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Some bilinear inequalities through a weighted Hardy operator

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In this paper, we give some new generalizations of the bilinear Hardy type inequality by using weighted mean operator $S := Sf_v$ and its dual $\widetilde{S} := \widetilde{S}f_v$, where f is a non-negative integrable function with two variables on $(0,+\infty) \times (0,+\infty)$ and v is a weight function.

Key words and phrases: Fubini theorem, Hölder's inequality, weight function, weighted Hardy operator.

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Introduction

The inequality

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^q dx \leq \left(\frac{q}{q-1}\right)^q \int_0^\infty f^q(x)dx, \tag{1}$$

where $F(x) = \int_0^x f(\tau)\,d\tau$, known as Hardy's inequality, holds for all non-negative and measurable functions f on $(0,\infty)$ with $q > 1$. The constant $\left(\frac{q}{q-1}\right)^q$ is the best possible. In 1928, G.H. Hardy [4] gave a generalization for (1). If $f \geq 0$, $q > 1$ and

$$F(x) = \begin{cases} \displaystyle\int_0^x f(\tau)\,d\tau, & \mu > 1, \\ \displaystyle\int_x^\infty f(\tau)\,d\tau, & \mu < 1, \end{cases}$$

then

$$\int_0^\infty x^{-\mu} F^q(x)\,dx \leq \left(\frac{q}{|\mu-1|}\right)^q \int_0^\infty x^{-\mu}\big(xf(x)\big)^q(x)\,dx. \tag{2}$$

The constant $\left(\frac{q}{|\mu-1|}\right)^q$ is the best possible.

Over the past three decades, the study of Hardy type inequalities has focused on the characterizations of weights, bilinear etc. These results are of interest and importance. Recently, some studies have been devoted to this inequality in dimension two. Hardy type inequalities for various integral operators in dimension two have been studied in [2,3,5–8] (see also references therein). The objective of this paper is to give some new generalizations of the classical bilinear Hardy inequality by using some elementary methods of analysis and Hardy operator.

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1 Main results

Throughout the paper, we will assume that functions f are non-negative integrable functions on Γ and the integrals everywhere are assumed to exist and are finite. At the beginning, we give some lemmas which will be used in proof of the main theorems.

Let $0 < a < b < +\infty$, $0 < c < d < +\infty$, $\Gamma = (0, +\infty) \times (0, +\infty)$ and

$$V(z) = \int_0^z v(s)\, ds \quad \text{for } z \in (0, +\infty).$$

Lemma 1. *Suppose that f is a non-negative integrable function on Γ and $q > 1$, $\mu > 1$. Let*

$$S(x,t) = \frac{1}{V(x)V(t)} \int_a^x \int_c^t v(\tau)v(s)f(\tau,s)\, ds\, d\tau,$$

and

$$\Phi(x,t) = \int_a^x v(\tau)f(\tau,t)\, d\tau.$$

Let us fix $x > a$. If $\eta \geq \frac{\mu-1}{q+\mu-1}$, then

$$\int_c^d \frac{v(t)}{V^m(t)} S^q(x,t)\, dt \leq \left(\frac{\eta\, q}{\mu - 1}\right)^q \int_c^d \frac{v(t)}{V^\mu(t)} \left(\frac{\Phi(x,t)}{V(x)}\right)^q dt. \tag{3}$$

Proof. Since x is fixed in (3) and using Fubini's Theorem, we get

$$S(x,t) = \frac{1}{V(t)} \int_c^t \frac{v(s)}{V(x)} \left(\int_a^x v(\tau)f(\tau,s)\, d\tau \right) ds,$$

therefore

$$S'(x,t) = \frac{\partial S(x,t)}{\partial t} = \frac{v(t)}{V(x)V(t)}\Phi(x,t) - \frac{v(t)}{V(t)}S(x,t).$$

Denote by $I(x)$ the left hand side of (3). Integrating it by parts, we obtain

$$\begin{aligned} I(x) = \left(- \frac{S^q(x,t)}{(\mu-1)V^{\mu-1}(t)} \right)\Bigg|_c^d &- \frac{q}{\mu-1} \int_c^d \frac{v(t)}{V^m(t)} S^q(x,t)\, dt \\ &+ \frac{q}{\mu-1} \int_c^d \frac{v(t)}{V(x)V^\mu(t)} \Phi(x,t) S^{q-1}(x,t)\, dt, \end{aligned}$$

this gives

$$\frac{q+\mu-1}{\mu-1} I(x) = \left(- \frac{S^q(x,d)}{(\mu-1)V^{\mu-1}(d)} \right) + \frac{q}{\mu-1} \int_c^d \frac{v(t)}{V^\mu(t)} \frac{\Phi(x,t)}{V(x)} S^{q-1}(x,t)\, dt.$$

Since $\mu > 1$ and $S(x,d) \geq 0$, we deduce

$$\frac{q+\mu-1}{\mu-1} I(x) \leq \frac{q}{\mu-1} \int_c^d \frac{v(t)}{V^\mu(t)} \frac{\Phi(x,t)}{V(x)} S^{q-1}(x,t)\, dt.$$

Using the assumption on η, we get

$$\frac{1}{\eta} I(x) \leq \frac{q}{\mu-1} \int_c^d \frac{v(t)}{V^\mu(t)} \frac{\Phi(x,t)}{V(x)} S^{q-1}(x,t)\, dt.$$

By Hölder's inequality for $\frac{1}{q}+\frac{1}{p}=1$, we obtain

$$I(x) \leq \frac{\eta\, q}{\mu-1}\left(\int_c^d \frac{v(t)}{V^{\mu}(t)}\left(\frac{\Phi(x,t)}{V(x)}\right)^q dt\right)^{\frac{1}{q}}\left(\int_c^d \frac{v(t)}{V^{\mu}(t)} S^q(x,t)\, dt\right)^{\frac{1}{p}}.$$

Thus, upon simplification, we get

$$\int_c^d \frac{v(t)}{V^{\mu}(t)} S^q(x,t)\, dt \leq \left(\frac{\eta\, q}{\mu-1}\right)^q \int_c^d \frac{v(t)}{V^{\mu}(t)}\left(\frac{\Phi(x,t)}{V(x)}\right)^q dt,$$

which completes the proof. □

Lemma 2. *Suppose f is a non-negative integrable function on Γ and $q>1$, $\mu>1$. Let*

$$\Phi(x,t)=\int_a^x v(\tau) f(\tau,t)\, d\tau.$$

Fix $t>c$. If $\eta \geq \frac{\mu-1}{q+\mu-1}$, then

$$\int_a^b \frac{v(x)}{V^{\mu+q}(x)} \Phi^q(x,t)\, dx \leq \left(\frac{\eta\, q}{\mu-1}\right)^q \int_a^b \frac{v(x)}{V^{\mu}(x)} f^q(x,t)\, dx. \tag{4}$$

Proof. Since t is fixed in (4), so

$$\Phi'(x,t)=\frac{\partial \Phi(x,t)}{\partial x}=v(x) f(x,t).$$

Denote by $I(t)$ the left hand side of (4). Integrating it by parts, we obtain

$$\begin{aligned} I(t) &= \left(-\frac{\Phi^q(x,t)}{(\mu+q-1)V^{\mu+q-1}(x)}\right)\Bigg|_a^b + \frac{q}{\mu+q-1}\int_a^b \frac{v(x)}{V^{\mu+q-1}(x)} f(x,t)\Phi^{q-1}(x,t)\, dx \\ &= \left(-\frac{\Phi^q(b,t)}{(\mu+q-1)V^{\mu+q-1}(b)}\right) + \frac{q}{\mu+q-1}\int_a^b \frac{v(x)}{V^{\mu+q}(x)}\big(V(x) f(x,t)\big)\Phi^{q-1}(x,t)\, dx. \end{aligned}$$

Since $\mu>1$ and $\Phi(b,t)\geq 0$, we get

$$I(y) \leq \frac{q}{\mu+q-1}\int_a^b \frac{v(x)}{V^{\mu+q}(x)}\big(V(x) f(x,t)\big)\Phi^{q-1}(x,t)\, dx.$$

Using the assumption on η, we deduce that

$$I(t) \leq \frac{\eta\, q}{\mu-1}\int_a^b \frac{v(x)}{V^{\mu+q}(x)}\big(V(x) f(x,t)\big)\Phi^{q-1}(x,t)\, dx.$$

Let us apply the Hölder inequality, it gives

$$I(y) \leq \frac{\eta\, q}{\mu-1}\left(\int_a^b \frac{v(x)}{V^{\mu+q}(x)}\big(V(x) f(x,t)\big)^q dx\right)^{\frac{1}{q}}\left(\int_a^b \frac{v(x)}{V^{\mu+q}(x)}\Phi^q(x,t)\, dx\right)^{\frac{1}{p}}.$$

By simplifying, we get

$$\int_a^b \frac{v(x)}{V^{\mu+q}(x)} \Phi^q(x,t)\, dx \leq \left(\frac{\eta\, q}{\mu-1}\right)^q \int_a^b \frac{v(x)}{V^{\mu}(x)} f^q(x,t)\, dx,$$

which completes the proof. □

Lemma 3. *Let f be a non-negative integrable function on Γ and $q > 1$, $\mu < 1$. Let*

$$\widetilde{S}(x,t) = \frac{1}{V(x)V(t)} \int_x^b \int_t^d v(\tau)v(s)f(\tau,s)\,ds\,d\tau,$$

and

$$\Psi(x,t) = \int_x^b v(\tau)f(\tau,t)\,d\tau.$$

Fix $x > a$. If $\eta \geq \frac{1-\mu}{1-\mu-q} > 0$, then

$$\int_c^d \frac{v(t)}{V^{\mu}(t)} \widetilde{S}^q(x,t)\,dt \leq \left(\frac{\eta\, q}{\mu - 1}\right)^q \int_c^d \frac{v(t)}{V^{\mu}(t)} \left(\frac{\Psi(x,t)}{V(x)}\right)^q dt. \tag{5}$$

Proof. Since x is fixed in (5) and using Fubini's Theorem, we have

$$\widetilde{S}(x,t) = \frac{1}{V(t)} \int_y^b \frac{v(s)}{V(x)} \left(\int_x^b v(\tau)f(\tau,s)d\tau\right) ds,$$

therefore

$$\widetilde{S}'(x,t) = \frac{\partial \widetilde{S}(x,t)}{\partial t} = -\frac{v(t)}{V(x)V(t)}\Psi(x,t) - \frac{v(t)}{V(t)}\widetilde{S}(x,t).$$

Denote by $J(x)$ the left hand side of (5). Integrating it by parts, we get

$$\begin{aligned} J(x) = \left(\frac{\widetilde{S}^q(x,t)}{(1-\mu)V^{\mu-1}(t)}\right)\Bigg|_c^d &+ \frac{q}{1-\mu} \int_c^d \frac{v(t)}{V^{\mu}(t)} \widetilde{S}^q(x,t)\,dt \\ &+ \frac{q}{1-\mu} \int_c^d \frac{v(t)}{V(x)V^{\mu}(t)} \Psi(x,t)\widetilde{S}^{q-1}(x,t)\,dt. \end{aligned}$$

Therefore,

$$\frac{1-\mu-q}{1-\mu} J(x) = \left(-\frac{\widetilde{S}^q(x,c)}{(1-\mu)V^{\mu-1}(c)}\right) + \frac{q}{1-\mu} \int_c^d \frac{v(t)}{V^{\mu}(t)} \frac{\Psi(x,t)}{V(x)} \widetilde{S}^{q-1}(x,t)\,dt.$$

Since $\mu < 1$ and $\widetilde{S}(x,c) \geq 0$, we deduce that

$$\frac{1-\mu-q}{1-\mu} J(x) \leq \frac{q}{1-\mu} \int_c^d \frac{v(t)}{V^{\mu}(t)} \frac{\Psi(x,t)}{V(x)} \widetilde{S}^{q-1}(x,t)\,dt.$$

Using the hypothesis on η, we get

$$\frac{1}{\eta} J(x) \leq \frac{q}{1-\mu} \int_c^d \frac{v(t)}{V^{\mu}(t)} \frac{\Psi(x,t)}{V(x)} \widetilde{S}^{q-1}(x,t)\,dt.$$

By applying the Hölder inequality, we obtain

$$J(x) \leq \frac{\eta\, q}{1-\mu} \left(\int_c^d \frac{v(t)}{V^{\mu}(t)} \left(\frac{\Psi(x,t)}{V(x)}\right)^q dt\right)^{\frac{1}{q}} \left(\int_c^d \frac{v(t)}{V^{\mu}(t)} \widetilde{S}^q(x,t)\,dt\right)^{\frac{1}{p}},$$

then

$$\int_c^d \frac{v(t)}{V^{\mu}(t)} \widetilde{S}^q(x,t)\,dt \leq \left(\frac{\eta\, q}{1-\mu}\right)^q \int_c^d \frac{v(t)}{V^{\mu}(t)} \left(\frac{\Psi(x,t)}{V(x)}\right)^q dt.$$

$\square$

Lemma 4. *Suppose f is a non-negative integrable function on Γ and $q>1$, $\mu<1$. Let*

$$\Psi(x,t)=\int_x^b w(\tau)f(\tau,t)\,d\tau.$$

Let $t>c$ be fixed. If $\eta \geq \frac{1-\mu}{1-\mu-q}>0$, then

$$\int_a^b \frac{v(x)}{V^{\mu+q}(x)}\Psi^q(x,t)\,dx \leq \left(\frac{\eta\, q}{1-\mu}\right)^q \int_a^b \frac{v(x)}{V^{\mu}(x)} f^q(x,t)\,dx. \tag{6}$$

Proof. Since t is fixed in (6), we get

$$\Psi'(x,t)=\frac{\partial \Psi(x,t)}{\partial x}=-v(x)f(x,t).$$

Denote by $J(t)$ the left hand side of (6). Integrating it by parts, we get

$$\begin{aligned} J(t) &= \left(\frac{\Psi^q(x,t)}{(1-\mu-q)V^{\mu+q-1}(x)}\right)\Bigg|_a^b + \frac{q}{1-\mu-q}\int_a^b \frac{v(x)}{V^{\mu+q-1}(x)} f(x,t)\Psi^{q-1}(x,t)\,dx \\ &= \left(-\frac{\Psi^q(a,t)}{(1-\mu-q)V^{\mu+q-1}(a)}\right) + \frac{q}{1-\mu-q}\int_a^b \frac{v(x)}{V^{\mu+q}(x)}\big(V(x)f(x,t)\big)\Psi^{q-1}(x,t)\,dx. \end{aligned}$$

Since $\mu<1$ and $\Psi(a,t)\geq 0$, we obtain

$$J(t) \leq \frac{q}{1-\mu-q}\int_a^b \frac{v(x)}{V^{\mu+q}(x)}\big(V(x)f(x,t)\big)\Psi^{q-1}(x,t)\,dx.$$

Using the assumption on η, we deduce

$$J(t) \leq \frac{\eta\, q}{1-\mu}\int_a^b \frac{v(x)}{V^{\mu+q}(x)}\big(V(x)f(x,t)\big)\Psi^{q-1}(x,t)\,dx.$$

It follows that

$$J(t) \leq \frac{\eta\, q}{1-\mu}\left(\int_a^b \frac{v(x)}{V^{\mu+q}(x)}\big(V(x)f(x,t)\big)^q\,dx\right)^{\frac{1}{q}}\left(\int_a^b \frac{v(x)}{V^{\mu+q}(x)}\Psi^q(x,t)\,dx\right)^{\frac{1}{p}}.$$

As a result of simplification, we get

$$\int_a^b \frac{v(x)}{V^{\mu+q}(x)}\Psi^q(x,t)\,dx \leq \left(\frac{\eta\, q}{1-\mu}\right)^q \int_a^b \frac{v(x)}{V^{\mu}(x)} f^q(x,t)\,dx.$$

□

We now present the principal results.

Theorem 1. *Suppose f is a non-negative integrable function on Γ and $q>1$, $\mu>1$. Let*

$$S(x,t)=\frac{1}{V(x)V(t)}\int_a^x\int_c^t v(\tau)v(s)f(\tau,s)\,ds\,d\tau.$$

If $\eta \geq \frac{\mu-1}{q+\mu-1}$, then

$$\int_a^b\int_c^d \frac{v(x)v(t)}{V^{\mu}(x)V^{\mu}(t)}S^q(x,t)\,dt\,dx \leq \left(\frac{\eta\, q}{\mu-1}\right)^{2q}\int_a^b\int_c^d \frac{v(x)v(t)}{V^{\mu}(x)V^{\mu}(t)}f^q(x,t)\,dt\,dx. \tag{7}$$

Proof. We denote by LHS the left-hand side of inequality (7). By using Lemma 1 and Lemma 2, we get

$$\begin{aligned} LHS &= \int_a^b \frac{v(x)}{V^{\mu}(x)} \left(\int_c^d \frac{v(t)}{V^{\mu}(t)} S^q(x,t)\, dt \right) dx \\ &\le \left(\frac{\eta\, q}{\mu-1} \right)^q \int_a^b \frac{v(x)}{V^{\mu}(x)} \left(\int_c^d \frac{v(t)}{V^{\mu}(t)} \left(\frac{\Phi(x,t)}{V(x)} \right)^q dt \right) dx \\ &= \left(\frac{\eta\, q}{\mu-1} \right)^q \int_c^d \frac{v(t)}{V^{\mu}(t)} \left(\int_a^b \frac{v(x)}{V^{\mu+q}(x)} \Phi^q(x,t)\, dx \right) dt \\ &\le \left(\frac{\eta\, q}{\mu-1} \right)^q \int_c^d \frac{v(t)}{V^{\mu}(t)} \left(\left(\frac{\eta\, q}{\mu-1} \right)^q \int_a^b \frac{v(x)}{V^{\mu}(x)} f^q(x,t)\, dx \right) dt \\ &= \left(\frac{\eta\, q}{\mu-1} \right)^{2q} \int_a^b \int_c^d \frac{v(x)v(t)}{V^{\mu}(x)V^{\mu}(t)} f^q(x,t)\, dt\, dx. \end{aligned}$$

□

Theorem 2. *Suppose f is a non-negative integrable function on Γ and $q > 1$, $\mu < 1$. Let*

$$\widetilde{S}(x,t) = \frac{1}{V(x)V(t)} \int_x^b \int_t^d v(\tau)v(s) f(\tau,s)\, ds\, d\tau.$$

If $\eta \ge \frac{1-\mu}{1-\mu-q} > 0$, then

$$\int_a^b \int_c^d \frac{v(x)v(t)}{V^{\mu}(x)V^{\mu}(t)} \widetilde{S}^q(x,t)\, dt\, dx \le \left(\frac{\eta\, q}{1-\mu} \right)^{2q} \int_a^b \int_c^d \frac{v(x)v(t)}{V^{\mu}(x)V^{\mu}(t)} f^q(x,t)\, dt\, dx.$$

Proof. Applying Lemma 3 and Lemma 4, the proof is similar to Theorem 1. □

2 Applications

In this section, we present some special cases according to the choice of the weight function v, the parameter η and the function $f(x,t)$.

2.1 Bilinear weighted Hardy integral inequality

Taking $v(x) = 1$ in Theorem 1 and Theorem 2, we get the following corollaries.

Corollary 1. *Let $q > 1$, $\mu > 1$ and f be a non-negative integrable function on Γ. Let*

$$F(x,t) = \frac{1}{x\, t} \int_a^x \int_c^t f(\tau,s)\, ds\, d\tau.$$

If $\eta \ge \frac{\mu-1}{q+\mu-1}$, then

$$\int_a^b \int_c^d (x\, t)^{-\mu}\, F^q(x,t)\, dt\, dx \le \left(\frac{\eta\, q}{\mu-1} \right)^{2q} \int_a^b \int_c^d (x\, t)^{-\mu}\, f^q(x,t)\, dt\, dx.$$

Corollary 2. *Let $q > 1$, $\mu < 1$ and f be a non-negative integrable function on Γ. Let*

$$\widetilde{F}(x,t) = \frac{1}{x\, t} \int_x^b \int_t^d f(\tau,s)\, ds\, d\tau.$$

If $\eta \ge \frac{1-\mu}{1-\mu-q} > 0$, then

$$\int_a^b \int_c^d (x\, t)^{-\mu}\, \widetilde{F}^q(x,t)\, dt\, dx \le \left(\frac{\eta\, q}{1-\mu} \right)^{2q} \int_a^b \int_c^d (x\, t)^{-\mu}\, f^q(x,t)\, dt\, dx.$$

Remark 1. *By setting $\eta = 1$ in the above Corollary 1 and Corollary 2, Hardy type integral bilinear inequality is obtained.*

2.2 Function with two independent variables

Taking $f(x,y) = f_1(x)f_2(t)$ where f_1, f_2 are non-negative integrable functions on $(0,\infty)$ in the Corollary 1 and the Corollary 2, we deduce the following results.

Corollary 3. *Let* $q > 1$, $\mu > 1$ *and*

$$F(x,t) = \left(\frac{1}{x}\int_a^x f_1(\tau)\,d\tau\right)\left(\frac{1}{t}\int_c^t f_2(s)\,ds\right).$$

If $\eta \geq \frac{\mu-1}{q+\mu-1}$, *then*

$$\int_a^b\int_c^d (x\,t)^{-\mu}\,F^q(x,t)\,dt\,dx \leq \left(\frac{\eta\,q}{\mu-1}\right)^{2q}\left(\int_a^b x^{-\mu}f_1^q(x)\,dx\right)\left(\int_c^d t^{-\mu}f_2^q(t)\,dt\right).$$

Corollary 4. *Let* $q > 1$, $\mu < 1$ *and*

$$\widetilde{F}(x,t) = \left(\frac{1}{x}\int_x^b f_1(\tau)\,d\tau\right)\left(\frac{1}{t}\int_t^d f_2(s)\,ds\right).$$

If $\eta \geq \frac{1-\mu}{1-\mu-q} > 0$, *then*

$$\int_a^b\int_c^d (x\,t)^{-\mu}\,\widetilde{F}^q(x,t)\,dt\,dx \leq \left(\frac{\eta\,q}{1-\mu}\right)^{2q}\left(\int_a^b x^{-\mu}f_1^q(x)\,dx\right)\left(\int_c^d t^{-\mu}f_2^q(t)\,dt\right).$$

2.3 Analogues of the initial inequality (2)

Putting now $f_1 = f_2$, $a = c$, $b = d$ in the Corollary 3 and the Corollary 4, we get the following results.

Corollary 5. *Let* $q > 1$, $\mu > 1$ *and* f *be a non-negative integrable function on* $(0,\infty)$. *Let*

$$F(x) = \frac{1}{x}\int_a^x f(\tau)\,d\tau.$$

If $\eta \geq \frac{\mu-1}{q+\mu-1}$, *then*

$$\int_a^b x^{-\mu}F^q(x)\,dx \leq \left(\frac{\eta\,q}{\mu-1}\right)^q\int_a^b x^{-\mu}f^q(x)\,dx.$$

Corollary 6. *Let* $q > 1$, $\mu < 1$ *and* f *be a non-negative integrable function on* $(0,\infty)$. *Let*

$$\widetilde{F}(x) = \frac{1}{x}\int_x^b f(\tau)\,d\tau.$$

If $\eta \geq \frac{1-\mu}{1-\mu-q} > 0$, *then*

$$\int_a^b x^{-m}\widetilde{F}^q(x)\,dx \leq \left(\frac{\eta\,q}{1-\mu}\right)^q\int_a^b x^{-\mu}f^q(x)\,dx.$$

Remark 2. *The Corollaries 5 and 6 are similar to Corollaries 1 and 2 obtained in [1], respectively.*

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У цій статті ми подаємо нові узагальнення білінійнійної нерівності типу Харді, використовуючи зважений оператор середнього $S := Sf_v$ та його спряжений $\widetilde{S} := \widetilde{S}f_v$, де f — невід'ємна інтегровна функція двох змінних, визначена на $(0,+\infty) \times (0,+\infty)$, а v — вагова функція.

Ключові слова і фрази: теорема Фубіні, нерівність Гельдера, вагова функція, зважений оператор Харді.