



On deferred harmonic summability of order ν

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The aim of this study is to introduce the concepts of deferred $(H, 1)$ -summability of order ν , deferred strongly harmonically summability of order ν and deferred statistical logarithmic convergence of order ν of sequences of real numbers. Besides we give some inclusion relations related to these concepts.

Key words and phrases: deferred statistical convergence, harmonic summability, density.

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1 Introduction

As usual we use \mathbb{N} to denote the set of all natural numbers. For any subset $S \subseteq \mathbb{N}$ let $S(n) = \{k \leq n : k \in S\}$. The natural density of S is defined as $\delta(S) = \lim_n \frac{1}{n} |S(n)|$ if $\lim_n \frac{1}{n} |S(n)|$ exists. By $|S|$, we denote the cardinality of the set S .

A sequence $x = (x_i)$ is said to be statistically convergent to ℓ if for every $\varepsilon > 0$ the set $S(\varepsilon) = \{i \leq n : |x_i - \ell| \geq \varepsilon\}$ has natural density zero. Recently statistical convergence was studied in [2–11, 13–18, 21–24].

In 1932, R.P. Agnew [1] defined the deferred Cesàro mean $D_{c,d}$ of the sequence $x = (x_i)$ by

$$(D_{c,d}x)_t = \frac{1}{d(t) - c(t)} \sum_{i=c(t)+1}^{d(t)} x_i,$$

where $\{c(t)\}$ and $\{d(t)\}$ are sequences of non-negative integers satisfying

$$c(t) < d(t), \quad \lim_{t \rightarrow \infty} d(t) = +\infty \quad \text{and} \quad \lim_{t \rightarrow \infty} (d(t) - c(t)) = +\infty.$$

We denote the set of all such (c, d) pairs by Λ . Some restrictions will be applied on (c, d) if necessary.

Let $S \subset \mathbb{N}$. Denote the set $\{s : c(t) < s \leq d(t), s \in S\}$ by $S_{c,d}(t)$. The *deferred density* of a set S is defined by

$$\delta_{c,d}(S) = \lim_{t \rightarrow \infty} \frac{1}{d(t) - c(t)} |S_{c,d}(t)|,$$

whenever the limit exists (finite or infinite).

The *harmonic means* of the sequence $x = (x_i)$ are defined by

$$\tau_n = \frac{1}{h_n} \sum_{i=1}^n \frac{x_i}{i}, \quad \text{where } h_n = \sum_{i=1}^n \frac{1}{i} \approx \log n \quad \text{for } n = 1, 2, \dots$$

A sequence $x = (x_i)$ is named $(H, 1)$ -*summable* to ℓ if

$$\lim_{n \rightarrow \infty} \frac{1}{h_n} \sum_{i=1}^n \frac{x_i}{i} = \ell.$$

The set of all $(H, 1)$ -summable sequences is denoted by H and the set of all real sequences which $(H, 1)$ -summable to 0 by H^0 . It is well known that ordinary convergence does imply harmonic summability. However, the converse implication holds only under additional conditions. Recently, harmonically summability was studied in [12, 19, 20, 25, 26].

We write $x_i \sim y_i$ if $\lim_{i \rightarrow \infty} x_i/y_i = 1$, where $x = (x_i)$ and $y = (y_i)$ are two sequences of positive numbers (except possibly a finite number of terms).

2 Main Results

Let us define deferred harmonic means of order ν of the sequence $x = (x_i)$ by

$$T_t^\nu = \frac{1}{(L_t)^\nu} \sum_{i=c(t)+1}^{d(t)} \frac{x_i}{i}, \quad \text{where } (L_t)^\nu = \left(\sum_{i=c(t)+1}^{d(t)} \frac{1}{i} \right)^\nu, \quad j = 1, 2, \dots$$

Definition 1. A sequence $x = (x_i)$ is said to be deferred $(H, 1)$ -summable of order ν to ℓ , if

$$\lim_{t \rightarrow \infty} \frac{1}{(L_t)^\nu} \sum_{i=c(t)+1}^{d(t)} \frac{x_i}{i} = \ell,$$

where ν is a real number such that $0 < \nu \leq 1$. In this case, we write $D_{(c,d)}^\nu\text{-}\lim x_i = \ell$.

The class of deferred $(H, 1)$ -summable sequences of order ν would be denoted by $D_{(c,d)}^\nu$. In the special case $\nu = 1$, we write $D_{(c,d)}$ instead of $D_{(c,d)}^\nu$. Furthermore, if $c(t) = 0$ and $d(t) = t$, we write D^ν instead of $D_{(c,d)}^\nu$.

Definition 1 reduces to $(H, 1)$ -summability, when $\nu = 1$, $d(t) = t$ and $c(t) = 0$.

Definition 2. A sequence $x = (x_i)$ of real numbers is said to be deferred strongly harmonically summable of order ν to ℓ , if

$$\lim_{t \rightarrow \infty} \frac{1}{(L_t)^\nu} \sum_{i=c(t)+1}^{d(t)} \frac{|x_i - \ell|}{i} = 0,$$

where ν be a real number such that $0 < \nu \leq 1$.

$$[D_{(c,d)}]^\nu\text{-}\lim x_i = \ell.$$

The class of deferred strongly harmonically summable sequences of order ν would be denoted by $[D_{(c,d)}]^\nu$. In the special case $\nu = 1$, we write $[D_{(c,d)}]$ instead of $[D_{(c,d)}]^\nu$. Also, if $c(t) = 0$ and $d(t) = t$, we write $[D]^\nu$ instead of $[D_{(c,d)}]^\nu$. If $\ell = 0$, then we write $[D_{(c,d)}^0]^\nu$ instead of $[D_{(c,d)}]^\nu$.

Theorem 1. $[D_{(c,d)}]^\nu$ and $D_{(c,d)}^\nu$ are linear spaces for $0 < \nu \leq 1$.

Proof. The result can be proved using standard arguments. □

Theorem 2. Let ν be a real number such that $0 < \nu \leq 1$. The sequence space $[D_{(c,d)}]^\nu$ is a Banach space equipped with the norm

$$\|x\| = \sup_t \frac{1}{L_t^\nu} \sum_{i=c(t)+1}^{d(t)} \frac{|x_i|}{i}.$$

Proof. Verifying the $[D_{(c,d)}]^\nu$ norm space is routine. Let (x^k) be a Cauchy sequence such that $x^k = (x_i^k)_i = (x_1^k, x_2^k, \dots) \in [D_{(c,d)}]^\nu$ for each $k \in \mathbb{N}$. Then we have $\|x^k - x^s\| \rightarrow 0$ as $k, s \rightarrow \infty$.

Let $\varepsilon > 0$ be given, then there exists a positive integer $m_0(\varepsilon)$ such that $\|x^k - x^s\| \leq \varepsilon$ for all $k, s > m_0$. So, we have

$$\frac{1}{L_t^\nu} \sum_{i=c(t)+1}^{d(t)} \frac{|x_i^k - x_i^s|}{i} \leq \varepsilon$$

for all $k, s > m_0$. Hence, we get $|x_i^k - x_i^s| \rightarrow 0$ as $k, s \rightarrow \infty$ for all $i \in \mathbb{N}$. It follows that the sequence $(x_i^k)_i = (x_i^1, x_i^2, \dots)$ is a Cauchy in \mathbb{R} for each $i \in \mathbb{N}$. Further, it is convergent, since \mathbb{R} is complete.

We claim that $\lim_k x_i^k = x_i$ for each $i \in \mathbb{N}$. As (x^k) is a Cauchy sequence, then for every $\varepsilon > 0$ there exists a number $m_0 = m_0(\varepsilon)$ such that $\|x^k - x^s\| < \varepsilon$ for all $k, s \geq m_0$. It follows that, for all $t \in \mathbb{N}$ and all $k, s \geq m_0$, we have

$$\frac{1}{L_t^\nu} \sum_{i=c(t)+1}^{d(t)} \frac{|x_i^k - x_i^s|}{i} < \varepsilon.$$

Passing to the limit as $t \rightarrow \infty$ in the above inequality, we obtain

$$\lim_t \frac{1}{L_t^\nu} \sum_{i=c(t)+1}^{d(t)} \frac{|x_i^k - x_i^s|}{i} = \frac{1}{L_t^\nu} \sum_{i=c(t)+1}^{d(t)} \frac{|x_i^k - x_i|}{i} < \varepsilon$$

for $k \geq m_0$ and all $t \in \mathbb{N}$. It follows that

$$\|x^k - x\| = \sup_t \frac{1}{L_t^\nu} \sum_{i=c(t)+1}^{d(t)} \frac{|x_i^k - x_i|}{i} \rightarrow 0,$$

and hence $x^k \rightarrow x$ as $k \rightarrow \infty$. Since

$$\frac{1}{L_t^\nu} \sum_{i=c(t)+1}^{d(t)} \frac{|x_i - a|}{i} \leq \frac{1}{L_t^\nu} \sum_{i=c(t)+1}^{d(t)} \frac{|x_i^{m_0} - a|}{i} + \frac{1}{L_t^\nu} \sum_{i=c(t)+1}^{d(t)} \frac{|x_i^{m_0} - x_i|}{i},$$

we get $x \in [D_{(c,d)}]^\nu$. □

Theorem 3. Let $0 < \nu \leq \gamma \leq 1$. Then $\liminf_t \frac{h_{d(t)}}{h_{c(t)}} > 1$ if and only if $[D]^\nu \subseteq [D_{(c,d)}]^\gamma$.

Proof. Let $\liminf_t \frac{h_{d(t)}}{h_{c(t)}} > 1$. Then there is an $a > 0$ such that $1 + a \leq \frac{h_{d(t)}}{h_{c(t)}}$, $t \in \mathbb{N}$. If $x \in [D^0]^\nu$, then for every $\varepsilon > 0$ and for sufficiently large t , we obtain

$$\begin{aligned} T_t^\gamma &= \frac{1}{(L_t)^\gamma} \sum_{i=c(t)+1}^{d(t)} \frac{|x_i|}{i} = \frac{1}{(L_t)^\gamma} \left(\sum_{i=1}^{d(t)} \frac{|x_i|}{i} - \sum_{i=1}^{c(t)} \frac{|x_i|}{i} \right) \\ &= \frac{h_{d(t)}^\nu}{(L_t)^\gamma} \left(\frac{1}{h_{d(t)}^\nu} \sum_{i=1}^{d(t)} \frac{|x_i|}{i} \right) - \frac{h_{c(t)}^\nu}{(L_t)^\gamma} \left(\frac{1}{h_{c(t)}^\nu} \sum_{i=1}^{c(t)} \frac{|x_i|}{i} \right). \end{aligned}$$

Since $L_t = h_{d(t)} - h_{c(t)}$, we have $\frac{h_{d(t)}}{L_t} \leq \frac{1+a}{a}$ and $\frac{h_{c(t)}}{L_t} \leq \frac{1}{a}$. This proves the sufficiency.

Now assume that $\liminf_t \frac{h_{d(t)}}{h_{c(t)}} = 1$ and $d(t) = k_t$, $c(t) = k_{t-1}$, where (k_t) is a lacunary sequence, $\nu = 1$. We can select a subsequence (k_{t_p}) of t satisfying

$$\frac{h_{k_{t_p}}}{h_{k_{t_p-1}}} < \frac{1}{p} \quad \text{and} \quad \frac{h_{k_{t_p-1}}}{h_{k_{t_p-1}}} > p,$$

where $t_p \geq t_{p-1} + 2$.

Define $x = (x_i)$ by

$$x_i = \begin{cases} 1, & \text{if } i \in I_{t_p} \text{ for some } p = 1, 2, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

Then for any real ℓ we get

$$\frac{1}{L_{t_p}} \sum_{i \in I_{t_p}} \frac{|x_i - \ell|}{i} = \frac{L_{t_p}}{L_{t_p}} |1 - \ell| \rightarrow \infty \quad \text{as } t_p \rightarrow \infty,$$

and for $t \neq t_p$ we obtain

$$\frac{1}{L_t} \sum_{i \in I_t} \frac{|x_i - \ell|}{i} = \frac{L_t}{L_t} |\ell| \rightarrow |\ell| \quad \text{as } t \rightarrow \infty.$$

Therefore $x \notin [D_{(c,d)}]^\nu$. However, $x \in [D^0]^\nu$. Since r is any sufficiently large integer, we can find a unique p for which $h_{s_{p-1}} < h_r \leq h_{s_{p+1-1}}$ and

$$\frac{1}{h_r} \sum_{i=1}^r \frac{|x_i|}{i} \leq \frac{h_{k_{t_p-1}} + L_{t_p}}{h_{k_{t_p-1}}} < \frac{h_{k_{t_p-1}}}{h_{k_{t_p-1}}} + \frac{h_{k_{t_p}}}{h_{k_{t_p-1}}} < \frac{2}{p}.$$

As $r \rightarrow \infty$, it follows that $p \rightarrow \infty$ as well. Hence $x \in [D^0]^\nu$. □

Theorem 4. Let $(c, d), (c', d') \in \Lambda$ be such that

$$c(t) < c'(t) < d'(t) < d(t) \tag{1}$$

for all $t \in \mathbb{N}$ and $0 < \nu \leq \gamma \leq 1$.

(i) If

$$\liminf_{t \rightarrow \infty} \frac{(L'_t)^\nu}{(L_t)^\gamma} > 0 \tag{2}$$

holds, then $[D_{(c,d)}]^\gamma \subset [D_{(c,d)'}]^\nu$.

(ii) If

$$\lim_{t \rightarrow \infty} \frac{L_t}{(L'_t)^\gamma} = 1 \tag{3}$$

holds and $x \in \ell_\infty$, then $[D_{(c,d)'}]^\nu \subset [D_{(c,d)}]^\gamma$.

Proof. (i) Let $x \in [D_{(c,d)}]^\gamma$ and suppose that (2) holds. We obtain

$$\begin{aligned} \frac{1}{(L_t)^\gamma} \sum_{i=c(t)+1}^{d(t)} \frac{|x_i - \ell|}{i} &= \frac{1}{(L_t)^\gamma} \left[\sum_{i=c(t)+1}^{c'(t)} + \sum_{i=c'(t)+1}^{d'(t)} + \sum_{i=d'(t)+1}^{d(t)} \right] \frac{|x_i - \ell|}{i} \\ &\geq \frac{1}{(L_t)^\gamma} \sum_{i=c'(t)+1}^{d'(t)} \frac{|x_i - \ell|}{i} \geq \frac{(L'_t)^\nu}{(L_t)^\gamma} \frac{1}{(L'_t)^\nu} \sum_{i=c'(t)+1}^{d'(t)} \frac{|x_i - \ell|}{i} \end{aligned}$$

for all $t \in \mathbb{N}$. Therefore $[D_{(c,d)}]^\gamma \subset [D_{(c,d)'}]^\nu$.

(ii) Let $x \in [D_{(c,d)'}]^\nu$ and assume that (3) holds. Since $x \in \ell_\infty$, there exists some $R > 0$ such that $\frac{|x_i - \ell|}{i} \leq |x_i - \ell| \leq R$ for all i . Now, since $L'_t \leq L_t$ for all $t \in \mathbb{N}$, we may write

$$\begin{aligned} \frac{1}{(L_t)^\gamma} \sum_{i=c(t)+1}^{d(t)} \frac{|x_i - \ell|}{i} &= \frac{1}{(L_t)^\gamma} \left[\sum_{i=c(t)+1}^{c'(t)} + \sum_{i=c'(t)+1}^{d'(t)} + \sum_{i=d'(t)+1}^{d(t)} \right] \frac{|x_i - \ell|}{i} \\ &\leq \frac{L_t - L'_t}{(L_t)^\gamma} R + \frac{1}{(L_t)^\gamma} \sum_{i=c'(t)+1}^{d'(t)} \frac{|x_i - \ell|}{i} \\ &\leq \frac{L_t - (L'_t)^\gamma}{(L'_t)^\gamma} R + \frac{1}{(L'_t)^\nu} \sum_{i=c'(t)+1}^{d'(t)} \frac{|x_i - \ell|}{i} \\ &\leq \left(\frac{L_t}{(L'_t)^\gamma} - 1 \right) R + \frac{1}{(L'_t)^\nu} \sum_{i=c'(t)+1}^{d'(t)} \frac{|x_i - \ell|}{i} \end{aligned}$$

for every $t \in \mathbb{N}$. Therefore $\ell_\infty \cap [D_{(c,d)'}]^\nu \subset [D_{(c,d)}]^\gamma$. □

Thus in the light of Theorem 4, we have the following result.

Corollary 1. Let $(c, d), (c', d') \in \Lambda$ be four sequences satisfying the condition (1).

If (2) holds, then

- (i) $[D_{(c,d)}] \subset [D_{(c,d)'}]^\nu$ for $0 < \nu \leq 1$,
- (ii) $[D_{(c,d)}] \subset [D_{(c,d)'}]$.

If (3) holds and $x \in \ell_\infty$, then

- (i) $[D_{(c,d)'}]^\nu \subset [D_{(c,d)}]$ for $0 < \nu \leq 1$,
- (ii) $[D_{(c,d)'}] \subset [D_{(c,d)}]$.

Definition 3. Let ν be a real number such that $0 < \nu \leq 1$. A sequence $x = (x_i)$ of real numbers is said to be deferred statistically logarithmic convergent of order ν to ℓ , if for every $\varepsilon > 0$ we have

$$\lim_{t \rightarrow \infty} \frac{1}{(L_t)^\nu} \left| \left\{ c(t) < i \leq d(t) : \frac{1}{i} |x_i - \ell| \geq \varepsilon \right\} \right| = 0.$$

In this case, we write $[SD_{(c,d)}]^\nu\text{-lim } x_i = \ell$.

The class of deferred statistically logarithmic convergent of order ν sequences would be denoted $[SD_{(c,d)}]^\nu$. In the special case $\nu = 1$, we write $[SD_{(c,d)}]$ instead of $[SD_{(c,d)}]^\nu$. Also, if $c(t) = 0$ and $d(t) = t$, then we write $[SD]^\nu$ instead of $[SD_{(c,d)}]^\nu$.

Theorem 5. Let $0 < \nu \leq 1$. Then $[SD_{(c,d)}]^\nu$ is a linear space.

Proof. The result can be proved using standard arguments. □

Theorem 6. Let $0 < \nu \leq \gamma \leq 1$. Then $[D_{(c,d)}]^\nu \subseteq [SD_{(c,d)}]^\gamma$.

Proof. Let $x = (x_i)$ be deferred strongly harmonically summable to ℓ sequence of order ν . We can write

$$\begin{aligned} \frac{1}{(L_t)^\nu} \sum_{i=c(t)+1}^{d(t)} \frac{|x_i - \ell|}{i} &= \frac{1}{(L_t)^\nu} \sum_{\substack{i=c(t)+1 \\ \frac{|x_i - \ell|}{i} \geq \varepsilon}}^{d(t)} \frac{|x_i - \ell|}{i} + \frac{1}{(L_t)^\nu} \sum_{\substack{i=c(t)+1 \\ \frac{|x_i - \ell|}{i} < \varepsilon}}^{d(t)} \frac{|x_i - \ell|}{i} \\ &\geq \frac{1}{(L_t)^\gamma} \sum_{\substack{i=c(t)+1 \\ \frac{|x_i - \ell|}{i} \geq \varepsilon}}^{d(t)} \frac{|x_i - \ell|}{i} \geq \frac{\varepsilon}{(L_t)^\gamma} \left| \left\{ c(t) < i \leq d(t) : \frac{1}{i} |x_i - \ell| \geq \varepsilon \right\} \right|. \end{aligned}$$

To complete the proof, it only remains to pass to the limit as $t \rightarrow \infty$. □

Theorem 7. Let $(c, d), (c', d') \in \Lambda$ be four sequences satisfying the condition (1), ν and γ be fixed real numbers such that $0 < \nu \leq \gamma \leq 1$. If (2) holds, then $[SD_{(c,d)}]^\gamma \subset [SD_{(c,d)'}]^\nu$.

Proof. We have

$$\left\{ c'(t) < i \leq d'(t) : \frac{|x_i - \ell|}{i} \geq \varepsilon \right\} \subseteq \left\{ c(t) < i \leq d(t) : \frac{|x_i - \ell|}{i} \geq \varepsilon \right\},$$

so

$$\begin{aligned} \frac{1}{(L_t)^\gamma} \left| \left\{ c(t) < i \leq d(t) : \frac{|x_i - \ell|}{i} \geq \varepsilon \right\} \right| &\geq \frac{1}{(L_t)^\gamma} \left| \left\{ c'(t) < i \leq d'(t) : \frac{|x_i - \ell|}{i} \geq \varepsilon \right\} \right| \\ &= \frac{(L'_t)^\nu}{(L_t)^\gamma} \frac{1}{(L'_t)^\nu} \left| \left\{ c'(t) < i \leq d'(t) : \frac{|x_i - \ell|}{i} \geq \varepsilon \right\} \right|. \end{aligned}$$

Passing to the limit as $t \rightarrow \infty$, we obtain $[SD_{(c,d)'}]^\nu\text{-lim } x_i = \ell$. □

Corollary 2. Let $(c, d), (c', d') \in \Lambda$ be four sequences satisfying the condition (1).

If (2) holds, then

- (i) $[SD_{(c,d)}] \subset [SD_{(c,d)'}]^\nu$ for $0 < \nu \leq 1$,
- (ii) $[SD_{(c,d)}] \subset [SD_{(c,d)'}]$.

Theorem 8. Let $(c, d), (c', d') \in \Lambda$ be four sequences satisfying the condition (1), ν and γ be fixed real numbers such that $0 < \nu \leq \gamma \leq 1$.

- (i) Let (2) be satisfied, then $[D_{(c,d)}]^\gamma \subset [SD_{(c,d)'}]^\nu$,
- (ii) Let (3) be satisfied and $x \in \ell_\infty$, then $[SD_{(c,d)'}]^\nu \subset [D_{(c,d)}]^\gamma$.

Proof. (i) Omitted.

(ii) Assume that $x \in [SD_{(c,d)'}]^\nu$ and $x \in \ell_\infty$, and there exists $K > 0$ such that $\frac{|x_i - \ell|}{i} \leq K$ for all i . Then for every $\varepsilon > 0$ we obtain

$$\begin{aligned} \frac{1}{(L_t)^\gamma} \sum_{i=c(t)+1}^{d(t)} \frac{|x_i - \ell|}{i} &= \frac{1}{(L_t)^\gamma} \left[\sum_{i=c(t)+1}^{c'(t)} + \sum_{i=c'(t)+1}^{d'(t)} + \sum_{i=d'(t)+1}^{d(t)} \right] \frac{|x_i - \ell|}{i} + \frac{1}{(L_t)^\gamma} \sum_{i=c'(t)+1}^{d'(t)} \frac{|x_i - \ell|}{i} \\ &\leq \frac{L_t - L'_t}{(L_t)^\gamma} K + \frac{1}{(L_t)^\gamma} \sum_{i=c'(t)+1}^{d'(t)} \frac{|x_i - \ell|}{i} \\ &\leq \frac{L_t - (L'_t)^\gamma}{(L'_t)^\gamma} K + \frac{1}{(L'_t)^\gamma} \sum_{i=c'(j)+1}^{d'(j)} \frac{|x_i - \ell|}{i} \\ &\leq \left(\frac{L_t}{(L'_t)^\gamma} - 1 \right) K + \frac{1}{(L'_t)^\gamma} \sum_{\substack{i=c'(t)+1 \\ \frac{|x_i - \ell|}{i} \geq \varepsilon}}^{d'(t)} \frac{|x_i - \ell|}{i} + \frac{1}{(L'_t)^\gamma} \sum_{\substack{i=c'(t)+1 \\ \frac{|x_i - \ell|}{i} < \varepsilon}}^{d'(t)} \frac{|x_i - \ell|}{i} \\ &\leq \left(\frac{L_t}{(L'_t)^\gamma} - 1 \right) K + \frac{K}{(L'_t)^\nu} \left| \left\{ c'(t) < i \leq d'(t) : \frac{|x_i - \ell|}{i} \geq \varepsilon \right\} \right| + \frac{L_j}{(L'_t)^\gamma} \varepsilon \\ &\leq \left(\frac{L_t}{(L'_t)^\gamma} - 1 \right) K + \frac{K}{(L'_t)^\nu} \left| \left\{ c'(t) < i \leq d'(t) : \frac{|x_i - \ell|}{i} \geq \varepsilon \right\} \right| + \frac{L_t}{(L'_t)^\gamma} \varepsilon. \end{aligned}$$

Consequently $[SD_{(c,d)'}]^\nu \subset [D_{(c,d)}]^\gamma$ for $0 < \nu \leq \gamma \leq 1$. □

Corollary 3. Let $(c, d), (c', d') \in \Lambda$ be four sequences satisfying the condition (1).

If (2) holds, then

- (i) $[D_{(c,d)}] \subset [SD_{(c,d)'}]^\nu$ for $0 < \nu \leq 1$,
- (ii) $[D_{(c,d)}] \subset [SD_{(c,d)'}]$.

If (3) holds, then

- (i) $[SD_{(c,d)'}]^\nu \subset [D_{(c,d)}]$ for $0 < \nu \leq 1$,
- (ii) $[SD_{(c,d)'}] \subset [D_{(c,d)}]$.

3 Conclusion

In the present study, using the concept of harmonic summability, we have introduced the new notations of deferred $(H, 1)$ -summability and deferred strongly harmonically summability of order ν . Further we give some inclusion relations related to these concepts. Finally deferred statistically logarithmic convergence of sequences of order ν have been introduced. We expect that the introduced notions and the results might be a reference for further studies in this field. For further studies one can investigate and generalize this results using neutrosophic normed spaces, multiplier sequences, sequence of modulus functions, etc.

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Арал Н.Д., Шенгюль Кандемір Г., Ет М. *Про відкладену гармонічну підсумовуваність порядку ν* // Карпатські матем. публ. — 2026. — Т.18, №1. — С. 189–197.

Метою цієї роботи є введення понять відкладеної $(H, 1)$ -підсумовуваності порядку ν , відкладеної сильної гармонічної підсумовуваності порядку ν та відкладеної статистичної логарифмічної збіжності порядку ν для послідовностей дійсних чисел. Крім того, встановлено деякі співвідношення включення, пов'язані з цими поняттями.

Ключові слова і фрази: відкладена статистична збіжність, гармонічна підсумовуваність, густина.