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Expanding the function $\ln(1+\mathrm{e}^x)$ into power series in terms of the Dirichlet eta function and the Stirling numbers of the second kind

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In the paper, using several approaches, the authors expand the composite function $\ln(1+\mathrm{e}^x)$ into power series around $x=0$, whose coefficients are expressed in terms of the Dirichlet eta function $\eta(1-n)$ and the Stirling numbers of the second kind $S(n,k)$.

Key words and phrases: Dirichlet eta function, composite function, power series expansion, Stirling number of the second kind, Riemann zeta function, partial Bell polynomial.

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1 Preliminaries

For fluently and smoothly proceeding, we prepare some notions and notations.

The Stirling numbers of the second kind $S(n,k)$ for $n \ge k \ge 0$ can be analytically generated [18, pp. 131–132] by

$$\frac{(\mathrm{e}^x-1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k)\frac{x^n}{n!}. \tag{1}$$

The equation (1) can be rearranged as power series expansions

$$\left(\frac{\mathrm{e}^x-1}{x}\right)^k = \sum_{n=0}^{\infty} \frac{S(n+k,k)}{\binom{n+k}{k}}\frac{x^n}{n!}, \quad k \ge 0.$$

Let $\zeta(z)$ denotes the Riemann zeta function (see [20, pp. 57–61, Section 3.5]). In [1, p. 807, Entries 23.2.14 and 23.2.15], the identity

$$\zeta(1-2n) = -\frac{B_{2n}}{2n}, \quad n \in \mathbb{N}, \tag{2}$$

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is collected, where B_n denotes the Bernoulli numbers which can be generated by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} B_{2n} \frac{z^{2n}}{(2n)!}, \quad |z| < 2\pi.$$

For recent results on the Bernoulli numbers B_n, please refer to the papers [4,6,13,14,19]. The Dirichlet eta function $\eta(s)$ and the Riemann zeta function $\zeta(s)$ have the relation

$$\eta(s) = \left(1 - 2^{1-s}\right)\zeta(s). \tag{3}$$

For new results on the Dirichlet eta function $\eta(s)$, please read the papers [9,10,12,16,17].

The Faà di Bruno formula (see [3, Theorem 11.4] and [5, p. 139, Theorem C]) can be described in terms of partial Bell polynomials $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ for $n \ge k \ge 0$ by

$$\frac{d^n}{d x^n} f \circ h(x) = \sum_{k=0}^{n} f^{(k)}(h(x)) B_{n,k}\left(h'(x), h''(x), \dots, h^{(n-k+1)}(x)\right). \tag{4}$$

The partial Bell polynomials $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ for $n \ge k \ge 0$ satisfy the identities

$$B_{n,k}\left(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\right) = a^k b^n B_{n,k}\left(x_1, x_2, \dots, x_{n-k+1}\right) \tag{5}$$

and

$$B_{n,k}(\underbrace{1, 1, \dots, 1}_{n-k+1 \text{ times}}) = S(n,k). \tag{6}$$

Last two identities can be found in [3, p. 412] and [5, p. 135].

2 Motivations

In 2013, at the site `https://math.stackexchange.com/q/307274`, G. Helms, whose profile is at the site `https://stackexchange.com/users/314145`, posed the conjecture

$$t(x) = \ln\left(1 + e^x\right) = \sum_{n=0}^{\infty} \frac{\eta(1-n)}{n!} x^n. \tag{7}$$

The motivation of this conjecture can be seen from the file at `https://go.helms-net.de/math/musings/UncompletingGamma.pdf`. There have been two answers to the conjecture (7) at the web sites `https://math.stackexchange.com/a/308100` and `https://math.stackexchange.com/a/308319`.

In this paper, we will provide several simple and alternative proofs of the above conjecture (7).

3 Power series expansions

We now start out to provide several proofs of the above conjecture (7).

3.1 First proof

Here, we recite the proof at the site `https://math.stackexchange.com/a/308319`. In this proof, the convergent radius of the power series expansion (7) is not discussed.

Theorem 1. *The power series expansion*

$$\ln(1+\mathrm{e}^x)=\sum_{n=0}^{\infty}\eta(1-n)\frac{x^n}{n!} \tag{8}$$

is valid.

Proof. This proof is extracted from `https://math.stackexchange.com/a/308319`.

The Dirichlet eta function $\eta(s)$ is given by $\eta(s)=\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}$, but this converges only for s with positive real part. A globally convergent series for $\eta(s)$ can be derived from the relation

$$\eta(s)=\left(1-2^{1-s}\right)\zeta(s)=\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{(k+1)^s}. \tag{9}$$

Using the expansion (9), we write $\eta(1-k)$ as

$$\eta(1-k)=\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\sum_{j=0}^{n}(-1)^j\binom{n}{j}(j+1)^{k-1}.$$

The power series expansion (8) is then

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{\eta(1-k)x^k}{k!}&=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\sum_{j=0}^{n}(-1)^j\binom{n}{j}(j+1)^{k-1}\frac{x^k}{k!}\\
&=\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\sum_{j=0}^{n}\frac{(-1)^j}{j+1}\binom{n}{j}\sum_{k=0}^{\infty}\frac{[x(j+1)]^k}{k!}\\
&=\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\sum_{j=0}^{n}\frac{(-1)^j}{j+1}\binom{n}{j}(\mathrm{e}^x)^{j+1}=\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\int_{-\mathrm{e}^x}^{0}\mathrm{d}\,y\sum_{j=0}^{n}\binom{n}{j}y^j\\
&=\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\int_{-\mathrm{e}^x}^{0}(1+y)^n\,\mathrm{d}\,y=\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\frac{(1+y)^{n+1}}{n+1}\bigg|_{-\mathrm{e}^x}^{0}\\
&=\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\frac{1-(1-\mathrm{e}^x)^{n+1}}{n+1}=f\left(\frac{1}{2}\right)-f\left(\frac{1-\mathrm{e}^x}{2}\right),
\end{aligned}$$

where

$$f(z)=\sum_{n=0}^{\infty}\frac{z^{n+1}}{n+1}=-\ln(1-z).$$

Putting this together, we find

$$\sum_{k=0}^{\infty}\frac{\eta(1-k)x^k}{k!}=-\ln\left(\frac{1}{2}\right)+\ln\left(\frac{1+\mathrm{e}^x}{2}\right)=\ln(1+\mathrm{e}^x).$$

Thus, the power series expansion (8) is proved. □

3.2 Second proof

In this subsection, we provide a simple, direct, and elementary proof of the conjecture (7) with a convergent range of the power series expansion (7).

Theorem 2. *For $x<0$, we have*

$$\ln(1+e^x)=\sum_{n=0}^{\infty}\eta(1-n)\frac{x^n}{n!}. \tag{10}$$

Proof. It is well known that

$$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}, \quad |x|<1.$$

Hence, it follows that

$$t(x)=\ln(1+e^x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{e^{nx}}{n}, \quad |e^x|<1.$$

From $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ for $|x|<\infty$, it follows that

$$t(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}, \quad x<0.$$

Interchanging the order of summations leads to

$$t(x)=\sum_{k=0}^{\infty}\left[\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{1-k}}\right]\frac{x^k}{k!}=\sum_{k=0}^{\infty}\eta(1-k)\frac{x^k}{k!}, \quad x<0.$$

The series expansion (10) is proved. □

3.3 Third proof

In this subsection, we provide a proof of the conjecture (7) with a convergent radius of the power series expansion (7).

Theorem 3. *For $|x|<\pi$, we have*

$$\ln(1+e^x)=\ln 2+\frac{x}{2}+\sum_{n=1}^{\infty}\eta(1-2n)\frac{x^{2n}}{(2n)!}. \tag{11}$$

Proof. Since the Euler polynomials $E_n(x)$ has the generating function

$$\frac{2e^{xy}}{e^y+1}=\sum_{n=0}^{\infty}E_n(x)\frac{y^n}{n!}, \quad |y|<\pi,$$

we easily see that

$$t'(x)=\left[\ln(1+e^x)\right]'=1-\frac{1}{1+e^x}=\frac{1}{2}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{E_n(0)}{n!}x^n, \quad |x|<\pi.$$

Integrating on both sides with respect to $x \in (0, y) \subseteq (-\pi, \pi)$ yields

$$t(y) - t(0) = \frac{y}{2} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{E_n(0)}{(n+1)!} y^{n+1}, \quad |y| < \pi.$$

Since

$$E_n(0) = -2\left(2^{n+1} - 1\right)\frac{B_{n+1}}{n+1}, \quad n \ge 1,$$

which can be found in [1, p. 805], we acquire

$$t(x) = \ln 2 + \frac{x}{2} + \sum_{n=1}^{\infty}\left(2^{n+1} - 1\right)\frac{B_{n+1}}{n+1}\frac{x^{n+1}}{(n+1)!}, \quad |x| < \pi.$$

Since $B_{2n+1} = 0$ for $n \ge 1$, we deduce

$$t(x) = \ln 2 + \frac{x}{2} + \sum_{n=1}^{\infty}\left(2^{2n} - 1\right)\frac{B_{2n}}{2n}\frac{x^{2n}}{(2n)!}, \quad |x| < \pi.$$

Making use of the equality (2) and the relation (3), we have

$$\frac{B_{2n}}{2n} = -\zeta(1-2n) = \frac{\eta(1-2n)}{2^{2n} - 1}, \quad n \ge 1.$$

Consequently, we derive the series expansion (11). □

Remark 1. *Since*

$$\eta(1) = \ln 2, \quad \eta(0) = \frac{1}{2}, \quad \eta(-2n) = 0$$

for $n \ge 1$, the series expansion (11) *is equivalent to the conjecture* (7).

3.4 Forth proof

By virtue of the Faà di Bruno formula (4) and the identities (5) and (6), we now provide an alternative and direct proof of the conjecture (7).

Theorem 4. *For $|x| < \pi$, we have*

$$\ln(1 + e^x) = \ln 2 + \sum_{n=1}^{\infty}\left[\sum_{k=1}^{n} (-1)^{k-1}\frac{(k-1)!}{2^k} S(n,k)\right]\frac{x^n}{n!}. \tag{12}$$

Proof. For $n \ge 0$, by virtue of the Faà di Bruno formula (4) and the identities (5) and (6), we obtain

$$\begin{aligned}
t^{(n)}(x) &= \sum_{k=0}^{n} (\ln u)^{(k)} B_{n,k}\Big(u'(x), u''(x), \dots, u^{(n-k+1)}(x)\Big) \\
&= (\ln u) B_{n,0}\left(e^x, e^x, \dots, e^x\right) + \sum_{k=1}^{n} (\ln u)^{(k)} B_{n,k}\left(e^x, e^x, \dots, e^x\right) \\
&= \begin{cases} \ln u, & n = 0, \\ \displaystyle\sum_{k=1}^{n} (\ln u)^{(k)} B_{n,k}\left(e^x, e^x, \dots, e^x\right), & n > 0, \end{cases} \\
&= \begin{cases} \ln(1+e^x), & n = 0, \\ \displaystyle\sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!}{(1+e^x)^k} e^{kx} B_{n,k}(1,1,\dots,1), & n > 0, \end{cases} \\
&= \begin{cases} \ln\left(1+e^x\right), & n = 0, \\ \displaystyle\sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!}{(1+e^x)^k} e^{kx} S(n,k), & n > 0, \end{cases} \\
&\to \begin{cases} \ln 2, & n = 0, \\ \displaystyle\sum_{k=1}^{n} (-1)^{k-1} \frac{(k-1)!}{2^k} S(n,k), & n > 0 \end{cases} \qquad \text{as } x \to 0,
\end{aligned}$$

where $u = u(x) = 1 + e^x$. Consequently, we derive the power series expansion (12).

Since the equation $1 + e^x = 0$ has infinitely many complex roots $(2k+1)\pi\, i$ for $k \in \mathbb{Z}$, by some knowledge in the theory of complex functions, we can determine that the convergent radius of the power series expansion (12) is π. The required proof is complete. □

Remark 2. *Comparing the above four theorems, we can conclude the identities*

$$\begin{aligned}
\sum_{k=1}^{n} (-1)^k \frac{(k-1)!}{2^k} S(n,k) &= \eta(1-n), \\
\sum_{k=1}^{n} (-1)^k \frac{(k-1)!}{2^k} S(n,k) &= (1-2^n)\zeta(1-n), \\
\sum_{k=1}^{2n} (-1)^k \frac{(k-1)!}{2^k} S(2n,k) &= \eta(1-2n), \\
\sum_{k=1}^{2n} (-1)^k \frac{(k-1)!}{2^k} S(2n,k) &= \left(1-2^{2n}\right)\zeta(1-2n), \\
\sum_{k=1}^{2n+1} (-1)^k \frac{(k-1)!}{2^k} S(2n+1,k) &= 0
\end{aligned}$$

for $n \ge 1$.

Remark 3. *In general and in practice, it is not always easy to expand a composite function into power series. For example, in the papers [2,7,8,15], the functions*

$$x^x, \qquad \left(\frac{\arcsin x}{x}\right)^{\alpha}, \qquad \left[\frac{(\arccos x)^2}{2(1-x)}\right]^{\alpha}$$

for $\alpha \in \mathbb{R}$ *are expanded into power series around* $x = 0$ *and* $x = 1$.

Remark 4. *After publishing [2], the third author of this paper found the preprint [11], in which the* n*th derivative of the power-exponential function* x^{ax} *was presented as*

$$[x^{ax}]^{(n)} = x^{ax} \sum_{k=1}^{n} \frac{n!}{k!} a^k x^{k-n} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \sum_{i=0}^{n} \binom{j}{i} \sum_{m=0}^{j} \frac{m!}{(n-i)!} \binom{j}{m} \begin{bmatrix} n-i \\ m \end{bmatrix} (\ln x)^{k-m}, \quad (13)$$

where $\begin{bmatrix} n-i \\ m \end{bmatrix}$ *is an alternative notation of the first kind Stirling numbers* $s(n-i,m)$.

In [2, Theorem 3], it was obtained that

$$\begin{aligned} \frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left[(1+x)^{t(1+x)}\right] =& n!(1+x)^{t(1+x)-n} \\ & \times \sum_{k=0}^{n} t^k (1+x)^k \sum_{j=0}^{k} \left[\sum_{q=0}^{n-k} \frac{s(q+j,j)}{(q+j)!} \binom{j}{n-k-q}\right] \frac{[\ln(1+x)]^{k-j}}{(k-j)!}. \end{aligned} \quad (14)$$

Replacing $1+x$ *by* x *in* (14) *yields*

$$\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}(x^{tx}) = n! x^{tx-n} \sum_{k=0}^{n} t^k x^k \sum_{j=0}^{k} \left[\sum_{q=0}^{n-k} \frac{s(q+j,j)}{(q+j)!} \binom{j}{n-k-q}\right] \frac{(\ln x)^{k-j}}{(k-j)!}. \quad (15)$$

It is clear that the formula (15) *is better and simpler than the one in* (13).

See also related comments on the answer at `https://mathoverflow.net/a/439188`.

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Вен-Гуї Лі, Донгкю Лім, Фенґ Кі. *Розвинення функції* $\ln(1+e^x)$ *у степеневий ряд через ета-функцію Діріхле та числа Стірлінга другого роду* // Карпатські матем. публ. — 2024. — Т.16, №1. — С. 320–327.

У статті, використовуючи кілька підходів, автори розвивають складену функцію $\ln(1+e^x)$ у степеневі ряди в околі точки $x=0$, коефіцієнти яких виражаються через ета-функцію Діріхле $\eta(1-n)$ і числа Стірлінга другого роду $S(n,k)$.

Ключові слова і фрази: ета-функція Діріхле, складена функція, розвинення в степеневий ряд, число Стірлінга другого роду, дзета-функція Рімана, частковий поліном Белла.