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On positive Cohen weakly nuclear multilinear operators

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In this article, we establish new relationships involving the class of Cohen positive strongly p-summing multilinear operators. Furthermore, we introduce a new class of multilinear operators on Banach lattices, called positive Cohen weakly nuclear multilinear operators. We establish a Pietsch domination-type theorem for this new class of multilinear operators. As an application, we show that every positive Cohen weakly p-nuclear multilinear operator is positive Dimant strongly p-summing and Cohen positive strongly p-summing. We conclude with a tensor representation of our class.

Key words and phrases: Banach lattice, Pietsch domination theorem, positive p-summing operator, tensor norm.

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1 Introduction

Great mathematicians such as A. Grothendieck in the 1950s and A. Pietsch in 1967 developed the theory of linear operators. Over the years, several authors have extended different summability concepts to multilinear operators between Banach spaces. Now we are interested in extending these concepts to positive multilinear mappings between Banach lattices. The multi-ideals $\mathcal{N}_p^m$ of Cohen p-nuclear multilinear operators between Banach spaces was defined by D. Achour and A. Alouani in [1] as a natural extension of the classical ideal of p-nuclear multilinear operators that were studied in [9]. These multi-ideals have excellent properties such as Pietsch domination theorem and Kwapień's factorization. In [7], the concept of positive Cohen p-nuclear multilinear operators between the spaces of the Banach lattice is discussed, and a natural analogue to Pietsch domination theorem for this class of operators is presented.

In this article, we extend the class of positive Cohen p-nuclear multilinear operators to a new class of positive multilinear operators between two Banach lattices, called positive Cohen weakly p-nuclear multilinear operators. We prove that this new class lies between the class of Cohen strongly p-summing multilinear operators and the class of Cohen p-nuclear multilinear operators and study some their properties. Furthermore, we obtain a new relationship between Cohen positive strongly p-summing multilinear operators and those involving multilinear operators resulting from the composition method.

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The letters $X, X_1, \dots, X_m, Y$ will be used for Banach spaces. By B_X we denote the closed unit ball of X. Let $E, E_1, \dots, E_m, F$ stand for Banach lattices. Our spaces are over the field of real scalars $\mathbb{R}$. Recall that all Banach lattices are Archimedean. We denote by $\mathcal{L}(X_1, \dots, X_m; Y)$ the Banach space of all bounded m-linear operators $T : X_1 \times \cdots \times X_m \to Y$ endowed with the norm

$$\|T\| = \sup_{\substack{x^j \in B_{X_j} \\ 1 \le j \le m}} \|T(x^1, \dots, x^m)\|.$$

When $m = 1$ we say that T is a bounded linear operator instead of saying that it is a bounded 1-linear operator.

Let X^* be the topological dual of X, that is $X^* = \mathcal{L}(X; \mathbb{R})$. The elements of the topological dual will be called linear functionals. Given $T \in \mathcal{L}(X_1, \dots, X_m; Y)$, it is known that there is a unique bounded linear operator $T_L : X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m \to Y$, where $X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m$ denotes the completed projective m-fold tensor product, such that $T = T_L \circ \delta_m$, where $\delta_m : X_1 \times \cdots \times X_m \to X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m$ is given by $\delta_m(x_1, \dots, x_m) = x_1 \otimes \cdots \otimes x_m$.

For a Banach lattice E, by E^+ we denote its positive cone, and for each $x \in E$, x^+ and x^- stand for the positive and the negative part of x, respectively.

Recall that a linear functional $x^* \in E^*$ is *positive* if $\langle x, x^* \rangle \ge 0$ for all $x \in E^+$ and x^* is called *regular* if such a functional is the difference of two positive linear functionals. Denote by E^{*+} the set of all positive functionals on E. The order dual $E^\sim$ of E is the Dedekind complete Banach lattice $E^\sim$ generated by E^{*+} and it is well-known that $E^\sim$ coincides with the topological dual E^* of E. We consider the unit ball in E^{*+} as the set $B^+_{E^*} := B_{E^*} \cap E^{*+} = \{\xi \in B_{E^*} : \xi \ge 0\}$. An m-linear operator $T : E_1 \times \cdots \times E_m \to F$, satisfying $|T(x_1, \dots, x_m)| = T(|x_1|, \dots, |x_n|)$ for all $x_k \in E_k$ with $k = 1, \dots, m$, is called a *lattice m-morphism*. In the case $m = 1$, we say that T is a *lattice homomorphism*.

Given the m-fold tensor product $E_1 \otimes \cdots \otimes E_m$ of Banach lattices $E_1, \dots, E_m$, in [10, 11] D.H. Fremlin introduced the vector lattice tensor product $E_1 \overline{\otimes} \cdots \overline{\otimes} E_m$ in such a way that $|x_1 \otimes \cdots \otimes x_m| = |x_1| \otimes \cdots \otimes |x_m|$ for all $x_i \in E_i$, $i = 1, \dots, m$. Moreover, if $\theta \in E_1 \overline{\otimes} \cdots \overline{\otimes} E_m$, then there exist $x_i \in E_i^+$, $i = 1, \dots, m$, such that $\|\theta\| \le x_1 \otimes \cdots \otimes x_m$, and $E_1 \otimes \cdots \otimes E_m$ is order dense in $E_1 \overline{\otimes} \cdots \overline{\otimes} E_m$, that is, for any $\theta > 0$ in $E_1 \overline{\otimes} \cdots \overline{\otimes} E_m$ there exist $x_i > 0$ in E_i, $i = 1, \dots, m$, such that $\theta \ge x_1 \otimes \cdots \otimes x_m > 0$. The positive projective tensor norm $\|\theta\|_{|\pi|}$ of an element $\theta \in E_1 \overline{\otimes} \cdots \overline{\otimes} E_m$ is defined by

$$\|\theta\|_{|\pi|} = \inf \left\{ \sum_{i=1}^n \prod_{j=1}^n \|x_i^j\| : x_i^j \in E_j^+, \ n \in \mathbb{N}, |\theta| \le \sum_{i=1}^n x_i^1 \otimes \cdots \otimes x_i^m \right\}.$$

It is well known that $\|\cdot\|_{|\pi|}$ is a lattice norm on $E_1 \overline{\otimes} \cdots \overline{\otimes} E_m$. By [11], the tensor product $E_1 \otimes \cdots \otimes E_m$ is norm dense in $E_1 \overline{\otimes} \cdots \overline{\otimes} E_m$ and the cone generated by the elements $\{x_1 \otimes \cdots \otimes x_m : x_i \in E_i^+, i = 1, \dots, m\}$ is norm dense in $(E_1 \overline{\otimes} \cdots \overline{\otimes} E_m)^+$. We call the space $E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m$ a *positive m-fold projective tensor product*, and such a space is the Banach lattice defined as the completion of $\left(E_1 \overline{\otimes} \cdots \overline{\otimes} E_m, \|.\|_{|\pi|}\right) = E_1 \overline{\otimes}_{|\pi|} \cdots \overline{\otimes}_{|\pi|} E_m$.

According to [11], $E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m$ is the completion of $\left(E_1 \otimes \cdots \otimes E_m, \|.\|_{|\pi|}\right)$. Moreover, $\otimes : E_1 \times \cdots \times E_m \to E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m$, with $\otimes(x_1, \dots, x_m) = x_1 \otimes \cdots \otimes x_m$, $x_i \in E_i$, $i = 1, \dots, m$, is a lattice m-morphism.

For any Archimedean vector lattice F there is a one-to-one correspondence between lattice m-morphisms $T : E_1 \times \cdots \times E_m \to F$ and lattice homomorphisms $T^{\otimes} : E_1 \overline{\otimes} \cdots \overline{\otimes} E_m \to F$ given by $T = T^{\otimes} \circ \otimes$.

The positive cone in $E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m$ is the closure in $E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m$ of the cone in $E_1 \otimes \cdots \otimes E_m$ generated by $\{x_1 \otimes \cdots \otimes x_m : x_i \in E_i^+, i = 1, \ldots, m\}$. Let us denote by $E_1^+ \otimes \cdots \otimes E_m^+$ such a cone, that is

$$E_1^+ \otimes \cdots \otimes E_m^+ := \left\{ \sum_{i=1}^{n} x_i^1 \otimes \cdots \otimes x_i^m : x_i^j \in E_j^+,\ j = 1, \ldots, m,\ n \in \mathbb{N} \right\}.$$

Recall that an m-linear operator $T : E_1 \times \cdots \times E_m \to F$ is called *positive* if $T(x^1, \ldots, x^m) \in F^+$, whenever $x^1 \in E_1^+, \ldots, x^m \in E_m^+$. We denote by $\mathcal{L}^+ (E_1, \ldots, E_m; F)$ the collection of all positive m-linear operators from $E_1 \times \cdots \times E_m$ to F, and $\mathcal{L}^r (E_1, \ldots, E_m; F)$ stands for the space of all *regular m-linear operators* from $E_1 \times \cdots \times E_m$ to F, that is, $T \in \mathcal{L}^r (E_1, \ldots, E_m; F)$ if T can be written as $T = T_1 - T_2$ with $T_1, T_2 \in \mathcal{L}^+ (E_1, \ldots, E_m; F)$. If F is Dedekind complete, then $\mathcal{L}^r (E_1, \ldots, E_m; F)$ is a Banach lattice with the norm $\|T\|_r = \||T|\|$, for each $T \in \mathcal{L}^r (E_1, \ldots, E_m; F)$. Note that $\mathcal{L}^r (E; \mathbb{R}) = E^\sim = E^*$.

Q. Bu and G. Buskes studied the relation between spaces of regular m-linear mappings (and spaces of regular m-homogeneous polynomials), as the spaces of linear operators defined on positive m-fold projective tensor products.

From [8, Proposition 3.3], it follows that if $E_1, \ldots, E_m$ and F are Banach lattices such that F is Dedekind complete, then for every $T \in \mathcal{L}^r (E_1, \ldots, E_m; F)$ there is an unique operator $T^{\otimes} \in \mathcal{L}^r \big(E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m; F\big)$ such that $T^{\otimes}\big(x^1 \otimes \cdots \otimes x^m\big) = T\big(x^1, \ldots, x^m\big)$.

The correspondence $T \to T^{\otimes}$ is an isometric isomorphism and lattice homomorphism between $\mathcal{L}^r (E_1, \ldots, E_m; F)$ and $\mathcal{L}^r \big(E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m; F\big)$. The linear mapping T_L is called the (*lattice* or *Riesz*) *linearization of* T. In particular, when $F = \mathbb{R}$ then $\mathcal{L}^r (E_1, \ldots, E_m; \mathbb{R})$ and $\Big(E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m\Big)^*$ are isometrically isomorphic and lattice homomorphic. In light of the above, we can consider

$$B^+_{\mathcal{L}^r(E_1,\ldots,E_m;\mathbb{R})} := \left\{\phi \in \mathcal{L}^+(E_1, \ldots, E_m; \mathbb{R}) : \|\phi\| \le 1\right\},$$

and identify it with $B^+_{\left(E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m\right)^*}$, which is compact for the weak*-topology. Given $\phi \in \big(E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m\big)^*$, we will denote by $\widetilde{\phi}$ the unique regular m-linear operator in $\mathcal{L}^r (E_1, \ldots, E_m; F)$ such that $(\widetilde{\phi})^{\otimes} = \phi$.

Now we briefly recall some classic sequence spaces that will be very useful for the theory. Considering $1 \le p < \infty$, we remind that

- $\ell_p(X)$ consist of absolutely p-summable X-valued sequences with the usual norm $\|\cdot\|_p$;
- $\ell_{p,weak}(X)$ consist of weakly p-summable X-valued sequences with the norm

$$w_p\big((x_i)_{i=1}^\infty\big) = \big\|(x_i)_{i=1}^\infty\big\|_{\ell_{p,weak}(X)} := \sup_{\varphi \in B_{X^*}} \left(\sum_{i=1}^{\infty} |\langle x_i, \varphi\rangle|^p \right)^{\frac{1}{p}};$$

- $\ell^n_{p,|weak|}(E) = \left\{(x_i)_{i=1}^n : \big(|x_i|\big)_{i=1}^n \in \ell^n_{p,weak}(E)\right\}$.

If $x_1, \ldots, x_n \geq 0$, with $x_i \in E$ for every $i = 1, \ldots, n$, then we have

$$\left\| (x_i)_{i=1}^n \right\|_{\ell^n_{p,|weak|}(E)} := \sup_{\xi \in B^+_{E^*}} \left(\sum_{i=1}^n \langle x_i, \xi \rangle^p \right)^{\frac{1}{p}}.$$

Recall that when the sequences are finite (with n terms) we write $\ell^n_p(X)$ and $\ell^n_{p,weak}(X)$ instead of $\ell_p(X)$ and $\ell_{p,weak}(X)$, respectively. When there is no ambiguity about the space worked, to simplify the notation, we will sometimes just use $\|\cdot\|_{\ell^n_{p,|weak|}}$, rather than $\|\cdot\|_{\ell^n_{p,|weak|}(E)}$.

Given $y_1, \ldots, y_n \in E$, we have (see [12])

$$\left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} = \sup \left\{ \sum_{i=1}^n \alpha_i y_i : \alpha_i \in \mathbb{R}, \sum_{i=1}^n |\alpha_i|^{p^*} \leq 1 \right\},$$

where $1 \leq p, p^* < \infty$ are such that $\frac{1}{p} + \frac{1}{p^*} = 1$.

In the case $p = \infty$, we have

$$\left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} = \sup_{1 \leq i \leq n} |y_i|.$$

We conclude this section by recalling the definitions of important operators, which we will use later. It is possible to show that all these spaces are Banach spaces with their respective norms. In all cases we consider $1 \leq p \leq \infty$, with the exception of item *(v)*, where p must always be finite.

i) Introduced by O. Blasco in [3], a continuous linear operator $u : E \longrightarrow Y$ is *positive p-summing* (notation $u \in \Pi^+_p(E, Y)$), if there exists a constant $C > 0$ such that the inequality such that for any $(x_i)_{i=1}^n \subset E$, we have

$$\left\| \left(u(x_i) \right)_{i=1}^n \right\|_p \leq C \left\| (x_i)_{i=1}^n \right\|_{\ell^n_{p,|weak|}(E)}.$$

For $p = \infty$,

$$\sup_{1 \leq i \leq n} \left\| u(x_i) \right\| \leq C \sup_{\xi \in B^+_{E^*}} \sup_{1 \leq i \leq n} \left\langle |x_i|, \xi \right\rangle.$$

The infimum of the constant $C > 0$, with respect to previous inequalities, defines a norm $\pi^+_p(\cdot)$ on $\Pi^+_p(E, Y)$.

ii) By [2], a linear operator $u : X \longrightarrow F$ is *positive strongly p-summing* if there exists a constant $C > 0$, such that for all $n \in \mathbb{N}$, $(x_i)_{i=1}^n \subset X$ and $(y^*_i)_{i=1}^n \subset F^*$, we have

$$\sum_{i=1}^n \left| \langle u(x_i), y^*_i \rangle \right| \leq C \left\| (x_i)_{i=1}^n \right\|_p \left\| (y^*_i)_{i=1}^n \right\|_{\ell^n_{p^*,|weak|}(F^*)}.$$

The class of all positive strongly p-summing operators between X and F will be denoted by $\mathcal{D}^+_p(X, F)$. The infimum of all the constant C with respect to the previous inequality defines the norm $d^+_p(\cdot)$ on $\mathcal{D}^+_p(X, F)$.

iii) According to [5], a multilinear operator $T \in \mathcal{L}(X_1, \ldots, X_m; F)$ is called *Cohen positive strongly p-summing* if there is a constant $C > 0$, such that for any $x_1^j, \ldots, x_n^j \in X_j$, with $1 \leq j \leq m$, and any $y_1^*, \ldots, y_n^* \in F^*$, we have

$$\left\| \left\langle \left(T\left(x_i^1, \ldots, x_i^m\right), y_i^* \right) \right\rangle_{i=1}^n \right\|_{\ell_1^n} \leq C \left(\sum_{i=1}^n \prod_{j=1}^m \left\| x_i^j \right\|_{X_j}^p \right)^{\frac{1}{p}} \left\| (y_i^*)_{i=1}^n \right\|_{\ell^n_{p^*,|weak|}(F^*)}.$$

Again, the class of all Cohen positive strongly p-summing m-linear operators from $X_1 \times \cdots \times X_m$ into F, will be denoted by $\mathcal{D}_p^{m+}(X_1, \ldots, X_m; F)$. The infimum of all the constants C with respect to the previous inequality defines the norm $d_p^{m+}(\cdot)$ on $\mathcal{D}_p^{m+}(X_1, \ldots, X_m; F)$.

iv) In line with [6], a multilinear operator $T \in \mathcal{L}(E_1, \ldots, E_m; F)$ is called *positive Dimant-strongly p-summing*, if there exists a constant $C > 0$ such that for every $x_1^j, \ldots, x_n^j \in E_j^+$, $j = 1, \ldots, m$, we have

$$\left(\sum_{i=1}^n \left\| T\left(x_i^1, \ldots, x_i^m\right) \right\|^p \right)^{\frac{1}{p}} \leq C \sup_{\phi \in B^+_{\mathcal{L}(E_1, \ldots, E_m)}} \left(\sum_{i=1}^n \left(\phi\left(x_i^1, \ldots, x_i^m\right)\right)^p \right)^{\frac{1}{p}}.$$

The space of all positive strongly p-summing m-linear operators from $E_1 \times \cdots \times E_m$ into F, will be denoted by $\mathcal{L}_{ss,p}^{m+}(E_1, \ldots, E_m; F)$. The infimum of all the constants C with respect to the previous inequality defines the norm $d_{s,p}^{m+}(\cdot)$ on $\mathcal{L}_{ss,p}^{m+}(E_1, \ldots, E_m; F)$.

v) According to [7], $T \in \mathcal{L}(E_1, \ldots, E_m; F)$ is called *positive Cohen p-nuclear*, if there is a constant $C > 0$, such that for any $x_1^j, \ldots, x_n^j \in E_j$, $1 \leq j \leq m$, and any $y_1^*, \ldots, y_n^* \in F^*$, we have

$$\left| \sum_{i=1}^n \left\langle T(x_i^1, \ldots, x_i^m), y_i^* \right\rangle \right| \leq C \left(\sup_{x^{j*} \in B^+_{E_j^*}, 1 \leq j \leq m} \sum_{i=1}^n \prod_{j=1}^m \left\langle |x_i^j|, x^{j*} \right\rangle^p \right)^{\frac{1}{p}} \left\| (y_i^*)_{i=1}^n \right\|_{\ell^n_{p^*,|weak|}(F^*)}.$$

The space of all positive Cohen p-nuclear m-linear operators from $E_1 \times \cdots \times E_m$ into F will be denoted by $\mathcal{N}_p^{m+}(E_1 \times \cdots \times E_m; F)$. If $m = 1$, such space consist of all the positive Cohen p-nuclear operators. The infimum of all the constants C with respect to the previous inequality defines the norm $n_p^{m+}(\cdot)$ on $\mathcal{N}_p^{m+}(E_1 \times \cdots \times E_m; F)$. For $p = \infty$, it is possible to show that $\mathcal{N}_\infty^{m+}(E_1 \times \cdots \times E_m; F) = \mathcal{D}_\infty^{m+}(E_1 \times \cdots \times E_m; F)$.

Our article is organized as follows. In the Section 2, we study the multilinear mappings generated by the class $\mathcal{D}_p^+$. We also show that the multilinear operators resulting from the composition method coincide with the space of Cohen positive strongly summing multilinear operators. In the Section 3, we introduce the notion of positive Cohen weakly nuclear multilinear operator and prove a natural analogue to Pietsch domination theorem for this class. As an application, we study its relationships with some classes of positive m-linear operators (Dimant positive p-summing and Cohen strongly p-summing). These definitions and properties are inspired by the work done by Q. Bu et al. in [13]. Finally, in the Section 4, we introduce a reasonable crossnorm $\widetilde{\omega}_{(p;p^*)}$ such that the space $\mathcal{N}_{w,p}^{m+}(E_1, \ldots, E_m; F)$ of positive Cohen p-nuclear m-linear mappings is isometric to the dual of $E_1 \otimes \cdots \otimes E_m \otimes F^*$ endowed with $\widetilde{\omega}_{(p;p^*)}$. The results presented in this article can be seen as applications of the Riesz linearizations to regular multilinear mappings and polynomials provided by Q. Bu and G. Buskes.

2 Factorization method for Cohen positive strongly p-summing multilinear operators

In this section, we study multilinear mappings generated by the class $\mathcal{D}_p^+$. We show that the multilinear operators resulting by the composition method coincides with the space of Cohen positive strongly p-summing multilinear operators.

Given an ideal $\mathcal{I}$ of linear operators, a multilinear operator $T \in \mathcal{L}(X_1, \dots, X_m; Y)$ is said to be of *type* $\mathcal{L}(\mathcal{I})$, in symbols $T \in \mathcal{L}(\mathcal{I})(X_1, \dots, X_m; Y)$, if there exist Banach spaces $G_1, \dots, G_m$, linear mappings $u_j \in \mathcal{I}\left(X_j; G_j\right)$, $j = 1, \dots, m$, and a multilinear operator $A \in \mathcal{L}(G_1, \dots, G_m; Y)$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\;T\;} & Y\,, \\ {\scriptstyle (u_1,\dots,u_m)}\big\downarrow & \nearrow_{A} & \\ G & & \end{array}$$

with $X = X_1 \times \cdots \times X_m$, $G = G_1 \times \cdots \times G_m$, $(u_1, \dots, u_m)(x_1, \dots, x_m) := \big(u_1(x_1), \dots, u_m(x_m)\big)$. In other words, $T = A \circ (u_1, \dots, u_m)$.

If $\mathcal{I}$ is a normed operator ideal and $T \in \mathcal{L}(\mathcal{I})(X_1, \dots, X_m; Y)$, we can define

$$\|A\|_{\mathcal{L}(\mathcal{I})} = \inf \|A\| \, \|u_1\|_{\mathcal{I}} \cdots \|u_m\|_{\mathcal{I}}\,,$$

where the infimum is taken over all possible factorizations $T = A \circ (u_1, \dots, u_m)$ with $u_j \in \mathcal{I}$. Let $\mathcal{I}_j$, $1 \le j \le m$, be ideals of the operator. We will denote by $\mathcal{L}\left(\mathcal{I}_1, \dots, \mathcal{I}_m\right)$ the *ideal of multilinear mappings generated by the factorization method*, i.e. in the factorization $T = A \circ (u_1, \dots, u_m)$ we have $u_j \in \mathcal{I}_j\left(X_j; G_j\right)$, $j = 1, \dots, m$.

In order to prove an important result related to the factorization method, it is necessary to resort to the following results.

Proposition 1 ([5, Proposition 3.1])**.** *Let $1 < p \le \infty$ and $T \in \mathcal{L}(E_1, \dots, E_m; F)$. Then the operator T is Cohen positive strongly p-summing if and only if its linearization T_L is positive strongly p-summing. In this case, $d_p^{m+}(T) = d_p^+(T_L)$.*

Corollary 1. *The class $\mathcal{D}_p^{m+}$ is generated by the composition method from $\mathcal{D}_p^+$, i.e.*

$$\mathcal{D}_p^{m+}(E_1, \dots, E_m; F) = \mathcal{D}_p^+ \circ \mathcal{L}(E_1, \dots, E_m; F)\,.$$

Proposition 2. *Consider $1 \le p, p_1, \dots, p_m \le +\infty$ such that $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$. Let $u_j \in \mathcal{D}_{p_j}^+(E_j; F_j)$. Then the positive projective tensor product operator*

$$u_1 \otimes_{|\pi|} \cdots \otimes_{|\pi|} u_m : E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m \to F_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} F_m$$

is Cohen positive strongly p-summing.

Proof. We prove the result for the case $m = 2$, since the other cases are analogous. Let $u_1 \in \mathcal{D}_{p_1}^+(E_1; F_1)$ and $u_2 \in \mathcal{D}_{p_2}^+(E_2; F_2)$, we define the bilinear operator $u_1 \times u_2$ by

$$\begin{array}{rccc} u_1 \times u_2 : & E_1 \times E_2 & \longrightarrow & F_1 \widehat{\otimes}_{|\pi|} F_2 \\ & (x_1, x_2) & \longmapsto & u_1(x_1) \otimes u_2(x_2)\,. \end{array}$$

Since $u_1 \widehat{\otimes}_{|\pi|} u_2$ is the linearization of the operator $u_1 \times u_2$ it sufficent to show that $u_1 \times u_2 \in \mathcal{D}_p^{2+}\left(E_1 \times E_2; F_1 \widehat{\otimes}_{|\pi|} F_2\right)$. Indeed, let $\left(x_i^1\right)_{i=1}^n \in E_1$, $\left(x_i^2\right)_{i=1}^n \in E_2$ and $\widetilde{\varphi}_i \in \left(F_1 \widehat{\otimes}_{|\pi|} F_2\right)^*$, $1 \leq i \leq n$. In this way, we can say that

$$\begin{aligned}\sum_{i=1}^n \left\langle u_1 \times u_2\left(x_i^1, x_i^2\right), \widetilde{\varphi}_i\right\rangle &= \sum_{i=1}^n \left\langle u_1\left(x_i^1\right) \otimes u_2\left(x_i^2\right), \widetilde{\varphi}_i\right\rangle \\ &= \sum_{i=1}^n \left\langle \varphi_i\Big(u_1\left(x_i^1\right), u_2\left(x_i^2\right)\Big)\right\rangle = \sum_{i=1}^n \left\langle u_1\left(x_i^1\right), \varphi_{i,u_2\left(x_i^2\right)}\right\rangle,\end{aligned}$$

where φ_i is the bilinear functional on $F_1 \times F_2$ associated with $\widetilde{\varphi}_i$ and $\varphi_{i,u_2\left(x_i^2\right)} : F_1 \to \mathbb{K}$ is defined by $\varphi_{i,u_2\left(x_i^2\right)}(y) = \varphi_i\big(y, u_2\left(x_i^2\right)\big)$. As $u_1 \in \mathcal{D}_{p_1}^+(E_1; F_1)$, we have

$$\sum_{i=1}^n \left\langle u_1\left(x_i^1\right), \varphi_{i,u_2\left(x_i^2\right)}\right\rangle \leq d_{p_1}^+\left(u_1\right) \left\|\left(x_i^1\right)\right\|_{\ell_{p_1}^n} \sup_{\|y_1\|=1} \left(\sum_{i=1}^n \left(\varphi_{i,u_2\left(x_i^2\right)}\left(y_1\right)\right)^{p_1^*}\right)^{1/p_1^*}.$$

By the functional calculus of Krivine, it follows that

$$\sum_{i=1}^n \left\langle u_1\left(x_i^1\right), \varphi_{i,u_2\left(x_i^2\right)}\right\rangle \leq d_{p_1}^+\left(u_1\right) \left\|\left(x_i^1\right)\right\|_{\ell_{p_1}^n} \sup_{\substack{\|y_1\|=1 \\ \left\|\left(\lambda_i\right)_{i=1}^n\right\|_{\ell_{p_1}^n}=1}} \left(\sum_{i=1}^n \lambda_i \varphi_i\left(y_1, u_2\left(x_i^2\right)\right)\right).$$

We therefore deduce that

$$\sum_{i=1}^n \left\langle u_1 \times u_2\left(x_i^1, x_i^2\right), \widetilde{\varphi}_i\right\rangle \leq d_{p_1}^+\left(u_1\right) \left\|\left(x_i^1\right)\right\|_{\ell_{p_1}^n} \sup_{\substack{\|y_1\|=1 \\ \left\|\left(\lambda_i\right)_{i=1}^n\right\|_{\ell_{p_1}^n}=1}} \left(\sum_{i=1}^n \lambda_i \varphi_i\left(y_1, u_2\left(x_i^2\right)\right)\right). \tag{1}$$

We are interested now in the quantity

$$\sum_{i=1}^n \lambda_i \varphi_i\Big(y_1, u_2\left(x_i^2\right)\Big) = \sum_{i=1}^n \left\langle u_2\left(x_i^2\right), \lambda_i \varphi_{i,y_1}\right\rangle,$$

where $\varphi_{i,y_1} : F_2 \to \mathbb{K}$ with $y \mapsto \varphi_{i,y_1}(y) = \varphi_i\left(y_1, y\right)$. Again, if $u_2 \in \mathcal{D}_{p_2}^+(E_2; F_2)$, then

$$\sum_{i=1}^n \lambda_i \varphi_i\Big(y_1, u_2\left(x_i^2\right)\Big) \leq d_{p_2}^+\left(u_2\right) \left\|\left(x_i^2\right)\right\|_{\ell_{p_2}^n(E_2)} \sup_{\|y_2\|=1} \left(\sum_{i=1}^n \left(\lambda_i \varphi_{i,y_1},\left(y_2\right)\right)^{p_2^*}\right)^{1/p_2^*}. \tag{2}$$

By the functional calculus of Krivine, we have

$$\sum_{i=1}^n \lambda_i \varphi_i\left(y_1, u_2\left(x_i^2\right)\right) \leq d_{p_2}^+\left(u_2\right) \left\|\left(x_i^2\right)\right\|_{\ell_{p_2}^n(E_2)} \sup_{\substack{\|y_2\|=1 \\ \left\|\left(\alpha_i\right)_{i=1}^n\right\|_{\ell_{p_2}^n}=1}} \sum_{i=1}^n \alpha_i \lambda_i \varphi_i\left(y_1, y_2\right).$$

As $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, we have

$$\left\{\left(\alpha_i \lambda_i\right) : \left(\lambda_i\right)_{i=1}^n \in B_{\ell_{p_1}^n}^+ \text{ and } \left(\alpha_i\right)_{i=1}^n \in B_{\ell_{p_2}^n}^+\right\} \subset B_{\ell_p^n}^+.$$

Then

$$\sup_{\|y_2\|=1}\sup_{\|(\alpha_i)\|_{\ell^n_{p_2}}=1}\sup_{\|\lambda_i\|_{\ell^n_{p_1}}\leq 1}\sum_{i=1}^{n}\alpha_i\lambda_i\varphi_i(y_1,y_2)\leq \sup_{\|y_2\|=1}\sup_{\left\|(\delta_i)_{i=1}^n\right\|_{\ell^n_p}=1}\sum_{i=1}^{n}\delta_i\varphi_i(y_1,y_2)$$
$$\leq \sup_{\|y_2\|=1}\left(\sum_{i=1}^{n}\left(\lambda_i\varphi_{i,y_1},(y_2)\right)^{p_2^*}\right)^{1/p_2^*}$$
$$\leq \sup_{\|y_2\|=1}\left(\sum_{i=1}^{n}\left(\varphi_i(y_1,y_2)\right)^{p^*}\right)^{1/p^*}.$$

However, as $B^+_{F_1}\otimes B^+_{F_2}:=\left\{y_1\otimes y_2 : y_i\in B^+_{F_i}\right\}\subseteq B^+_{F_1\widehat{\otimes}_{|\pi|}F_2}$, we have

$$\sup_{\substack{\|y_j\|=1\\ j=1,2}}\left(\sum_{i=1}^{n}\left(\varphi_i(y_1,y_2)\right)^{p^*}\right)^{1/p^*}=\sup_{\|y_1\otimes y_2\|_{B^+_{F_1}\otimes B^+_{F_2}}=1}\left(\sum_{i=1}^{n}\left(\widetilde{\varphi}_i(y_1\otimes y_2)\right)^{p^*}\right)^{1/p^*}$$
$$\leq \sup_{\|y_1\otimes y_2\|_{B^+_{F_1\widehat{\otimes}_{|\pi|}F_2}}=1}\left(\sum_{i=1}^{n}\left(\widetilde{\varphi}_i(y_1\otimes y_2)\right)^{p^*}\right)^{1/p^*}.$$

Consequently, we obtain

$$\sup_{\substack{\|y_j\|=1\\ j=1,2}}\left(\sum_{i=1}^{n}\left(\varphi_i(y_1,y_2)\right)^{p^*}\right)^{1/p^*}\leq \sup_{v\in B^+_{F_1\widehat{\otimes}_{|\pi|}F_2}}\left(\sum_{i=1}^{n}\left(\widetilde{\varphi}_i(v)\right)^{p^*}\right)^{1/p^*}. \tag{3}$$

By (1), (2) and (3), we obtain

$$\sum_{i=1}^{n}\left\langle u_1\times u_2(x_i^1,x_i^2),\widetilde{\varphi}_i\right\rangle\leq d^+_{p_1}(u_1)\,d^+_{p_2}(u_2)\left\|(x_i^1)\right\|_{\ell^n_{p_1}}\left\|(x_i^2)\right\|_{\ell^n_{p_2}}\sup_{v\in B^+_{F_1\widehat{\otimes}_{|\pi|}F_2}}\|\widetilde{\varphi}_i(v)\|_{\ell^n_{p^*}}.$$

□

To prove the next result, it suffices to combine Proposition 2 and Corollary 1 with [4, Proposition 3.4].

Theorem 1. *Let $1\leq p,p_1,\ldots,p_m\leq+\infty$ are such that $\frac{1}{p_1}+\cdots+\frac{1}{p_m}=\frac{1}{p}$. Then*

$$\mathcal{L}\left(\mathcal{D}^+_{p_1},\ldots,\mathcal{D}^+_{p_m}\right)(E_1,\ldots,E_m;F)\subseteq\mathcal{D}^{m+}_p(E_1,\ldots,E_m;F).$$

3 Positive Cohen weakly nuclear multilinear operators

In this section, we present a new class of determinate operators on Banach lattice. We also discuss excellent properties of such operators. This new concept was inspired by the work done by Q. Bu et al. in [13].

Definition 1. *Let $1 \leq p < +\infty$. An m-linear operator $T \in \mathcal{L}(E_1, \ldots, E_m; F)$ is called positive Cohen weakly p-nuclear if there exists a positive constant $C > 0$ such that for any $\left(x_i^j\right)_{i=1}^n$ in E_j^+, $1 \leq j \leq m$, and $\left(y_i^*\right)_{i=1}^n$ in F^{*+}, we have*

$$\sum_{i=1}^n \left\langle T(x_i^1, \ldots, x_i^m), y_i^* \right\rangle \leq C \left(\sup_{\varphi \in B^+_{\mathcal{L}^r(E_1,\ldots,E_m;\mathbb{R})}} \sum_{i=1}^n \left(\varphi(x_i^1, \ldots, x_i^m) \right)^p \right)^{\frac{1}{p}} w_{p^*}\left((y_i^*)_{i=1}^n \right).$$

The class of all positive Cohen weakly p-nuclear m-linear operators from $E_1 \times \cdots \times E_m$ into F will be denoted by $\mathcal{N}_{w,p}^{m+}(E_1, \ldots, E_m; F)$. This space is a Banach space with the norm $n_{w,p}^{m+}(\cdot)$, which is defined as the infimum of the constants C, that satisfies the previous inequality. Without too much difficulty, it is possible to show that $\mathcal{N}_{w,p}^{1+}(E; F) = \mathcal{N}_p^+(E; F)$.

Remark 1. *Through simple calculations it is possible to obtain the property of ideal for multilinear operators p-nuclear weakly positive Cohen, that is, if $T \in \mathcal{N}_{w,p}^{m+}(F_1, \ldots, F_m; F)$, $u_j \in \mathcal{L}^+\left(E_j, F_j\right)$, $j = 1, \ldots, m$, and $v \in \mathcal{L}^+\left(F, G\right)$, then*

$$v \circ T \circ (u_1, \ldots, u_m) \in \mathcal{N}_{w,p}^{m+}(E_1, \ldots, E_m; G).$$

For the proof of Pietsch domination theorem in relation to the new class of operators we use the abstract notion presented by D. Pellegrino et al. in [14]. Let $E_1, \ldots, E_m, F_1, \ldots, F_k$ and F be (arbitrary) non-void sets, $\mathcal{H}$ be a family of mappings from $E_1 \times \cdots \times E_m$ to F. Also consider compact Hausdorff topological spaces $K_1, \ldots, K_t$, Banach spaces $G_1, \ldots, G_t$ and suppose that the maps

$$\begin{cases} R_j : & K_j \times F_1 \times \cdots \times F_k \times G_j & \longrightarrow & [0, +\infty) \\ S : & \mathcal{H} \times F_1 \times \cdots \times F_k \times G_1 \times \ldots \times G_t & \longrightarrow & [0, +\infty) \end{cases}$$

with $j \in \{1, \ldots, t\}$, satisfy:

1) for each $x^l \in F_l$ and $b \in G_j$, with $(j, l) \in \{1, \ldots, t\} \times \{1, \ldots, k\}$, the mapping $\left(R_j\right)_{x^1,\ldots,x^k,b} : K_j \longrightarrow [0, +\infty)$ defined by

$$\left(R_j\right)_{x^1,\ldots,x^k,b}(\varphi_j) = R_j\left(\varphi_j, x^1, \ldots, x^k, b\right)$$

   is continuous;

2) the following inequalities hold:

$$R_j\left(\varphi_j, x^1, \ldots, x^k, \eta_j b^j\right) \leq \eta_j R_j\left(\varphi_j, x^1, \ldots, x^k, b^j\right),$$
$$S\left(f, x^1, \ldots, x^k, \alpha_1 b^1, \ldots, \alpha^t b^t\right) \geq \alpha_1 \cdots \alpha_t S\left(f, x^1, \ldots, x^k, b^1, \ldots, b^t\right).$$

Definition 2 ([14]). *Let $0 \leq p_1, \ldots, p_t, q < \infty$ be such that $\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_t}$. A mapping $f : E_1 \times \cdots \times E_m \to F$ in $\mathcal{H}$ is said to be $R_1, \ldots, R_t$-S-abstract $(p_1, \ldots, p_t)$- summing if there is a constant $C > 0$ such that*

$$\left(\sum_{i=1}^n S\left(f, x_i^1, \ldots, x_i^k, b_i^1, \ldots, b_i^t\right)^q \right)^{\frac{1}{q}} \leq C \prod_{j=1}^t \sup_{\varphi_j \in K_j} \left(\sum_{i=1}^n R_j\left(\varphi_j, x_i^1, \ldots, x_i^k, b_i^j\right)^{p_j} \right)^{\frac{1}{p_j}},$$

for all $\left(x_i^s\right)_{i=1}^n$ in F_s, $\left(b_i^j\right)_{i=1}^n$ in G_j, $n \in \mathbb{N}$ and $(s, j) \in \{1, \ldots, k\} \times \{1, \ldots, t\}$.

Theorem 2 ([14]). *A map $f \in \mathcal{H}$ is $R_1,\ldots,R_t$-S-abstract $(p_1,\ldots,p_t)$-summing if and only if there is a constant $C>0$ and Borel probability measures μ_j on K_j with the weak star topology such that*

$$S\left(f,x^1,\ldots,x^k,b^1,\ldots,b^t\right) \leq C\prod_{j=1}^{t}\left(\int_{K_j} R_j\left(\varphi_j,x^1,\ldots,x^k,b^j\right)^{p_j} d\mu_j\right)^{\frac{1}{p_j}},$$

for all $x^l \in E_l, l \in \{1,\ldots,k\}$ and $b^j \in G_j$ with $j=1,\ldots,t$.

We now use this abstract approach to obtain the desired result.

Theorem 3. *An m-linear operators $T \in \mathcal{L}(E_1,\ldots,E_m;F)$ is positive Cohen weakly p-nuclear, $1 \leq p < \infty$, if and only if there exist a positive constant $C>0$, a Radon probability measures μ on $B^+_{\left(E_1\widehat{\otimes}_{|\pi|}\cdots\widehat{\otimes}_{|\pi|}E_m\right)^*}$ and λ on $B^+_{F^{**}}$ with the weak star topology such that for all $\left(x^1,\ldots,x^m\right) \in E_1^+ \times \cdots \times E_m^+$ and $y^* \in F^{*+}$, we have*

$$\left\langle T(x^1,\ldots,x^m),y^*\right\rangle \leq C\left(\int_{B^+_{(E_1\widehat{\otimes}_{|\pi|}\cdots\widehat{\otimes}_{|\pi|}E_m)^*}}\left(\varphi(x^1,\ldots,x^m)\right)^p d\lambda\left(\varphi\right)\right)^{\frac{1}{p}} \|y^*\|_{L_{p^*}\left(B^+_{F^{**}},\lambda\right)}. \quad (4)$$

Proof. By choosing the parameters

$$\begin{cases} t=2,\ k=m, \\ F_l = E_l^+,\ l=1,\ldots,m, \\ K_1 = B^+_{\mathcal{L}^r(E_1,\ldots,E_m;\mathbb{R})}, \\ K_2 = B^+_{F^{**}}, \\ G_1 = \mathbb{K},\ G_2 = F^{*+}, \\ \mathcal{H} = \mathcal{L}(E_1,\ldots,E_m;F), \\ q=1,\ p_1=p,\ p_2=p^*, \\ S\left(T,x^1,\ldots,x^m,b,y^*\right) = b\left\langle T\left(x^1,\ldots,x^m\right),y^*\right\rangle, \\ R_1\left(\varphi_1,x^1,\ldots,x^m,b\right) = b\varphi_1\left(x^1,\ldots,x^m\right), \\ R_2\left(\varphi_2,x^1,\ldots,x^k,y^*\right) = \left\langle y^*,\varphi_2\right\rangle, \end{cases}$$

we can easily conclude that $T: E_1 \times \cdots \times E_m \longrightarrow F$ is positive Cohen weakly p-nuclear if and only if T is R_1,R_2-S-abstract (p,p^*)-summing. Theorem 2 tells us that T is R_1,R_2-S-abstract (p,p^*)-summing if and only if there is a $C>0$ and there are probability measures μ_1 on K_1 and μ_2 on K_2 with the weak star topology such that

$$S\left(T,x^1,\ldots,x^m,b,y^*\right) \leq C\left(\int_{K_1}\left(\varphi_1(x^1,\ldots,x^m)\right)^p d\mu_1\right)^{\frac{1}{p}}\left(\int_{K_2}\langle y^*,\varphi_2\rangle^{p^*} d\mu_2\right)^{\frac{1}{p^*}}$$

and we recover (4). □

Corollary 2. *Let $1 \leq p < \infty$. Then*

$$\mathcal{N}_p^{m+}(E_1,\ldots,E_m;F) \subseteq \mathcal{N}_{w,p}^{m+}(E_1,\ldots,E_m;F) \subseteq \mathcal{D}_p^{m+}\left(X_1,\ldots,X_m;F\right).$$

Proof. If $\left(x_i^1 \otimes \cdots \otimes x_i^m\right)_{i=1}^{\infty}$ is a weakly p-summable sequence in space $E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m$, then $\sum\limits_{i=1}^{\infty} \left(\varphi\left(x_i^1, \ldots, x_i^m\right)\right)^p = \sum\limits_{i=1}^{\infty} \left(\varphi^{\otimes}\left(x_i^1 \otimes \cdots \otimes x_i^m\right)\right)^p < \infty$ for every $\varphi \in \mathcal{L}^+(E_1, \ldots, E_m)$. By definition, $T : E_1 \times \cdots \times E_m \to F$ is positive Cohen weakly p-nuclear if and only if $\left(T^{\otimes}\left(x_i^1 \otimes \cdots \otimes x_i^m\right)\right)_{i=1}^{\infty}$ is a strongly p-summable sequence in F, whenever $\left(x_i^1 \otimes \cdots \otimes x_i^m\right)_i$ is a weakly p-summable sequence in $E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m$. Also note that for any $\varphi \in \mathcal{L}^r(E_1, \ldots, E_m)$, we have $\left|\varphi\left(x^1, \ldots, x^m\right)\right| \leq \|\varphi\| \left\|x^1\right\| \cdots \|x^m\|$. For any $x^{j*} \in B^+_{E_j^*}$, $j = 1, \ldots, m$, let $\varphi_1 = x^{1*} \otimes \cdots \otimes x^{m*}$. Then $\varphi_1 \in B^+_{\mathcal{L}^r(E_1,\ldots,E_m;\mathbb{R})}$ and $\varphi\left(x^1, \ldots, x^m\right) = \prod\limits_{j=1}^{m} \left\langle x^j, x^{j*}\right\rangle$. Thus, the inclusions of this corollary are taken directly from the definitions. $\square$

Combining Theorem 3 with the Pietsch domination theorem for positive Dimant-strongly p-summing operators (see [6, Theorem 1]), we have the following inclusion.

Proposition 3. *Let* $1 \leq p < \infty$. *Then*

$$\mathcal{N}^{m+}_{w,p}(E_1, \ldots, E_m; F) \subseteq \mathcal{L}^{m+}_{ss,p}(E_1, \ldots, E_m; F).$$

Theorem 4. *Let* $T \in \mathcal{L}(E_1, \ldots, E_m; F)$ *and* $1 \leq p < \infty$. *Consider the following statements:*

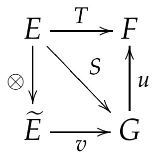
(i) $T^{\otimes} : E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m \to F$ *is positive Cohen* p*-nuclear;*

(ii) there exist a Banach lattice space G, *a positive Dimant-strongly* p*-summing* m*-linear operator* $S : E_1 \times \cdots \times E_m \to G$ *and a Cohen positive strongly* p*-summing linear operator* $u : G \to F$ *such that* $T = u \circ S$*;*

(iii) T *is positive Cohen weakly* p*-nuclear.*

Then, the statement (i) implies (ii), which implies (iii).

Proof. *(i)* $\Rightarrow$ *(ii)* Since $T^{\otimes}$ is positive Cohen p-nuclear, by [7, Theorem 3.2] there exist a Banach lattice space G, an positive p-summing linear operator $v : E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m \to G$ and a Cohen positive strongly p-summing linear operator $u : G \to F$ such that $T = u \circ v$. Let $S = v \circ \otimes$. Then $T = u \circ v \circ \otimes = u \circ S$ and the following diagram



commutes, with $E = E_1 \times \cdots \times E_m$ and $\widetilde{E} = E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m$. Since $S^{\otimes} = v$, it follows that S is positive Dimant-strongly p-summing (see [6]) and hence *(ii)* is checked.

(ii) $\Rightarrow$ *(iii)* Given any $x^1 \in E_1^+, \ldots, x^m \in E_m^+$ and $y^* \in F^{*+}$, by Pietsch domination theorems for Cohen positive strongly p-summing operators and for positive Dimant-strongly p-summing operators, we have

$$\begin{aligned}\Big\langle T\big(x^1, \ldots, x^m\big), y^*\Big\rangle &= \Big\langle u\Big(S\big(x^1, \ldots, x^m\big)\Big), y^*\Big\rangle \leq K_1 \Big\|S\big(x^1, \ldots, x^m\big)\Big\|_G \|y^*\|_{L_{p^*}\left(B^+_{F^{**}}, \mu\right)} \\ &\leq K_1 K_2 \Bigg(\int_{B^+_{(E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m)^*}} \Big(\varphi\big(x^1, \ldots, x^m\big)\Big)^p d\lambda\,(\varphi) \Bigg)^{\frac{1}{p}} \|y^*\|_{L_{p^*}\left(B^+_{F^{**}}, \lambda\right)}.\end{aligned}$$

By Theorem 3, T is positive Cohen weakly p-nuclear. $\square$

Open problem. In Theorem 4, is it possible to prove that *(iii)* implies *(ii)*?

4 Tensor norms related to positive Cohen weakly p-nuclear mappings

In this final section, we introduce a reasonable cross norm in $E_1 \otimes \cdots \otimes E_m \otimes F^*$, so that the topological dual of the resulting space is isometric to $\left(\mathcal{N}^{m+}_{w,p}(E_1, \ldots, E_m; F), n^{m+}_{w,p}(\cdot)\right)$. For $1 \leq p < +\infty$ and $u \in E_1 \otimes \cdots \otimes E_m \otimes F^*$, we consider

$$\widetilde{\omega}_{(p;p^*)}(u) := \inf \left\|(\lambda_i)_{i=1}^n\right\|_\infty \left(\sup_{\varphi \in B^+_{\mathcal{L}^r(E;\mathbb{R})}} \sum_{i=1}^n \left(\varphi(x_i^1, \ldots, x_i^m)\right)^p \right)^{\frac{1}{p}} \left\|(y_i^*)_{1\leq i\leq n}\right\|_{\ell^n_{p^*,|weak|}},$$

where $E = E_1 \times \cdots \times E_m$ and the infimum is taken over all representations of u of the form $u = \sum_{i=1}^n \lambda_i x_i^1 \otimes \cdots \otimes x_i^m \otimes y_i^*$, with $x_i^j \in E_j^+$, $y_i^* \in F^{*+}$, $\lambda_i \in \mathbb{K}$, $1 \leq i \leq n$ and $1 \leq j \leq m$.

Proposition 4. *Denoting by ε the norm of the injective tensor in $E_1 \otimes \cdots \otimes E_m \otimes F^*$, we have $\widetilde{\omega}_{(p;p^*)}$ is a reasonable crossnorm on $E_1 \otimes \cdots \otimes E_m \otimes F^*$ and $\varepsilon \leq \widetilde{\omega}_{(p;p^*)}$.*

Proof. If $u = \sum_{i=1}^n \lambda_i x_i^1 \otimes \cdots \otimes x_i^m \otimes y_i^*$ and $E = E_1 \times \cdots \times E_m$, then

$$\begin{aligned}
\varepsilon(u) &= \sup \left\{ \sum_{i=1}^n \lambda_i \varphi(x_i^1, \ldots, x_i^m) \psi(y_i^*) ; \varphi \in B^+_{\mathcal{L}^r(E;\mathbb{R})}, \psi \in B^+_{F^{**}} \right\} \\
&\leq \left\|(\lambda_i)_{1\leq i\leq n}\right\|_\infty \left\| \sup_{\substack{\varphi \in B^+_{\mathcal{L}^r(E;\mathbb{R})} \\ \psi \in B^+_{F^{**}}}} \sum_{i=1}^n \lambda_i \varphi\left(x_i^1, \ldots, x_i^m\right) \psi(y_i^*) \right\| \\
&\leq \left\|(\lambda_i)_{1\leq i\leq n}\right\|_\infty \left(\sup_{\varphi \in B^+_{\mathcal{L}^r(E;\mathbb{R})}} \sum_{i=1}^n \left(\varphi(x_i^1, \ldots, x_i^m)\right)^p \right)^{\frac{1}{p}} \sup_{\psi \in B^+_{F^{**}}} \left\|(\psi(y_i^*))_{1\leq i\leq n}\right\|_{\ell^n_{p^*}} \\
&\leq \left\|(\lambda_i)_{1\leq i\leq n}\right\|_\infty \left(\sup_{\varphi \in B^+_{\mathcal{L}^r(E;\mathbb{R})}} \sum_{i=1}^n \left(\varphi(x_i^1, \ldots, x_i^m)\right)^p \right)^{\frac{1}{p}} \left\|(y_i^*)_{1\leq i\leq n}\right\|_{\ell^n_{p^*,|weak|}(F^{**})}.
\end{aligned}$$

Hence $\varepsilon(u) \leq \widetilde{\omega}_{(p;p^*)}(u)$.

To show that $\widetilde{\omega}_{(p;p^*)}$ is a norm, let us consider $u, v \in E_1 \otimes \cdots \otimes E_m \otimes F^*$. For any $\delta > 0$, we can find a representations of u and v of the form

$$u = \sum_{i=1}^n \lambda_i x_i^1 \otimes \cdots \otimes x_i^m \otimes y_i^* \qquad \text{and} \qquad v = \sum_{i=1}^n \alpha_i z_i^1 \otimes \cdots \otimes z_i^m \otimes b_i^*,$$

so that the following conditions are satisfied:

1) $\left\|(\lambda_i)_{i=1}^n\right\|_\infty = 1$ and $\left\|(\alpha_i)_{i=1}^n\right\|_\infty = 1$;

2) $\sup_{\varphi \in B^+_{\mathcal{L}^r(E;\mathbb{R})}} \sum_{i=1}^n \left(\varphi(x_i^1, \ldots, x_i^m)\right)^p \leq \left(\widetilde{\omega}_{(p;p^*)} + \delta\right)$;

3) $\left\|(y_i^*)_{i=1}^n\right\|_{\ell^n_{p^*,|weak|}(F^{**})} \leq \left(\widetilde{\omega}_{(p;p^*)} + \delta\right)^{1/p^*}$;

4) $\sup_{\varphi \in B^{+}_{\mathcal{L}^r(E;\mathbb{R})}} \sum_{i=1}^{n} \left(\varphi\left(z_i^1, \ldots, z_i^m\right) \right)^p \leq \left(\widetilde{\omega}_{(p;p^*)} + \delta \right);$

5) $\left\| \left(b_i^*\right)_{1 \leq i \leq n} \right\|_{\ell^n_{p^*,|weak|}(F^{**})} \leq \left(\widetilde{\omega}_{(p;p^*)} + \delta \right)^{1/p^*}.$

Then it follows that

$$\begin{aligned}\widetilde{\omega}_{(p;p^*)}(u+v) &\leq \left(\widetilde{\omega}_{(p;p^*)}(u) + \widetilde{\omega}_{(p;p^*)}(v) + 2\delta \right)^{\frac{1}{p}} \left(\widetilde{\omega}_{(p;p^*)}(u) + \widetilde{\omega}_{(p;p^*)}(v) + 2\delta \right)^{\frac{1}{p^*}} \\ &= \widetilde{\omega}_{(p;p^*)}(u) + \widetilde{\omega}_{(p;p^*)}(v) + 2\delta.\end{aligned}$$

Therefore,

$$\widetilde{\omega}_{(p;p^*)}(u+v) \underset{\delta \to 0}{\longrightarrow} \widetilde{\omega}_{(p;p^*)}(u) + \widetilde{\omega}_{(p;p^*)}(v),$$

guaranteeing the triangular inequality. The other conditions are easily verified. Thus, $\widetilde{\omega}_{(p;p^*)}$ is a norm on $E_1 \otimes \cdots \otimes E_m \otimes F^*$. Let us consider $\left(x_i^j\right)_{1=i}^n$ in E_j^+ and $y^* \in F^{*+}$ with $1 \leq j \leq m$. If $\phi \neq 0$, $\phi \in \left(E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m \right)^*$, $\psi \in F^{**+}$ and $u = \sum_{i=1}^{n} \lambda_i x_i^1 \otimes \cdots \otimes x_i^m \otimes y_i^*$, then by Hölder's inequality we get

$$\begin{aligned}\phi \otimes \psi(u) &= \phi \otimes \psi \left(\sum_{i=1}^{n} \lambda_i x_i^1 \otimes \cdots \otimes x_i^m \otimes y_i^* \right) = \sum_{i=1}^{n} \lambda_i \phi\left(x_i^1, \ldots, x_i^m\right) \psi\left(y_i^*\right) \\ &\leq \left\| (\lambda_i)_{1 \leq i \leq n} \right\|_\infty \|\phi\| \, \|\psi\| \left\| \frac{\phi\left(x_i^1, \ldots, x_i^m\right)}{\|\phi\|} \frac{\psi\left(y_i^*\right)}{\|\psi\|} \right\|_1 \leq \|\phi\| \, \|\psi\| \, \widetilde{\omega}_{(p;p^*)}(u).\end{aligned}$$

Therefore we obtain that $\phi \otimes \psi(u) \leq \|\phi\| \, \|\psi\| \, \widetilde{\omega}_{(p;p^*)}(u)$ and we have shown that $\widetilde{\omega}_{(p;p^*)}$ is a reasonable crossnorm. □

The next result characterizes the space of positive Cohen weakly p-nuclear m-linear operators as the topological dual of the space of the tensor product $E_1 \otimes \cdots \otimes E_m \otimes F^*$ equipped with the norm $\widetilde{\omega}_{(p;p^*)}$ via an isometric isomorphism.

Theorem 5. *The space* $\left(\mathcal{N}^{m+}_{w,p}(E_1, \ldots, E_m; F), n^{m+}_{w,p} \right)$ *is isometrically isomorphic to the space* $\left(E_1 \otimes \cdots \otimes E_m \otimes F^*, \widetilde{\omega}_{(p;p^*)} \right)^*$ *through the mapping* $T \to \psi_T$*, where*

$$\psi_T\left(x^1 \otimes \cdots \otimes x^m \otimes y^*\right) = T\left(x^1, \ldots, x^m\right)(y^*),$$

for every $x^j \in E_j^+$ *with* $1 \leq j \leq m$ *and* $y^* \in F^{*+}$.

Proof. Without too much difficulty, it is possible to verify that the correspondence $T \longrightarrow \psi_T$ is a linear and injective map. To show the surjectivity, we consider $E = E_1 \times \ldots \times E_m$, $\psi \in \left(E \otimes F^*, \widetilde{\omega}_{(p;p^*)} \right)^*$ and the corresponding m-linear mapping $T_\psi \in \mathcal{L}(E_1, \ldots, E_m; F)$ defined by $T_\psi\left(x^1, \ldots, x^m\right)(y^*) = \psi\left(x^1 \otimes \cdots \otimes x^m \otimes y^*\right)$ for all $x^j \in E_j^+$ and $y^* \in F^{*+}$. Let us also consider $x_i^j \in E_j^+$ and $y_i^* \in F^{*+}$ with $1 \leq i \leq n$ and $1 \leq j \leq m$. For an appropriate $\lambda_i \in \mathbb{K}$ with $|\lambda_i| = 1$, $i = 1, \ldots, n$, we have that

$$\sum_{i=1}^{n}\left\langle T_{\psi}\left(x_{i}^{1},\ldots,x_{i}^{m}\right),y_{i}^{*}\right\rangle = \sum_{i=1}^{n}\left|\lambda_{i}\right|\psi\left(x_{i}^{1}\otimes\cdots\otimes x_{i}^{m}\otimes y_{i}^{*}\right) = \psi\left(\sum_{i=1}^{n}\left|\lambda_{i}\right| x_{i}^{1}\otimes\cdots\otimes x_{i}^{m}\otimes y_{i}^{*}\right)$$
$$\leq \|\psi\|\,\widetilde{\omega}_{(p;p^{*})}\left|\sum_{i=1}^{n}\lambda_{i}x_{i}^{1}\otimes\cdots\otimes x_{i}^{m}\otimes y_{i}^{*}\right| \leq \|\psi\|\,\widetilde{\omega}_{(p;p^{*})}\left(u\right)$$
$$\leq \|\psi\|\left(\sup_{\varphi\in B^{+}_{\mathcal{L}^{r}(E;\mathbb{R})}}\sum_{i=1}^{n}\left(\varphi\left(x_{i}^{1},\ldots,x_{i}^{m}\right)\right)^{p}\right)^{\frac{1}{p}} w_{p^{*}}\left((y_{i}^{*})_{i=1}^{n}\right),$$

which shows that $T_{\psi}\in\mathcal{N}^{m+}_{w,p}(E_1,\ldots,E_m;F)$ and $n^{m+}_{w,p}\left(T_{\psi}\right)\leq\|\psi\|$.

Conversely, if T is positive Cohen weakly p-nuclear from E into F, we define a linear functional on $E\otimes F^{*}$ by

$$\psi_{T}\left(u\right)=\sum_{i=1}^{n}\lambda_{i}T\left(x_{i}^{1},\ldots,x_{i}^{m}\right)\left(y_{i}^{*}\right),$$

for $u=\sum\limits_{i=1}^{n}\lambda_{i}x_{i}^{1}\otimes\cdots\otimes x_{i}^{m}\otimes y_{i}^{*}$, where $m\in\mathbb{N},\lambda_{i}\in\mathbb{K},x_{i}^{j}\in E_{j}^{+},y_{i}^{*}\in F^{*+},1\leq i\leq n,1\leq j\leq m$. Hence, by Hölder's inequality we obtain

$$\left|\psi_{T}\left(u\right)\right|\leq\left\|(\lambda_{i})_{i=1}^{n}\right\|_{\infty}\left|\sum_{i=1}^{n}\left\langle T\left(x_{i}^{1},\ldots,x_{i}^{m}\right),y_{i}^{*}\right\rangle\right|$$
$$\leq n^{m+}_{w,p}\left(T\right)\left\|(\lambda_{i})_{i=1}^{n}\right\|_{\infty}\left(\sup_{\varphi\in B^{+}_{\mathcal{L}^{r}(E;\mathbb{R})}}\sum_{i=1}^{n}\left(\varphi(x_{i}^{1},\ldots,x_{i}^{m})\right)^{p}\right)^{\frac{1}{p}} w_{p^{*}}\left((y_{i}^{*})_{i=1}^{n}\right).$$

This shows that T is $\widetilde{\omega}_{(p;p^{*})}$-continuous and $\|\psi_{T}\|\leq n^{m+}_{w,p}\left(T\right)$. □

Corollary 3. *The space* $\left(\mathcal{N}^{m+}_{w,p}(E_1,\ldots,E_m),n^{m+}_{w,p}\right)$ *is isometrically isomorphic to the space* $\left(E_1\otimes\cdots\otimes E_m,\widetilde{\omega}_{(p;p^{*})}\right)^{*}$ *through the mapping* $T\rightarrow\psi_{T}$, *where*

$$\psi_{T}\left(x^{1}\otimes\cdots\otimes x^{m}\otimes y^{*}\right)=T\left(x^{1},\ldots,x^{m}\right)\left(y^{*}\right)$$

for every $x^{j}\in E_{j}^{+}$ *with* $1\leq j\leq m$ *and* $y^{*}\in F^{*+}$.

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References

[1] Achour D., Alouani A. *On multilinear generalizations of the concept of nuclear operators*. Colloq. Math. 2010, **120** (1), 85–102. doi:10.4064/cm120-1-7

[2] Achour D., Belacel A. *Domination and factorization theorems for positive strongly p-summing operators*. Positivity 2014, **18**, 785–804. doi:10.1007/s11117-014-0276-6

[3] Blasco O. *A class of operators from a Banach lattice into a Banach space*. Collect. Math. 1986, **37** (1), 13–22.

[4] Berrios S., Botelho G. *Approximation properties determined by operator ideals and approximability of homogeneous polynomials and holomorphic functions*. Studia Math. 2012, **208** (2), 97–116. doi:10.4064/SM208-2-1

[5] Bougoutaia A., Belacel A. *Cohen positive strongly p-summing and p-convex multilinear operators*. Positivity 2019, **23**, 379–395. doi:10.1007/s11117-018-0612-3

[6] Bougoutaia A., Belacel A., Hamdi H. *On the positive Dimant strongly p-summing multilinear operators*. Carpathian Math. Publ. 2020, **12** (2), 401–411. doi:10.15330/cmp.12.2.401-411

[7] Bougoutaia A., Belacel A., Hamdi H. *Domination and Kwapien's factorization theorems for positive Cohen p-nuclear m-linear operators*. Moroccan J. of Pure and Appl. Anal. (MJPAA) 2021, **7** (1), 100–115. doi:10.2478/mjpaa-2021-0010

[8] Bu Q., Buskes G. *Polynomials on Banach lattices and positive tensor products*. J. Math. Anal. Appl. 2012, **388** (2), 845–862. doi:10.1016/j.jmaa.2011.10.001

[9] Cohen J.S. *Absolutely p-summing, p-nulear operators and thier conjugates*. Math. Ann. 1973, **201**, 177–200.

[10] Fremlin D.H. *Tensor products of Archimedean vector lattices*. Amer. J. Math. 1972, **94** (3), 777–798. doi:10.2307/2373758

[11] Fremlin D.H. *Tensor products of Banach lattices*. Math. Ann. 1974, **211**, 87–106.

[12] Krivine J.L. *Théorèmes de factorisation dans les espaces réticulés*. Séminaire Maurey-Schwartz 1973–1974, exposes 22–23. École Polytechnique, Paris, 1974.

[13] Li Y., Liu Q., Bu Q. *On Cohen weakly nuclear multilinear operators*. Sci. Sin. Math. 2020, **50** (2), 1783–1792. doi:10.1360/SSM-2020-0177

[14] Pellegrino D., Santos J., Seoane-Sepúlveda J. *Some techniques on nonlinear analysis and applications*. Adv. Math. 2012, **229**, 1235–1265. doi:10.48550/arXiv.1006.0536

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Бугутая А., Беласел А., Македо Р., Хамді Х. *Про додатні слабко ядерні мультилінійні оператори Коена* // Карпатські матем. публ. — 2023. — Т.15, №2. — C. 396–410.

У цій статті ми встановлюємо нові співвідношення із залученням класу додатних сильно p-сумовних мультилінійних операторів Коена. Крім того, ми вводимо новий клас мультилінійних операторів на банахових ґратках і називаємо їх додатними слабко ядерними мультилінійними операторами Коена. Ми встановлюємо теорему Піча мажорантного типу для цього нового класу мультилінійних операторів. В якості застосування ми показуємо, що кожен додатний слабко p-ядерний мультилінійний оператор Коена є додатним строгим за Дімант p-сумовним та додатним строго p-сумовним оператором Коена. Ми завершуємо тензорним представленням введеного класу операторів.

Ключові слова і фрази: банахова ґратка, мажорантна теорема Піча, додатний p-сумовний оператор, тензорна норма.