|\sigma|\big)}d\sigma \leq \int_{\sigma_0}^{0} \frac{\alpha\big(\ln\mu(\sigma,F)\big)}{|\sigma|^2\beta\big(1/|\sigma|\big)}d\sigma+\text{const}<+\infty,$$

i.e. all $F_j \in C^0_{\alpha,\beta}$. The proof of Theorem 4 is complete. □

3 Pseudostarlikeness and pseudoconvexity

Here we consider Dirichlet series of the form

$$F(s) = e^{sh} - \sum_{n=1}^{\infty} a_n \exp\{s\lambda_n\}, \qquad a_n > 0, \quad \lambda_1 > h > 0, \tag{18}$$

absolutely convergent in half-plane $\Pi_0 = \{s : \operatorname{Re} s < 0\}$.

A conformal function (18) in Π_0 is said [16] to be pseudostarlike of the order $\alpha \in [0, h)$ and the type $\beta > 0$ if

$$\left|\frac{F'(s)}{F(s)} - h\right| < \beta\left|\frac{F'(s)}{F(s)} - (2\alpha - h)\right|, \qquad s \in \Pi_0, \tag{19}$$

and is said to be pseudoconvex of the order $\alpha \in [0, h)$ and the type $\beta > 0$ if

$$\left|\frac{F''(s)}{F'(s)} - h\right| < \beta\left|\frac{F''(s)}{F'(s)} - (2\alpha - h)\right|, \qquad s \in \Pi_0.$$

If $\beta = 1$ we obtain the definition of the pseudostarlikeness and the pseudoconvexity of the order $\alpha \in [0, h)$ and if $\beta = 1$ and $\alpha = 0$ we obtain the definition [7,17] of the pseudostarlikeness and the pseudoconvexity of function (18).

We need the following lemma.

Lemma 7 ([16]). *Function* (18) *is pseudostarlike of the order* $\alpha \in [0, h)$ *and the type* $\beta > 0$ *if and only if*

$$\sum_{n=1}^{\infty} \big\{(1+\beta)\lambda_n - 2\beta\alpha - h(1-\beta)\big\} a_n \le 2\beta(h-\alpha), \tag{20}$$

and is pseudoconvex of the order $\alpha \in [0, h)$ *and the type* $\beta > 0$ *if and only if*

$$\sum_{n=1}^{\infty} \lambda_n \big\{(1+\beta)\lambda_n - 2\beta\alpha - h(1-\beta)\big\} a_n \le 2h\beta(h-\alpha). \tag{21}$$

Let Dirichlet series

$$F_j(s) = e^{sh} - \sum_{n=1}^{\infty} a_{n,j} \exp\{s\lambda_n\}, \quad a_{n,j} > 0 (1 \le j \le p), \quad \lambda_1 > h > 0, \tag{22}$$

be absolutely convergent in $\Pi_0 = \{s : \operatorname{Re} s < 0\}$. We say that function (18) is a positive Hadamard composition of genus m of Dirichlet series (22) if $a_n = P(a_{n,1}, \dots, a_{n,p})$, where $P(x_1, \dots, x_p)$ is a homogeneous polynomial of degree $m \ge 1$ with non-negative coefficients.

Using Lemma 7, we can find conditions under which the pseudostarlikeness (pseudoconvexity) of all functions F_j implies the pseudostarlikeness (pseudoconvexity) of the function F. We will have to distinguish between cases $m = 1$ and $m \ge 2$. Remark that a homogeneous polynomial of the degree $m = 1$ has the form $P(x_1, \dots, x_p) = c_1 x_1 + \dots + c_p x_p$.

Theorem 5. *Let functions* (22) *are pseudostarlike of the order* $\alpha_j \in [0, h)$ *for all* $1 \le j \le p$ *and function* (18) *is a positive Hadamard composition of genus* m *of functions* (22). *If either* $m = 1$ *and* $0 < c_1 + \dots + c_p \le 1$ *or* $m \ge 2$ *and* $\sum\limits_{n=1}^{\infty} (h/\lambda_n)^{m-1} \sum\limits_{k_1+\dots+k_p=m} c_{k_1\dots k_p} \le 1$*, then function* (18) *is pseudostarlike of the order* $\alpha = \min\limits_{1\le j\le p} \alpha_j$.

Proof. If the function F_j is pseudostarlike of the order $\alpha_j \in [0, h)$ then by Lemma 7 we have $\sum_{n=1}^{\infty} \{\lambda_n - \alpha_j\} a_{n,j} \le h - \alpha_j$ and, thus,

$$a_{n,j} \le \frac{h-\alpha_j}{\lambda_n - \alpha_j} < 1, \quad j = 1, 2. \tag{23}$$

Therefore, if F is the positive Hadamard composition of genus $m = 1$ of the functions F_j, that is $a_n = c_1 a_{n,1} + \dots + c_p a_{n,p}$, then

$$\begin{aligned}
\sum_{n=1}^{\infty} \{\lambda_n - \alpha\} a_n &= c_1 \sum_{n=1}^{\infty} \{\lambda_n - \alpha\} a_{n,1} + \dots + c_p \sum_{n=1}^{\infty} \{\lambda_n - \alpha\} a_{n,p} \\
&= c_1 \sum_{n=1}^{\infty} \frac{\lambda_n - \alpha}{\lambda_n - \alpha_1} (\lambda_n - \alpha_1) a_{n,1} + \dots + c_p \sum_{n=1}^{\infty} \frac{\lambda_n - \alpha}{\lambda_n - \alpha_p} (\lambda_n - \alpha_p) a_{n,p} \\
&\le c_1 \sum_{n=1}^{\infty} \frac{h-\alpha}{h-\alpha_1} (\lambda_n - \alpha_1) a_{n,1} + \dots + c_p \sum_{n=1}^{\infty} \frac{h-\alpha}{h-\alpha_p} (\lambda_n - \alpha_p) a_{n,p} \\
&= c_1 \frac{h-\alpha}{h-\alpha_1} \sum_{n=1}^{\infty} (\lambda_n - \alpha_1) a_{n,1} + \dots + c_p \frac{h-\alpha}{h-\alpha_p} \sum_{n=1}^{\infty} (\lambda_n - \alpha_p) a_{n,p} \\
&\le c_1 (h-\alpha) + \dots + c_p (h-\alpha) = (c_1 + \dots + c_p)(h-\alpha) \le h - \alpha,
\end{aligned}$$

i.e. the function F is pseudostarlike of the order $\alpha = \min\limits_{1\le j\le p} \alpha_j$.

Now let $m \geq 2$. Then $a_n = \sum\limits_{k_1+\cdots+k_p=m} c_{k_1\ldots k_p} a_{n,1}^{k_1} \cdots a_{n,p}^{k_p}$ and in view of (23) we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\{\lambda_n-\alpha\}a_n &\leq \sum_{n=1}^{\infty}\{\lambda_n-\alpha\}\sum_{k_1+\cdots+k_p=m} c_{k_1\ldots k_p}\left(\frac{h-\alpha_1}{\lambda_n-\alpha_1}\right)^{k_1}\cdots\left(\frac{h-\alpha_p}{\lambda_n-\alpha_p}\right)^{k_p}\\
&\leq \sum_{n=1}^{\infty}\{\lambda_n-\alpha\}\sum_{k_1+\cdots+k_p=m} c_{k_1\ldots k_p}\left(\frac{h-\alpha}{\lambda_n-\alpha}\right)^{k_1}\cdots\left(\frac{h-\alpha}{\lambda_n-\alpha}\right)^{k_p}\\
&= \sum_{n=1}^{\infty}\{\lambda_n-\alpha\}\sum_{k_1+\cdots+k_p=m} c_{k_1\ldots k_p}\left(\frac{h-\alpha}{\lambda_n-\alpha}\right)^{m}\\
&= \sum_{n=1}^{\infty}\left(\frac{h-\alpha}{\lambda_n-\alpha}\right)^{m-1}(h-\alpha)\sum_{k_1+\cdots+k_p=m} c_{k_1\ldots k_p}\\
&\leq \sum_{n=1}^{\infty}\left(\frac{h}{\lambda_n}\right)^{m-1}(h-\alpha)\sum_{k_1+\cdots+k_p=m} c_{k_1\ldots k_p} \leq h-\alpha,
\end{aligned}$$

i.e. the function F is pseudostarlike of the order $\alpha = \min\limits_{1\leq j\leq p} \alpha_j$. Theorem 5 is proved. □

Using (21) and repeating the proof of Theorem 5, it is easy to prove the following assertion.

Proposition 1. *Let functions* (22) *are pseudoconvex of the order* $\alpha_j \in [0, h)$ *for all* $1 \leq j \leq p$, *and function* (18) *is a positive Hadamard composition of genus* m *of functions* (22). *If either* $m = 1$ *and* $0 < c_1 + \cdots + c_p \leq 1$ *or* $m \geq 2$ *and* $\sum\limits_{n=1}^{\infty} (h/\lambda_n)^{2(m-1)} \sum\limits_{k_1+\cdots+k_p=m} c_{k_1\ldots k_p} \leq 1$, *then function* (18) *is pseudoconvex of the order* $\alpha = \min\limits_{1\leq j\leq p} \alpha_j$.

For pseudostarlike functions of the order $\alpha \in [0, h)$ and the type $\beta \geq 0$ the following theorem is true.

Theorem 6. *Let functions* (22) *are pseudostarlike of the order* $\alpha \in [0, h)$ *and the type* $\beta_j \geq 0$ *for all* $1 \leq j \leq p$, *and function* (18) *is a positive Hadamard composition of genus* m *of functions* (22). *If either* $m = 1$ *and* $0 < c_1 + \cdots + c_p \leq 1$ *or* $m \geq 2$ *and*

$$\left(\sum_{k_1+\cdots+k_p=m} c_{k_1\ldots k_p}\right)\sum_{n=1}^{\infty}\left(\frac{2h}{\lambda_n-h}\right)^{m-1} \leq 1$$

then function (18) *is pseudostarlike of the order* α *and the type* $\beta = \max\limits_{1\leq j\leq p} \beta_j$.

Proof. Let the functions F_j are pseudostarlike of the order $\alpha_j \in [0, h)$ and the type $\beta_j \geq 0$. If F is a positive Hadamard composition of genus $m = 1$ of the functions F_j, then in view of (20) we get

$$\begin{aligned}
&\sum_{n=1}^{\infty}\big\{(1+\beta)\lambda_n-2\beta\alpha-h(1-\beta)\big\}a_n\\
&\quad= c_1\sum_{n=1}^{\infty}\big\{(1+\beta)\lambda_n-2\beta\alpha-h(1-\beta)\big\}a_{n,1}+\cdots+c_p\sum_{n=1}^{\infty}\big\{(1+\beta)\lambda_n-2\beta\alpha-h(1-\beta)\big\}a_{n,p}\\
&\quad= c_1\sum_{n=1}^{\infty}\frac{(1+\beta)\lambda_n-2\beta\alpha-h(1-\beta)}{(1+\beta_1)\lambda_n-2\beta_1\alpha-h(1-\beta_1)}\big\{(1+\beta_1)\lambda_n-2\beta_1\alpha-h(1-\beta_1)\big\}a_{n,1}+\ldots\\
&\qquad+c_p\sum_{n=1}^{\infty}\frac{(1+\beta)\lambda_n-2\beta\alpha-h(1-\beta)}{(1+\beta_p)\lambda_n-2\beta_p\alpha-h(1-\beta_p)}\big\{(1+\beta_p)\lambda_n-2\beta_p\alpha-h(1-\beta_p)\big\}a_{n,p}
\end{aligned}$$

$$\le c_1\sum_{n=1}^{\infty}\frac{\beta}{\beta_1}\{(1+\beta_1)\lambda_n-2\beta_1\alpha-h(1-\beta_1)\}a_{n,1}+\ldots$$
$$+c_p\sum_{n=1}^{\infty}\frac{\beta}{\beta_p}\{(1+\beta_p)\lambda_n-2\beta_p\alpha-h(1-\beta_p)\}a_{n,p}$$
$$\le c_1\frac{\beta}{\beta_1}2\beta_1(h-\alpha)+\cdots+c_p\frac{\beta}{\beta_p}2\beta_p(h-\alpha)=2(c_1+\cdots+c_p)\beta(h-\alpha)\le 2\beta(h-\alpha),$$

i.e. the function F is pseudostarlike of the order α and the type $\beta=\max\limits_{1\le j\le p}\beta_j$.

If F_j is the positive Hadamard composition of genus $m\ge 2$ of the functions F_j, then in view of (20) we get $a_{n,j}\le\frac{2\beta_j(h-\alpha)}{(1+\beta_j)\lambda_n-2\beta_j\alpha-h(1-\beta_j)}$, $1\le j\le p$, and, therefore,

$$a_n\le\sum_{k_1+\cdots+k_p=m}c_{k_1\ldots k_p}\left(\frac{2\beta_1(h-\alpha)}{(1+\beta_1)\lambda_n-2\beta_1\alpha-h(1-\beta_1)}\right)^{k_1}\cdots\left(\frac{2\beta_p(h-\alpha)}{(1+\beta_p)\lambda_n-2\beta_p\alpha-h(1-\beta_p)}\right)^{k_p}$$
$$\le\sum_{k_1+\cdots+k_p=m}c_{k_1\ldots k_p}\left(\frac{2\beta(h-\alpha)}{(1+\beta)\lambda_n-2\beta\alpha-h(1-\beta)}\right)^{k_1}\cdots\left(\frac{2\beta(h-\alpha)}{(1+\beta)\lambda_n-2\beta\alpha-h(1-\beta)}\right)^{k_p}$$
$$=\left(\frac{2\beta(h-\alpha)}{(1+\beta)\lambda_n-2\beta\alpha-h(1-\beta)}\right)^{m}\sum_{k_1+\cdots+k_p=m}c_{k_1\ldots k_p}.$$

Hence, it follows that

$$\sum_{n=1}^{\infty}\{(1+\beta)\lambda_n-2\beta\alpha-h(1-\beta)\}a_n$$
$$\le 2\beta(h-\alpha)\sum_{k_1+\cdots+k_p=m}c_{k_1\ldots k_p}\sum_{n=1}^{\infty}\left(\frac{2\beta(h-\alpha)}{(1+\beta)\lambda_n-2\beta\alpha-h(1-\beta)}\right)^{m-1}$$
$$\le 2\beta(h-\alpha)\left(\sum_{k_1+\cdots+k_p=m}c_{k_1\ldots k_p}\right)\sum_{n=1}^{\infty}\left(\frac{2h}{\lambda_n-h}\right)^{m-1}\le 2\beta(h-\alpha),$$

i.e. the function F is pseudostarlike of the order α and the type $\beta=\max\limits_{1\le j\le p}\beta_j$. Theorem 6 is proved. □

Using (21) and repeating the proof of Theorem 6, it is easy to prove the following assertion.

Proposition 2. *Let functions* (22) *are pseudoconvex of the order* $\alpha\in[0,h)$ *and the type* $\beta_j\ge 0$ *for all* $1\le j\le p$ *and function* (18) *is a positive Hadamard composition of genus* m *of functions* (22). *If either* $m=1$ *and* $0<c_1+\cdots+c_p\le 1$ *or* $m\ge 2$ *and*

$$\left(\sum_{k_1+\cdots+k_p=m}c_{k_1\ldots k_p}\right)\sum_{n=1}^{\infty}\left(\frac{2h}{\lambda_n-h}\right)^{2(m-1)}\le 1$$

then function (18) *is pseudoconvex of the order* α *and the type* $\beta=\max\limits_{1\le j\le p}\beta_j$.

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Нехай $F(s) = \sum\limits_{n=1}^{\infty} a_n \exp\{s\lambda_n\}$ і $F_j(s) = \sum\limits_{n=1}^{\infty} a_{n,j} \exp\{s\lambda_n\}$, $j = \overline{1,p}$, — ряди Діріхле з показниками $0 \le \lambda_n \uparrow +\infty$, $n \to \infty$, і абсцисами абсолютної збіжності рівними 0. Функція F називається адамаровою композицією роду $m \ge 1$ функцій F_j, якщо $a_n = P(a_{n,1}, \dots, a_{n,p})$, де $P(x_1, \dots, x_p) = \sum\limits_{k_1+\dots+k_p=m} c_{k_1 \dots k_p} x_1^{k_1} \cdots x_p^{k_p}$ — однорідний поліном степеня m. У термінах узагальненого порядку, узагальнених типів і узагальнених класів збіжності досліджено зв'язок між зростанням функцій F_j і зростанням їхньої адамарової композиції роду $m \ge 1$. Вивчено псевдозірковість і псевдоопуклість адамарової композиції роду $m \ge 1$.

Ключові слова і фрази: ряд Діріхле, композиція Адамара, узагальнений порядок, узагальнений тип, узагальнений клас збіжності, псевдозірковість, псевдоопуклість.