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Relative r-noncommuting graph of finite rings

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Let S be a subring of a finite ring R and $r \in R$. The relative r-noncommuting graph of R relative to S, denoted by $\Gamma^r_{S,R}$, is a simple undirected graph whose vertex set is R and two vertices x and y are adjacent if and only if $x \in S$ or $y \in S$ and $[x,y] \neq r$, $[x,y] \neq -r$. In this paper, we determine degree of any vertex in $\Gamma^r_{S,R}$ and characterize all finite rings such that $\Gamma^r_{S,R}$ is a star, lollipop or a regular graph. We derive connections between relative r-noncommuting graphs of two isoclinic pairs of rings. We also derive certain relations between the number of edges in $\Gamma^r_{S,R}$ and various generalized commuting probabilities of R. Finally, we conclude the paper by studying an induced subgraph of $\Gamma^r_{S,R}$.

Key words and phrases: finite ring, non-commuting graph, isoclinism.

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1 Introduction

Throughout the paper, S denotes a subring of a finite ring R and $r \in R$. Let us denote $Z(S,R) = \{z \in S : zr = rz \text{ for all } r \in R\}$. Then $Z(R,R) = Z(R)$ is the center of R. For any element $x \in R$, we write $C_S(x)$ to denote the set $\{y \in S : xy = yx\}$. Clearly, $Z(S,R) = \bigcap_{x \in R} C_S(x)$. Let the additive commutator of $x, y \in R$ be denoted by $[x,y]$. Then $[x,y] = xy - yx$. Let $K(S,R) = \{[x,y] : x \in S \text{ and } y \in R\}$ and $[S,R]$ be the additive subgroup of $(R,+)$ generated by the set $K(S,R)$.

In [18], R.K. Nath et. al. introduced the notion of r-noncommuting graph of a finite ring motivated by the work of B. Tolue, A. Erfanian and A. Jafarzadeh [23]. Recall that the r-noncommuting graph of R is an undirected graph whose vertex set is R and two distinct vertices x and y are adjacent if and only if $[x,y] \neq r$ and $[x,y] \neq -r$. Note that this graph is a generalization of the non-commuting graph of a finite ring introduced and studied by A. Erfanian, K. Khashyarmanesh and Kh. Nafar [9] in the year 2015. Recent results on non-commuting graph (along with its complement and generalizations) of finite rings can be found in [4,5,10,11,13,15,17,19,21]. Non-commuting graph of finite groups is also well-explored over the years (see [12] and the references therein).

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In this paper, we generalize the notion of r-noncommuting graph of a finite ring. More precisely, we consider a graph called relative r-noncommuting graph of R relative to S, which is denoted by $\Gamma^r_{S,R}$ and defined as a simple undirected graph with vertex set R and two vertices x and y are adjacent if and only if $x \in S$ or $y \in S$ and $[x,y] \neq r$, $[x,y] \neq -r$. Clearly, $\Gamma^r_{R,R}$ is the r-noncommuting graph of R. Further, if $r = 0$, then the induced subgraph of $\Gamma^r_{S,R}$ with vertex set $R \setminus C_R(S)$, where $C_R(S) = \{r \in R : sr = rs \text{ for all } s \in S\}$, is nothing but the relative non-commuting graph of R studied in [2]. Group theoretic analogues of this newly introduced graph were considered in [20,22].

Let K_n be the complete graph on n vertices and let $\overline{\mathcal{G}}$ be the complement of any graph $\mathcal{G}$. We write $\mathcal{G} + \mathcal{H}$ to denote the join of the graphs $\mathcal{G}$ and $\mathcal{H}$.

Observation 1. *Let S be a subring of a finite ring R and $r \in R$. Then we have the following.*

(a) *If $r \notin K(S,R)$, then $\Gamma^r_{S,R} = K_{|S|} + \overline{K_{|R|-|S|}}$ and so*

$$\deg(x) = \begin{cases} |R| - 1, & \text{if } x \in S, \\ |S|, & \text{if } x \in R \setminus S. \end{cases}$$

(b) *If $K(S,R) = \{0\}$ and $r = 0$, then $\Gamma^r_{S,R} = \overline{K_{|R|}}$.*

It follows that if $r \notin K(S,R)$, then

(i) $\Gamma^r_{S,R}$ is a tree if and only if $S = \{0\}$ or $|S| = |R| = 2$,

(ii) $\Gamma^r_{S,R}$ is a star graph if and only if $S = \{0\}$,

(iii) $\Gamma^r_{S,R}$ is a complete graph if and only if $S = R$.

Note that if R is commutative or $S = Z(S,R)$, then $K(S,R) = \{0\}$. Therefore, throughout this paper, in view of Observation 1, we shall consider R to be non-commutative, S to be a subring of R such that $S \neq Z(S,R)$ and $r \in K(S,R)$.

In Section 2, we derive formula for degree of any vertex in $\Gamma^r_{S,R}$ and characterize all finite rings such that $\Gamma^r_{S,R}$ is a star, lollipop or a regular graph. In Section 3, we show that $\Gamma^r_{S_1,R_1}$ is isomorphic to $\Gamma^{\psi(r)}_{S_2,R_2}$ if (ϕ, ψ) is an isoclinism between the pairs of finite rings (S_1, R_1) and (S_2, R_2) such that $|Z(S_1,R_1)| = |Z(S_2,R_2)|$. In Section 4, we obtain certain relations between the number of edges in $\Gamma^r_{S,R}$ and $\Pr_r(S,K)$. Recall that $\Pr_r(S,K)$ is the probability that the commutator of a randomly chosen pair of elements $(x,y) \in S \times K$ equals r, where S and K are two subrings of R. The probability $\Pr_r(S,K)$ is a generalization of commutativity degree of finite rings and it was introduced in [6]. Finally, we conclude the paper by deriving certain results on the induced subgraph of $\Gamma^r_{S,R}$ with vertex set $R \setminus Z(S,R)$.

2 Vertex degree and consequences

We write $\deg(x)$ to denote the degree of a vertex x in $\Gamma^r_{S,R}$. For any two given elements $x, r \in R$ we write $C^r_S(x)$ to denote the set $\{s \in S : [x,s] = r\}$. Note that $C^r_S(x)$ is the centralizer of x in R if $S = R$ and $r = 0$. The following theorem gives degree of any vertex in $\Gamma^r_{S,R}$.

Theorem 1. *Let x be any vertex in $\Gamma^r_{S,R}$.*

(a) *If $r = 0$, then* $\deg(x) = \begin{cases} |R| - |C_R(x)|, & \text{if } x \in S, \\ |S| - |C_S(x)|, & \text{if } x \in R \setminus S. \end{cases}$

(b) *If $r \neq 0$ and $2r = 0$, then* $\deg(x) = \begin{cases} |R| - \left|C^r_R(x)\right| - 1, & \text{if } x \in S, \\ |S| - \left|C^r_S(x)\right|, & \text{if } x \in R \setminus S. \end{cases}$

(c) *If $r \neq 0$ and $2r \neq 0$, then* $\deg(x) = \begin{cases} |R| - 2|C^r_R(x)| - 1, & \text{if } x \in S, \\ |S| - 2|C^r_S(x)|, & \text{if } x \in R \setminus S. \end{cases}$

Proof. (a) Let $r = 0$. If $x \in S$, then $\deg(x)$ is the number of $s \in R$ such that $sx \neq xs$. Hence, $\deg(x) = |R| - |C_R(x)|$. If $x \in R \setminus S$, then $\deg(x)$ is the number of $s \in S$ such that $sx \neq xs$. Hence, $\deg(x) = |S| - |C_S(x)|$.

(b) Let $r \neq 0$ and $2r = 0$. In this case, $r = -r$. If $x \in S$, then $s \in R$ is not adjacent to x if and only if $s = x$ or $s \in C^r_R(x)$. Hence, $\deg(x) = |R| - \left|C^r_R(x)\right| - 1$. If $x \in R \setminus S$, then $s \in S$ is not adjacent to x if and only if $s \in C^r_S(x)$. Hence, $\deg(x) = |S| - \left|C^r_S(x)\right|$.

(c) Let $r \neq 0$ and $2r \neq 0$. In this case, $r \neq -r$. Also, $C^r_S(x) \cap C^{-r}_S(x) = \varnothing$ and $s \in C^r_S(x)$ if and only if $-s \in C^{-r}_S(x)$. Therefore, $C^r_S(x)$ and $C^{-r}_S(x)$ have same cardinality. Further, if $x \in S$, then $s \in R$ is not adjacent to x if and only if $s = x$, $s \in C^r_R(x)$ or $s \in C^{-r}_R(x)$. Hence, $\deg(x) = |R| - \left|C^r_R(x)\right| - \left|C^{-r}_R(x)\right| - 1$. If $x \in R \setminus S$, then $s \in S$ is not adjacent to x if and only if $s \in C^r_S(x)$ or $s \in C^{-r}_S(x)$. Hence, $\deg(x) = |S| - \left|C^r_S(x)\right| - \left|C^{-r}_S(x)\right|$. Hence, the result follows. □

The next lemma shows that for all $x, r \in R$ the cardinality of $C^r_S(x)$ is either zero or $|C_S(x)|$.

Lemma 1. *If $C^r_S(x)$ is non-empty, then $\left|C^r_S(x)\right| = |C_S(x)|$ for all $x, r \in R$.*

Proof. Let $t \in C^r_S(x)$ and $p \in t + C_S(x)$. Then $p = t + m$ for some $m \in C_S(x)$. We have

$$[x, p] = [x, t + m] = x(t + m) - (t + m)x = [x, t] = r$$

and so $p \in C^r_S(x)$. Therefore, $t + C_S(x) \subseteq C^r_S(x)$. Again, if $y \in C^r_S(x)$, then $[x, t] = [x, y]$, which implies $(y - t)x = x(y - t)$. Therefore $(y - t) \in C_S(x)$ and so $y \in t + C_S(x)$. Thus, $C^r_S(x) \subseteq t + C_S(x)$. Hence, $C^r_S(x) = t + C_S(x)$ and the result follows. □

By Lemma 1 and Theorem 1, we have the following two corollaries.

Corollary 1. *Let $x \in S$ be a vertex in $\Gamma^r_{S,R}$.*

(a) *If $r \neq 0$ and $2r = 0$, then* $\deg(x) = \begin{cases} |R| - |C_R(x)| - 1, & \text{if } C^r_R(x) \neq \varnothing, \\ |R| - 1, & \text{otherwise.} \end{cases}$

(b) *If $r \neq 0$ and $2r \neq 0$, then* $\deg(x) = \begin{cases} |R| - 2\,|C_R(x)| - 1, & \text{if } C^r_R(x) \neq \varnothing, \\ |R| - 1, & \text{otherwise.} \end{cases}$

Corollary 2. *Let $x \in R \setminus S$ be a vertex in $\Gamma^r_{S,R}$.*

(a) *If $r \neq 0$ and $2r = 0$, then* $\deg(x) = \begin{cases} |S| - |C_S(x)|, & \text{if } C^r_S(x) \neq \varnothing, \\ |S|, & \text{otherwise.} \end{cases}$

(b) *If $r \neq 0$ and $2r \neq 0$, then* $\deg(x) = \begin{cases} |S| - 2\,|C_S(x)|, & \text{if } C^r_S(x) \neq \varnothing, \\ |S|, & \text{otherwise.} \end{cases}$

In the next few results we discuss some properties of $\Gamma^r_{S,R}$. The following lemma shows that $\Gamma^r_{S,R}$ is a disconnected graph if $r = 0$.

Lemma 2. *If $x \in Z(S, R)$, then* $\deg(x) = \begin{cases} 0, & \text{if } r = 0, \\ |R| - 1, & \text{if } r \neq 0. \end{cases}$

Proof. The result follows from Theorem 1 noting that $x \in S$ and

$$C^r_R(x) = \begin{cases} C_R(x) = R, & \text{if } r = 0, \\ \varnothing, & \text{if } r \neq 0. \end{cases}$$

□

Lemma 3. *Let S be a subring of a non-commutative ring R with unity 1 and $r \neq 0$. If $1 \in S$, then $\deg(x) \geq 2$ for all $x \in R$.*

Proof. The result follows from the fact that $[x, 0] = [x, 1] \neq r$ and $[x, 0] = [x, 1] \neq -r$ for all $x \in R$. □

Proposition 1. *Let S be a subring of a non-commutative ring R and $r \in R$.*

(a) *If $r = 0$, then $\Gamma^r_{S,R}$ is not a tree, star graph, lollipop graph and complete graph.*

(b) *If $r \neq 0$ and R has unity $1 \in S$, then $\Gamma^r_{S,R}$ is not a tree and a star graph.*

Proof. The results follow from Lemma 2 and Lemma 3. □

Theorem 2. *Let S be a subring of a non-commutative ring R and $r \neq 0$. Then $\Gamma^r_{S,R}$ is a star if and only if $2r = 0$, $S \neq \{0\}$ and R is isomorphic to*

$$\langle a, b : 2a = 2b = 0, a^2 = a, b^2 = b, ab = a, ba = b \rangle \tag{1}$$

or

$$\langle x, y : 2x = 2y = 0, x^2 = x, y^2 = y, xy = y, yx = x \rangle. \tag{2}$$

Proof. If R is isomorphic to (1) or (2), then it is easy to see that $\Gamma^r_{S,R}$ is a star graph for any subring S.

Suppose that $\Gamma^r_{S,R}$ is a star graph. Clearly, $\deg(0) = |R| - 1$. Also, $\deg(x) = 1$ for all $0 \neq x \in R$. Since $r \neq 0$ and $r \in K(S, R)$, we have $S \neq \{0\}$. Let $0 \neq y \in R$. Then consider the following cases.

Case 1: $y \in S$.

Note that $\deg(y) \neq |R| - 1$. Therefore, if $2r = 0$, then, by Corollary $1(a)$, we have $1 = \deg(y) = |R| - |C_R(y)| - 1$. Thus, $|R| - |C_R(y)| = 2$. We have $0, y \in C_R(y)$. Since $C_R(y)$ is a subring of R, $|C_R(y)|$ divides $|R| - |C_R(y)|$. Therefore, $|C_R(y)| = 2$ and hence $|R| = 4$.

If $2r \neq 0$, then, by Corollary $1(b)$, we have $1 = \deg(y) = |R| - 2\,|C_R(y)| - 1$. Therefore, $|C_R(y)| = 2$ and hence $|R| = 6$, a contradiction since R is non-commutative.

Hence, $2r = 0$ and $|R| = 4$.

Case 2: $y \in R \setminus S$.

Note that $\deg(y) \neq |S|$, otherwise $|S| = 1$, a contradiction. Therefore, if $2r = 0$, then, by Corollary $2(a)$, we have $1 = \deg(y) = |S| - |C_S(y)|$. Note that $0 \in C_S(y)$. Since $C_S(y)$ is a subring of S, $|C_S(y)|$ divides $|S| - |C_S(y)|$. Therefore, $|C_S(y)| = 1$ and hence $|S| = 2$. Thus, S has a non-zero element and so, by Case 1, we have $|R| = 4$.

If $2r \neq 0$, then, by Corollary $2(b)$, we have $1 = \deg(y) = |S| - 2\,|C_S(y)|$. Therefore, $|C_S(y)| = 1$ and hence $|S| = 3$. It follows that S has a non-zero element and so, by Case 1, we have $2r = 0$ and $|R| = 4$, a contradiction.

Hence, $2r = 0$ and R is isomorphic to (1) or (2). Thus, the result follows. □

Theorem 3. *Let S be a non-commutative subring of R. Then $\Gamma^r_{S,R}$ is not a lollipop graph.*

Proof. If $r = 0$, then the result follows from Proposition $1(a)$. Let $r \neq 0$ and $\Gamma^r_{S,R}$ be a lollipop graph. Then there exits an element $x \in R$ such that $\deg(x) = 1$.

Case 1: $x \in S$.

By Corollary 1, we have $\deg(x) = |R| - 1 = 1$ or $\deg(x) = |R| - |C_R(x)| - 1 = 1$ or $\deg(x) = |R| - 2\,|C_R(x)| - 1 = 1$. Therefore, $|R| - |C_R(x)| = 2$ or $|R| - 2\,|C_R(x)| = 2$ since $|R| \neq 2$. Thus, $|C_R(x)| = 2$ and so $|R| = 4$ since $|R| \neq 6$. Hence, by Theorem 2, $\Gamma^r_{S,R}$ is a star graph, a contradiction.

Case 2: $x \in R \setminus S$.

By Corollary 2, we have $\deg(x) = |S| - |C_S(x)| = 1$ or $\deg(x) = |S| - 2\,|C_S(x)| = 1$ since $|S| \neq 1$. Therefore $|C_S(x)| = 1$ and so $|S| = 2$ or $|S| = 3$. Hence, S is commutative, a contradiction. □

Note that Theorem 3 is a generalization of [18, Proposition 2.4]. We conclude this section with the following result.

Theorem 4. *Let S be a subring of a non-commutative ring R. Then $\Gamma^r_{S,R}$ is regular if and only if $K(S,R) = \{0\}$.*

Proof. If $K(S,R) = \{0\}$, then $r = 0$. Therefore, by Observation $1(b)$, it follows that $\Gamma^r_{S,R}$ is regular.

Suppose that $\Gamma^r_{S,R}$ is regular. If $r = 0$, then, by Lemma 2, we have $\deg(0) = 0$. Therefore, $\Gamma^r_{S,R} = \overline{K_{|R|}}$ and so $K(S,R) = \{0\}$. If $r \neq 0$, then, by Lemma 2, we obtain $\deg(0) = |R| - 1$. Therefore, $\Gamma^r_{S,R}$ is a complete graph and so $S = R$. That is, $\Gamma^r_{R,R}$ is regular, which is a contradiction by [18, Proposition 4]. □

3 $\Gamma^r_{S,R}$ of isoclinic pairs

Following P. Hall [14], S.M. Buckley et. al. [1] have defined isoclinism between two rings. Recently, J. Dutta et. al. [3] have generalized the notion of isoclinism between two rings to isoclinism between two pairs of rings. Let S_1 and S_2 be two subrings of the rings R_1 and R_2. A pair (ϕ, ψ) of additive group isomorphisms $\phi : \frac{R_1}{Z(S_1,R_1)} \to \frac{R_2}{Z(S_2,R_2)}$ and $\psi : [S_1, R_1] \to [S_2, R_2]$, such that $\phi\left(\frac{S_1}{Z(S_1,R_1)}\right) = \frac{S_2}{Z(S_2,R_2)}$ and $\psi([x_1, y_1]) = [x_2, y_2]$, whenever $x_i \in S_i$, $y_i \in R_i$, $i = 1, 2$, and $\phi(x_1 + Z(S_1, R_1)) = x_2 + Z(S_2, R_2)$, $\phi(y_1 + Z(S_1, R_1)) = y_2 + Z(S_2, R_2)$, is called an isoclinism between the pairs (S_1, R_1) and (S_2, R_2). Two pairs of rings (S_1, R_1) and (S_2, R_2) are called isoclinic if there exists an isoclinism between them. In this section, we mainly prove the following result.

Theorem 5. *Let R_1 and R_2 be two finite rings. Let S_1 and S_2 be two subrings of R_1 and R_2, respectively, such that $|Z(S_1, R_1)| = |Z(S_2, R_2)|$. If $r \in [S_1, R_1]$ and (ϕ, ψ) is an isoclinism between the pairs (S_1, R_1) and (S_2, R_2), then $\Gamma^r_{S_1,R_1} \cong \Gamma^{\psi(r)}_{S_2,R_2}$.*

Proof. We have $\phi : \frac{R_1}{Z(S_1,R_1)} \to \frac{R_2}{Z(S_2,R_2)}$ is an isomorphism such that $\phi\left(\frac{S_1}{Z(S_1,R_1)}\right) = \frac{S_2}{Z(S_2,R_2)}$. Therefore, $\left|\frac{R_1}{Z(S_1,R_1)}\right| = \left|\frac{R_2}{Z(S_2,R_2)}\right|$ and $\left|\frac{S_1}{Z(S_1,R_1)}\right| = \left|\frac{S_2}{Z(S_2,R_2)}\right|$. Let $\left|\frac{S_1}{Z(S_1,R_1)}\right| = m$ and $\left|\frac{R_1}{Z(S_1,R_1)}\right| = n$. Let $\{s_1, \ldots, s_m, r_{m+1}, \ldots, r_n\}$ and $\{s'_1, \ldots, s'_m, r'_{m+1}, \ldots, r'_n\}$ be two transversals of $\frac{R_1}{Z(S_1,R_1)}$ and $\frac{R_2}{Z(S_2,R_2)}$, respectively, such that $\{s_1, \ldots, s_m\}$ and $\{s'_1, \ldots, s'_m\}$ are transversals of $\frac{S_1}{Z(S_1,R_1)}$ and $\frac{S_2}{Z(S_2,R_2)}$, respectively.

Let ϕ be defined as $\phi(s_i + Z(S_1, R_1)) = s'_i + Z(S_2, R_2)$, $\phi(r_j + Z(S_1, R_1)) = r'_j + Z(S_2, R_2)$ for $1 \le i \le m$ and $m + 1 \le j \le n$. Let $\theta : Z(S_1, R_1) \to Z(S_2, R_2)$ be a one-to-one correspondence. Let us define a map $\alpha : R_1 \to R_2$ such that $\alpha(s_i + z) = s'_i + \theta(z)$, $\alpha(r_j + z) = r'_j + \theta(z)$ for $z \in Z(S_1, R_1)$, $1 \le i \le m$ and $m + 1 \le j \le n$. Then α is a bijection.

Suppose, u, v are adjacent in $\Gamma^r_{S_1,R_1}$. Then $u \in S_1$ or $v \in S_1$ and $[u, v] \neq r$, $[u, v] \neq -r$. Without any loss of generality, let us assume that $u \in S_1$. Then $u = s_i + z$ for $1 \le i \le m$ and $v = t + z_1$, where $z, z_1 \in Z(S_1, R_1)$, $t \in \{s_1, \ldots, s_m, r_{m+1}, \ldots, r_n\}$. Therefore, for some $t' \in \{s'_1, \ldots, s'_m, r'_{m+1}, \ldots, r'_n\}$, we have

$$\begin{aligned}
[s_i + z, t + z_1] \neq r, -r \Rightarrow\ & \psi([s_i + z, t + z_1]) \neq \psi(r), -\psi(r) \\
\Rightarrow\ & [s'_i + \theta(z), t' + \theta(z_1)] \neq \psi(r), -\psi(r) \\
\Rightarrow\ & [\alpha(s_i + z), \alpha(t + z_1)] \neq \psi(r), -\psi(r) \\
\Rightarrow\ & [\alpha(u), \alpha(v)] \neq \psi(r), -\psi(r).
\end{aligned}$$

This shows that $\alpha(u)$ and $\alpha(v)$ are adjacent in $\Gamma^{\psi(r)}_{S_2,R_2}$ noting that $\alpha(u) \in S_2$. Hence, α is an isomorphism between the graphs $\Gamma^r_{S_1,R_1}$ and $\Gamma^{\psi(r)}_{S_2,R_2}$. This completes the proof. □

4 Connecting $\Gamma^r_{S,R}$ with $\Pr_r(S, R)$

The commutativity degree of a finite ring R is the probability that a randomly chosen pair of elements of R commute and it is denoted by $\Pr(R)$. The study of this probability was first considered by D. MacHale [16]. Certain generalizations of this notion are also considered in [3, 6–8].

One such generalization, introduced in [6], is given by

$$\Pr_r(S,K) := \frac{\left|\{(x,y) \in S \times K : [x,y] = r\}\right|}{|S||K|},$$

where $r \in R$ and S, K are two subrings of R. That is, $\Pr_r(S,K)$ is the probability that the commutator of a randomly chosen pair of elements $(x,y) \in S \times K$ equals a given element $r \in R$. We have

$$\Pr_r(S,K) = \begin{cases} \Pr(S,R), & \text{if } r = 0 \text{ and } K = R, \\ \Pr_r(S,R), & \text{if } K = R, \\ \Pr_r(S), & \text{if } S = K, \\ \Pr(R), & \text{if } r = 0 \text{ and } S = K = R. \end{cases} \tag{3}$$

It is worth mentioning that $\Pr(S,R), \Pr_r(S,R)$ and $\Pr_r(S)$ are studied in [3,7,8] respectively.

In this section, we derive some connections between $\Gamma^r_{S,R}$ and $\Pr_r(S,R)$. Let $\left|E(\Gamma^r_{S,R})\right|$ denotes the number of edges in $\Gamma^r_{S,R}$. If $r \notin K(S,R)$, then it follows from Observation 1, that

$$\left|E(\Gamma^r_{S,R})\right| = |S||R| - \frac{|S|^2 + |S|}{2}.$$

The following theorem gives the number of edges in $\Gamma^r_{S,R}$ in terms of $\Pr_r(S,R)$ and $\Pr_r(S)$.

Theorem 6. *Let S be a subring of a finite ring R.*

(a) *If $r = 0$, then*

$$2\left|E(\Gamma^r_{S,R})\right| = 2|S||R|\big(1 - \Pr(S,R)\big) - |S|^2\big(1 - \Pr(S)\big).$$

(b) *If $r \neq 0$ and $2r = 0$, then*

$$2\left|E(\Gamma^r_{S,R})\right| = \begin{cases} 2|S||R|\left(1 - \Pr_r(S,R)\right) - |S|^2\left(1 - \Pr_r(S)\right) - |S|, & \text{if } r \in S, \\ 2|S||R|\left(1 - \Pr_r(S,R)\right) - |S|^2 - |S|, & \text{if } r \in R \setminus S. \end{cases}$$

(c) *If $r \neq 0$ and $2r \neq 0$, then*

$$2\left|E(\Gamma^r_{S,R})\right| = \begin{cases} 2|S||R|\big(1 - \sum\limits_{u=r,-r} \Pr_u(S,R)\big) - |S|^2\big(1 - \sum\limits_{u=r,-r} \Pr_u(S)\big) - |S|, & \text{if } r \in S, \\ 2|S||R|\big(1 - \sum\limits_{u=r,-r} \Pr_u(S,R)\big) - |S|^2 - |S|, & \text{if } r \in R \setminus S. \end{cases}$$

Proof. Let

$$\mathbb{I} = \big\{(x,y) \in S \times R : x \neq y, [x,y] \neq r \text{ and } [x,y] \neq -r\big\}$$

and

$$\mathbb{J} = \big\{(x,y) \in R \times S : x \neq y, [x,y] \neq r \text{ and } [x,y] \neq -r\big\}.$$

Then $\mathbb{I} \cap \mathbb{J} = \big\{(x,y) \in S \times S : x \neq y, [x,y] \neq r \text{ and } [x,y] \neq -r\big\}$. It is easy to see that $(x,y) \mapsto (y,x)$ defines a bijective map from $\mathbb{I}$ to $\mathbb{J}$ and so $|\mathbb{I}| = |\mathbb{J}|$. Also, $2\left|E(\Gamma^r_{S,R})\right| = |\mathbb{I} \cup \mathbb{J}|$. Therefore,

$$2\left|E(\Gamma^r_{S,R})\right| = 2|\mathbb{I}| - |\mathbb{I} \cap \mathbb{J}|. \tag{4}$$

(a) If $r = 0$, then, by (3), we have

$$\begin{aligned}|\mathbb{I}| &= \big|\{(x,y) \in S \times R : [x,y] \neq 0\}\big| \\ &= |S||R| - \big|\{(x,y) \in S \times R : [x,y] = 0\}\big| = |S||R|\big(1 - \Pr(S,R)\big)\end{aligned}$$

and

$$\begin{aligned}|\mathbb{I} \cap \mathbb{J}| &= \big|\{(x,y) \in S \times S : [x,y] \neq 0\}\big| \\ &= |S|^2 - \big|\{(x,y) \in S \times S : [x,y] = 0\}\big| = |S|^2\big(1 - \Pr(S)\big).\end{aligned}$$

Hence, the result follows from (4).

(b) If $r \neq 0$ and $2r = 0$, then $r = -r$. Therefore, by (3), we have

$$\begin{aligned}|\mathbb{I}| &= \big|\{(x,y) \in S \times R : x \neq y, [x,y] \neq r\}\big| \\ &= |S||R| - \big|\{(x,y) \in S \times R : [x,y] = r\}\big| - \big|\{(x,y) \in S \times S : x = y\}\big| \\ &= |S||R|\left(1 - \Pr_r(S,R)\right) - |S|.\end{aligned}$$

If $r \in S$, then, by (3), we have

$$\begin{aligned}|\mathbb{I} \cap \mathbb{J}| &= \big|\{(x,y) \in S \times S : x \neq y, [x,y] \neq r\}\big| \\ &= |S|^2 - \big|\{(x,y) \in S \times S : [x,y] = r\}\big| - \big|\{(x,y) \in S \times S : x = y\}\big| \\ &= |S|^2\left(1 - \Pr_r(S)\right) - |S|.\end{aligned}$$

If $r \in R \setminus S$, then we have $|\mathbb{I} \cap \mathbb{J}| = |S|^2 - |S|$, noting that $\big\{(x,y) \in S \times S : [x,y] = r\big\}$ is empty. Therefore,

$$|\mathbb{I} \cap \mathbb{J}| = \begin{cases} |S|^2\left(1 - \Pr_r(S)\right) - |S|, & \text{if } r \in S, \\ |S|^2 - |S|, & \text{if } r \in R \setminus S. \end{cases}$$

Hence, the result follows from (4).

(c) If r $\neq 0$ and $2r \neq 0$, then, by (3), we have

$$\begin{aligned}|\mathbb{I}| &= \big|\{(x,y) \in S \times R : x \neq y, [x,y] \neq r \ \text{ and } \ [x,y] \neq -r\}\big| \\ &= |S||R| - \big|\{(x,y) \in S \times R : [x,y] = r\}\big| \\ &\qquad - \big|\{(x,y) \in S \times R : [x,y] = -r\}\big| - \big|\{(x,y) \in S \times S : x = y\}\big| \\ &= |S||R|\Big(1 - \sum_{u=r,-r} \Pr_u(S,R)\Big) - |S|.\end{aligned}$$

If $r \in S$, then, by (3), we have

$$\begin{aligned}|\mathbb{I} \cap \mathbb{J}| &= \big|\{(x,y) \in S \times S : x \neq y, [x,y] \neq r \ \text{ and } \ [x,y] \neq -r\}\big| \\ &= |S|^2 - \big|\{(x,y) \in S \times S : [x,y] = r\}\big| \\ &\qquad - \big|\{(x,y) \in S \times S : [x,y] = -r\}\big| - \big|\{(x,y) \in S \times S : x = y\}\big| \\ &= |S|^2\Big(1 - \sum_{u=r,-r} \Pr_u(S)\Big) - |S|.\end{aligned}$$

If $r \in R \setminus S$, then we have $|\mathbb{I} \cap \mathbb{J}| = |S|^2 - |S|$, noting that $\big\{(x,y) \in S \times S : [x,y] = r\big\}$ and $\big\{(x,y) \in S \times S : [x,y] = -r\big\}$ are empty. Therefore,

$$|\mathbb{I} \cap \mathbb{J}| = \begin{cases} |S|^2\Big(1 - \sum\limits_{u=r,-r} \Pr_u(S)\Big) - |S|, & \text{if } r \in S, \\ |S|^2 - |S|, & \text{if } r \in R \setminus S. \end{cases}$$

Hence, the result follows from (4). $\square$

As an application of Theorem 6, in the following two propositions, we compute the number of edges in $\Gamma^r_{S,R}$ if $[S,R]$ has prime order.

Proposition 2. *Let S be a commutative subring of a finite ring R such that $\big|[S,R]\big| = p$, where p is a prime.*

(a) If $r = 0$, then

$$\big|E(\Gamma^r_{S,R})\big| = \frac{(p-1)|R|\big(|S| - |Z(S,R)|\big)}{p}.$$

(b) If $r \neq 0$ and $2r = 0$, then

$$\big|E(\Gamma^r_{S,R})\big| = \frac{2|R|\big((p-1)|S| + |Z(S,R)|\big) - p|S|^2 - p|S|}{2p}.$$

(c) If $r \neq 0$ and $2r \neq 0$, then

$$\big|E(\Gamma^r_{S,R})\big| = \frac{2|R|\big((p-2)|S| + 2|Z(S,R)|\big) - p|S|^2 - p|S|}{2p}.$$

Proof. If $\big|[S,R]\big| = p$, then, by [7, Corollary 2], we have

$$\Pr_r(S,R) = \begin{cases} \frac{1}{p}\Big(1 + \frac{p-1}{|S:Z(S,R)|}\Big), & \text{if } r = 0,\\ \frac{1}{p}\Big(1 - \frac{1}{|S:Z(S,R)|}\Big), & \text{if } r \neq 0. \end{cases}$$

Since S is commutative, we have $[S,S] = \{0\}$. Therefore, by (3), we have

$$\Pr_r(S) = \begin{cases} 1, & \text{if } r = 0,\\ 0, & \text{if } r \neq 0. \end{cases}$$

Hence, the results follows from Theorem 6. □

Proposition 3. *Let S be a non-commutative subring of a finite ring R such that $\big|[S,R]\big| = p$, where p is a prime.*

(a) If $r = 0$, then

$$\big|E(\Gamma^r_{S,R})\big| = \frac{(p-1)\big[2|R|\big(|S| - |Z(S,R)|\big) - |S|\big(|S| - |Z(S)|\big)\big]}{2p}.$$

(b) If $r \neq 0$ and $2r = 0$, then

$$\big|E(\Gamma^r_{S,R})\big| = \begin{cases} \dfrac{2|R|\big((p-1)|S| + |Z(S,R)|\big) - |S|\big((p-1)|S| + |Z(S)|\big) - p|S|}{2p}, & \text{if } r \in S,\\ \dfrac{2|R|\big((p-1)|S| + |Z(S,R)|\big) - p|S|^2 - p|S|}{2p}, & \text{if } r \in R \setminus S. \end{cases}$$

(c) If $r \neq 0$ and $2r \neq 0$, then

$$\big|E(\Gamma^r_{S,R})\big| = \begin{cases} \dfrac{2|R|\big((p-2)|S| + 2|Z(S,R)|\big) - |S|\big((p-2)|S| + 2|Z(S)|\big) - p|S|}{2p}, & \text{if } r \in S,\\ \dfrac{2|R|\big((p-2)|S| + 2|Z(S,R)|\big) - p|S|^2 - p|S|}{2p}, & \text{if } r \in R\setminus S. \end{cases}$$

Proof. If $\left|[S,R]\right| = p$, then, by [7, Corollary 2], we have

$$\Pr_r(S,R) = \begin{cases} \frac{1}{p}\left(1 + \frac{p-1}{|S:Z(S,R)|}\right), & \text{if } r = 0, \\ \frac{1}{p}\left(1 - \frac{1}{|S:Z(S,R)|}\right), & \text{if } r \neq 0. \end{cases}$$

If S is non-commutative, then $\left|[S,S]\right| = \left|[S,R]\right| = p$. Therefore, by [8, Lemma 3.2], we have

$$\Pr_r(S) = \begin{cases} \frac{1}{p}\left(1 + \frac{p-1}{|S:Z(S)|}\right), & \text{if } r = 0, \\ \frac{1}{p}\left(1 - \frac{1}{|S:Z(S)|}\right), & \text{if } r \neq 0. \end{cases}$$

Hence, the results follows from Theorem 6. □

Corollary 3. *Let* $R = E(p^2) = \langle a, b : pa = pb = 0, a^2 = a, b^2 = b, ab = a, ba = b\rangle$ *for a prime* p. *Let* S *be a subring of* R.

(*a*) *If* $|S| = p$, *then*

$$\left|E(\Gamma^r_{S,R})\right| = \begin{cases} p(p-1)^2, & \text{if } r = 0, \\ \frac{p(p-1)(2p-1)}{2}, & \text{if } r \neq 0 \text{ and } 2r = 0, \\ \frac{p(p-1)(2p-3)}{2}, & \text{if } r \neq 0 \text{ and } 2r \neq 0. \end{cases}$$

(*b*) *If* $S = R$, *then*

$$\left|E(\Gamma^r_{S,R})\right| = \begin{cases} \frac{p(p-1)^2(p+1)}{2}, & \text{if } r = 0, \\ \frac{p(p-1)^2(p+1)}{2}, & \text{if } r \neq 0 \text{ and } 2r = 0, \\ \frac{p(p-1)(p-2)(p+1)}{2}, & \text{if } r \neq 0 \text{ and } 2r \neq 0. \end{cases}$$

Proof. We have $[S,R] = \{ma + (p-m)b : 1 \leq m \leq p\}$ and $Z(S,R) = \{0\}$. Therefore, $\left|[S,R]\right| = p$ and $\left|Z(S,R)\right| = 1$. Hence, the result follows from Propositions 2 and 3 noting that $\left|Z(S)\right| = 1$ if $S = R$. □

The following two corollaries of Theorem 6 give certain lower bounds and upper bounds respectively for the number of edges in $\Gamma^r_{S,R}$, if $r \neq 0$.

Corollary 4. *Let* p *be the smallest prime dividing* $|R|$ *and* $r \neq 0$. *Then for a non-commutative subring* S *of* R *we have the following lower bounds for* $\left|E(\Gamma^r_{S,R})\right|$.

(*a*) *If* $2r = 0$, *then*

$$\left|E(\Gamma^r_{S,R})\right| \geq \begin{cases} \frac{2(p-1)|R||S|+2|R||Z(S,R)|-p|S|^2+6p|Z(S)|^2-p|S|)}{2p}, & \text{if } r \in S, \\ \frac{2(p-1)|R||S|+2|R||Z(S,R)|-p|S|^2-p|S|}{2p}, & \text{if } r \in R \setminus S. \end{cases}$$

(*b*) *If* $2r \neq 0$, *then*

$$\left|E(\Gamma^r_{S,R})\right| \geq \begin{cases} \frac{2(p-2)|R||S|+4|R||Z(S,R)|-p|S|^2+12p|Z(S)|^2-p|S|)}{2p}, & \text{if } r \in S, \\ \frac{2(p-2)|R||S|+4|R||Z(S,R)|-p|S|^2-p|S|}{2p}, & \text{if } r \in R \setminus S. \end{cases}$$

Proof. By [7, Proposition 3], we have

$$1 - \mathrm{Pr}_r(S, R) \geq \frac{(p-1)|S| + |Z(S,R)|}{p|S|} \tag{5}$$

and

$$1 - \sum_{u=r,-r} \mathrm{Pr}_u(S, R) \geq \frac{(p-2)|S| + 2|Z(S,R)|}{p|S|}. \tag{6}$$

By [8, Theorem 2.10], we have

$$\mathrm{Pr}_r(S) \geq \frac{6|Z(S)|^2}{|S|^2}. \tag{7}$$

(a) We have $2r = 0$. Therefore, if $r \in S$, then, using Theorem 6(b), (5) and (7), we get

$$2\left|E(\Gamma^r_{S,R})\right| + |S|^2 + |S| \geq 2|S||R| \left(\frac{(p-1)|S| + |Z(S,R)|}{p|S|}\right) + |S|^2 \left(\frac{6|Z(S)|^2}{|S|^2}\right).$$

Hence the result follows.

If $r \in R \setminus S$, then, using Theorem 6(b) and (5), we get

$$2\left|E(\Gamma^r_{S,R})\right| + |S|^2 + |S| \geq 2|S||R| \left(\frac{(p-1)|S| + |Z(S,R)|}{p|S|}\right).$$

Hence the result follows.

(b) We have $2r \neq 0$. Therefore, if $r \in S$, then, using Theorem 6(c), (6) and (7), we get

$$2\left|E(\Gamma^r_{S,R})\right| + |S|^2 + |S| \geq 2|S||R| \left(\frac{(p-2)|S| + 2|Z(S,R)|}{p|S|}\right) + |S|^2 \left(\frac{12|Z(S)|^2}{|S|^2}\right).$$

Hence the result follows.

If $r \in R \setminus S$, then, using Theorem 6(c) and (6), we get

$$2\left|E(\Gamma^r_{S,R})\right| + |S|^2 + |S| \geq 2|S||R| \left(\frac{(p-2)|S| + 2|Z(S,R)|}{p|S|}\right).$$

Hence the result follows. □

Corollary 5. *Let p be the smallest prime dividing $|R|$ and*

$$Z(R, S) = \{t \in R : ts = st \text{ for all } s \in S\}$$

for any non-commutative subring S of R. If $r \neq 0$, then we have the following upper bounds for $\left|E(\Gamma^r_{S,R})\right|$.

(a) *If $2r = 0$, then*

$$\left|E(\Gamma^r_{S,R})\right| \leq \begin{cases} \frac{2p|R||S| - 4p|Z(S,R)||Z(R,S)| - (p-1)|S|^2 - |S||Z(S)| - p|S|}{2p}, & \text{if } r \in S, \\ \frac{2|R||S| - 4|Z(S,R)||Z(R,S)| - |S|^2 - |S|}{2p}, & \text{if } r \in R \setminus S. \end{cases}$$

(b) *If $2r \neq 0$, then*

$$\left|E(\Gamma^r_{S,R})\right| \leq \begin{cases} \frac{2p|R||S| - 8p|Z(S,R)||Z(R,S)| - (p-2)|S|^2 - 2|S||Z(S)| - p|S|}{2p}, & \text{if } r \in S, \\ \frac{2|R||S| - 8|Z(S,R)||Z(R,S)| - |S|^2 - |S|}{2p}, & \text{if } r \in R \setminus S. \end{cases}$$

Proof. By [8, Theorem 2.9], we have

$$\Pr_r(S) \leq \frac{|S| - |Z(S)|}{p|S|}. \tag{8}$$

By [6, Proposition 3.1 (2)], we have

$$1 - \Pr_r(S, R) \leq \frac{|S||R| - 2|Z(S, R)||Z(R, S)|}{|S||R|} \tag{9}$$

and

$$1 - \sum_{u=r,-r} \Pr_u(S, R) \leq \frac{|S||R| - 4|Z(S, R)||Z(R, S)|}{|S||R|}. \tag{10}$$

(a) We have $2r = 0$. Therefore, if $r \in S$, then, using Theorem 6(b), (8) and (9), we get

$$2\left|E(\Gamma^r_{S,R})\right| + |S|^2 + |S| \leq 2|S||R| \left(\frac{|S||R| - 2|Z(S, R)||Z(R, S)|}{|S||R|} \right) + |S|^2 \left(\frac{|S| - |Z(S)|}{p|S|} \right).$$

Hence the result follows.

If $r \in R \setminus S$, then, using Theorem 6(b) and (9), we get

$$2\left|E(\Gamma^r_{S,R})\right| + |S|^2 + |S| \leq 2|S||R| \left(\frac{|S||R| - 2|Z(S, R)||Z(R, S)|}{|S||R|} \right).$$

Hence the result follows.

(b) We have $2r \neq 0$. Therefore, if $r \in S$, then, using Theorem 6(c), (8) and (10), we get

$$2\left|E(\Gamma^r_{S,R})\right| + |S|^2 + |S| \leq 2|S||R| \left(\frac{|S||R| - 4|Z(S, R)||Z(R, S)|}{|S||R|} \right) + 2|S|^2 \left(\frac{|S| - |Z(S)|}{p|S|} \right).$$

Hence the result follows.

If $r \in R \setminus S$, then, using Theorem 6(c) and (10), we get

$$2\left|E(\Gamma^r_{S,R})\right| + |S|^2 + |S| \leq 2|S||R| \left(\frac{|S||R| - 4|Z(S, R)||Z(R, S)|}{|S||R|} \right).$$

Hence the result follows. □

We conclude the section by noting that if $r = 0$, then using Theorem 6(a) and various bounds for $\Pr(S, R)$ and $\Pr(S)$ obtained in [3], we may derive several bounds for $\left|E(\Gamma^r_{S,R})\right|$.

5 An induced subgraph

In this section, we consider the induced subgraph $\Delta^r_{S,R}$ of $\Gamma^r_{S,R}$ with vertex set $R \setminus Z(S, R)$.

Theorem 7. *Let x be any vertex in $\Delta^r_{S,R}$.*

(a) *If $r = 0$, then* $\deg(x) = \begin{cases} |R| - |C_R(x)|, & \text{if } x \in S \setminus Z(S, R), \\ |S| - |C_S(x)|, & \text{if } x \in R \setminus S. \end{cases}$

(b) *If $r \neq 0$ and $2r = 0$, then*

$$\deg(x) = \begin{cases} |R| - |Z(S, R)| - |C^r_R(x)| - 1, & \text{if } x \in S \setminus Z(S, R), \\ |S| - |Z(S, R)| - |C^r_S(x)|, & \text{if } x \in R \setminus S. \end{cases}$$

(c) *If $r \neq 0$ and $2r \neq 0$, then*

$$\deg(x) = \begin{cases} |R| - |Z(S,R)| - 2|C_R^r(x)| - 1, & \text{if } x \in S \setminus Z(S,R), \\ |S| - |Z(S,R)| - 2|C_S^r(x)|, & \text{if } x \in R \setminus S. \end{cases}$$

Proof. Let x be a vertex in $\Delta_{S,R}^r$. If $x \in S \setminus Z(S,R)$, then $\deg(x)$ is the number of $y \in R \setminus Z(S,R)$ such that $xy \neq yx$. Hence, $\deg(x) = |R| - |Z(S,R)| - \big(|C_R(x)| - |Z(S,R)|\big) = |R| - |C_R(x)|$. If $x \in R \setminus S$, then $\deg(x)$ is the number of $s \in S \setminus Z(S,R)$ such that $sx \neq xs$. Hence, $\deg(x) = |S| - |Z(S,R)| - \big(|C_S(x)| - |Z(S,R)|\big) = |S| - |C_S(x)|$. Hence, part (a) follows.

The proofs of parts (b) and (c) follow from Theorem 1 (parts (b), (c)) noting that the vertex set of $\Delta_{S,R}^r$ is $R \setminus Z(S,R)$. □

By Lemma 1 and Theorem 7, we have the following two corollaries.

Corollary 6. *Let $x \in S$ be a vertex in $\Delta_{S,R}^r$.*

(a) *If $r \neq 0$ and $2r = 0$, then* $\deg(x) = \begin{cases} |R| - |Z(S,R)| - |C_R(x)| - 1, & \text{if } C_R^r(x) \neq \varnothing, \\ |R| - |Z(S,R)| - 1, & \text{otherwise.} \end{cases}$

(b) *If $r \neq 0$ and $2r \neq 0$, then* $\deg(x) = \begin{cases} |R| - |Z(S,R)| - 2|C_R(x)| - 1, & \text{if } C_R^r(x) \neq \varnothing, \\ |R| - |Z(S,R)| - 1, & \text{otherwise.} \end{cases}$

Corollary 7. *Let $x \in R \setminus S$ be a vertex in $\Delta_{S,R}^r$.*

(a) *If $r \neq 0$ and $2r = 0$, then* $\deg(x) = \begin{cases} |S| - |Z(S,R)| - |C_S(x)|, & \text{if } C_S^r(x) \neq \varnothing, \\ |S| - |Z(S,R)|, & \text{otherwise.} \end{cases}$

(b) *If $r \neq 0$ and $2r \neq 0$, then* $\deg(x) = \begin{cases} |S| - |Z(S,R)| - 2|C_S(x)|, & \text{if } C_S^r(x) \neq \varnothing, \\ |S| - |Z(S,R)|, & \text{otherwise.} \end{cases}$

Theorem 8. *Let S be a subring of a non-commutative ring R with unity 1 such that $|R| \neq 8$ and $1 \in S$. Then $\Delta_{S,R}^r$ is not a tree.*

Proof. Suppose that $\Delta_{S,R}^r$ is a tree. Therefore, there exists $x \in R \backslash Z(S,R)$ such that $\deg(x) = 1$.

Case 1: $r = 0$. If $x \in S \setminus Z(S,R)$, then, by Theorem 7(a), we have $\deg(x) = |R| - |C_R(x)| = 1$. Therefore, $|C_R(x)| = 1$, a contradiction.

If $x \in R \setminus S$, then, by Theorem 7(a), we also have $\deg(x) = |S| - |C_S(x)| = 1$. Therefore, $|C_S(x)| = 1$, a contradiction.

Case 2: $r \neq 0$ and $2r = 0$.

Subcase 2.1. Let $x \in S \setminus Z(S,R)$. Then, by Corollary 6(a), we have

$$\deg(x) = |R| - \big|Z(S,R)\big| - 1 = 1 \qquad \text{or} \qquad \deg(x) = |R| - \big|Z(S,R)\big| - |C_R(x)| - 1 = 1.$$

That is,

$$|R| - \big|Z(S,R)\big| = 2 \tag{11}$$

or

$$|R| - \big|Z(S,R)\big| - |C_R(x)| = 2. \tag{12}$$

Note that $Z(S,R)$ is a subring of R as well as $C_R(x)$ containing 0 and 1. Therefore, $|Z(S,R)|$ divides the left hand sides of (11) and (12). It follows that $|Z(S,R)| = 2$. Thus (11) gives $|R| = 4$, which is a contradiction since there is no non-commutative ring with unity having order 4. Again, (12) gives $|R| - |C_R(x)| = 4$ and so $|C_R(x)| = 4$. Therefore, $|R| = 8$, which contradicts our assumption.

Case 2.2. Let $x \in R \setminus S$. Then, by Corollary $7(a)$, we have $\deg(x) = |S| - |Z(S,R)| = 1$ or $\deg(x) = |S| - |Z(S,R)| - |C_S(x)| = 1$. Note that $Z(S,R)$ is a subring of S as well as $C_S(x)$ containing 0 and 1. Therefore, $|Z(S,R)|$ divides $|S| - |Z(S,R)|$ and $|S| - |Z(S,R)| - |C_S(x)|$. It follows that $|Z(S,R)| = 1$, a contradiction.

Case 3: $r \neq 0$ and $2r \neq 0$.

Subcase 3.1. Let $x \in S \setminus Z(S,R)$. Then, by Corollary $6(b)$, we have

$$\deg(x) = |R| - |Z(S,R)| - 1 = 1 \quad \text{or} \quad \deg(x) = |R| - |Z(S,R)| - 2\,|C_R(x)| - 1 = 1.$$

That is,

$$|R| - |Z(S,R)| = 2 \tag{13}$$

or

$$|R| - |Z(S,R)| - 2\,|C_R(x)| = 2. \tag{14}$$

Therefore, $|Z(S,R)| = 2$. Thus, (13) leads to the same contradiction that we get in the first part of Subcase 2.1. By (14), we have $|R| - 2\,|C_R(x)| = 4$ and so $|C_R(x)| = 4$. Therefore, $|R| = 12$ and so $\frac{R}{Z(S,R)}$ is cyclic. Hence, R is commutative, a contradiction.

Subcase 3.2. Let $x \in R \setminus S$. Then, by Corollary $7(b)$, we have $\deg(x) = |S| - |Z(S,R)| = 1$ or $\deg(x) = |S| - |Z(S,R)| - 2\,|C_S(x)| = 1$. Therefore, $|Z(S,R)| = 1$, a contradiction. □

The proof of Theorem 8 also tells that there is no vertex in the graph $\Delta^r_{S,R}$ having degree 1 if R is a non-commutative ring with unity 1 such that $|R| \neq 8$ and S is any subring of R with the same unity. We conclude this paper by obtaining conditions such that $\Delta^r_{S,R}$ has no vertex having degree 2.

Theorem 9. *Let S be a non-commutative subring of a ring R with unity 1 such that $1 \in S$. Then $\Delta^r_{S,R}$ has no vertex having degree 2 if $|R| \neq 12$ and $|S| \neq 8$.*

Proof. Suppose that $\Delta^r_{S,R}$ has a vertex x such that $\deg(x) = 2$.

Case 1: $r = 0$. If $x \in S \setminus Z(S,R)$, then, by Theorem $7(a)$, we have $\deg(x) = |R| - |C_R(x)| = 2$. Therefore, $|C_R(x)| = 2$, a contradiction.

If $x \in R \setminus S$, then, by Theorem $7(a)$, we also have $\deg(x) = |S| - |C_S(x)| = 2$. Therefore, $|C_S(x)| = 2$ and so $|S| = 4$, a contradiction since there is no non-commutative ring with unity having order 4.

Case 2: $r \neq 0$ and $2r = 0$.

Subcase 2.1. Let $x \in S \setminus Z(S,R)$. Then, by Corollary $6(a)$, we have

$$\deg(x) = |R| - |Z(S,R)| - 1 = 2 \quad \text{or} \quad \deg(x) = |R| - |Z(S,R)| - |C_R(x)| - 1 = 2.$$

That is,

$$|R| - |Z(S,R)| = 3 \tag{15}$$

or

$$|R| - |Z(S,R)| - |C_R(x)| = 3. \tag{16}$$

Therefore, $\left|Z(S,R)\right| = 3$. Thusm (15) gives $|R| = 6$, which is a contradiction since R is non-commutative. Again, (16) gives $|R| - |C_R(x)| = 6$ and so $|C_R(x)| = 6$ since $|C_R(x)| \neq 3$. Therefore $|R| = 12$, which contradicts our assumption.

Subcase 2.2. Let $x \in R \setminus S$. Then, by Corollary $7(a)$, we have $\deg(x) = |S| - \left|Z(S,R)\right| = 2$ or $\deg(x) = |S| - \left|Z(S,R)\right| - |C_S(x)| = 2$. Therefore, $\left|Z(S,R)\right| = 2$ and so $|S| - |C_S(x)| = 4$ since $|S| \neq 4$. We have $|C_S(x)| = 4$ since $|C_S(x)| \neq 2$. Therefore $|S| = 8$ which contradicts our assumption.

Case 3: $r \neq 0$ and $2r \neq 0$.

Subcase 3.1. Let $x \in S \setminus Z(S,R)$. Then, by Corollary $6(b)$, we have

$$\deg(x) = |R| - \left|Z(S,R)\right| - 1 = 2 \qquad \text{or} \qquad \deg(x) = |R| - \left|Z(S,R)\right| - 2\left|C_R(x)\right| - 1 = 2.$$

That is,

$$|R| - \left|Z(S,R)\right| = 3 \tag{17}$$

or

$$|R| - \left|Z(S,R)\right| - 2\left|C_R(x)\right| = 3. \tag{18}$$

Therefore, $\left|Z(S,R)\right| = 3$. Thus, (17) leads to the same contradiction that we get in the first part of Subcase 2.1. By (18), we have $|R| - 2\left|C_R(x)\right| = 6$ and so $|C_R(x)| = 6$ since $|C_R(x)| \neq 3$. Therefore, $|R| = 18$ and so $\frac{R}{Z(S,R)}$ is cyclic. Hence, R is commutative, a contradiction.

Subcase 3.2. Let $x \in R \setminus S$. Then, by Corollary $7(b)$, we have $\deg(x) = |S| - \left|Z(S,R)\right| = 2$ or $\deg(x) = |S| - \left|Z(S,R)\right| - 2\left|C_S(x)\right| = 2$. Therefore, $\left|Z(S,R)\right| = 2$ and so $|S| - 2\left|C_S(x)\right| = 4$ since $|S| \neq 4$. It follows that $|C_S(x)| = 4$. Thus $|S| = 12$ and so $\frac{S}{Z(S,R)}$ is cyclic. Hence, S is commutative, a contradiction. □

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Нехай S — підкільце скінченного кільця R та $r \in R$. Відносним r-некомутативним графом кільця R відносно S, що позначається як $\Gamma^r_{S,R}$, називають простий неорієнтований граф, множиною вершин якого є R, і дві вершини x та y є суміжними тоді і тільки тоді, коли $x \in S$ або $y \in S$ та $[x,y] \neq r$, $[x,y] \neq -r$. У цій статті ми визначаємо степінь будь-якої вершини в графі $\Gamma^r_{S,R}$ і характеризуємо всі скінченні кільця, для яких $\Gamma^r_{S,R}$ є зірковим, льодяниковим або регулярним графом. Ми встановлюємо зв'язки між відносними r-некомутативними графами двох ізоклінних пар кілець. Також виводимо певні співвідношення між кількістю ребер у графі $\Gamma^r_{S,R}$ та різними узагальненими ймовірностями комутативності для R. Нарешті, ми завершуємо статтю вивченням індукованого підграфа графа $\Gamma^r_{S,R}$.

Ключові слова і фрази: скінченне кільце, некомутативний граф, ізоклінізм.