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Some new Simpson type integral inequalities for (s, P)-functions

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In this paper, we establish some new Simpson type inequalities for functions, whose first derivative in absolute value is (s, P)-function by using Hölder, power-mean and Hölder-İşcan inequalities. After that, we compare the results obtained with both Hölder, Hölder-İşcan integral inequalities and prove that the Hölder-İşcan integral inequality gives a better approximation, than the Hölder inequality. Also, some applications to special means of real numbers are also given.

Key words and phrases: (s, P)-function, Simpson type inequality, Hermite-Hadamard inequality, Hölder-İşcan inequality.

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1 Preliminaries and fundamentals

Let $I \subset \mathbb{R}$ be an interval. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f\left(tx + (1-t)y\right) \le tf\left(x\right) + (1-t)\, f\left(y\right)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If the above inequality reverses, then the function f is said to be concave on interval $I \neq \varnothing$. Convexity theory provides important powerful principles and techniques to study a wide class of problems in both pure and applied mathematics and the other sciences. Let $f : I \to \mathbb{R}$ be a convex function. Then the following chain of integral inequalities hold

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f\left(a\right)+f(b)}{2}$$

for all $a, b \in I$ with $a < b$. Both inequalities hold in the reversed direction if f is concave. This double integral inequality is well known as the Hermite-Hadamard integral inequality, and it is one of the most studied result in inequalities via convex functions. The Hermite-Hadamard integral inequality provides a necessary and sufficient condition for a function to be convex. Additionally, readers can refer to [6–10, 14, 16] to obtain more information on convexity.

Suppose $f : [a, b] \to \mathbb{R}$ is a four times continuously differentiable mapping on (a, b) and $\|f^{(4)}\|_\infty := \sup_{x\in(a,b)} |f^{(4)}(x)| < \infty$. The following inequality

$$\left|\frac{1}{3}\left[\frac{f\left(a\right)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \le \frac{1}{2880}\|f^{(4)}\|_\infty (b-a)^4 \qquad (1)$$

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holds. The above inequality is well known in the literature as Simpson's inequality. In recent years, many authors and researchers have studied error estimations for the Simpson's inequality. For refinements, counterparts, generalizations and new Simpson-type inequalities, for some related papers see [2–5,9,11,19–21]. Some of the important results about the Simpson type integral inequality are presented below.

In [5], S.S. Dragomir et. al. pointed out some recent developments on Simpson's inequality for which the remainder is expressed in terms of lower derivatives than the fourth.

Theorem 1 ([5]). *Suppose $f:[a,b]\to\mathbb{R}$ is a differentiable mapping whose derivative is continuous on (a,b) and $f'\in L[a,b]$. Then the following inequality*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{b-a}{3}\left\|f'\right\|_1$$

holds, where $\|f'\|_1=\int_a^b|f'(x)|\,dx$.

Theorem 2 ([5]). *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping on $[a,b]$ whose derivative belongs to $L_p[a,b]$. Then the inequality*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right](b-a)^{\frac{1}{q}}\left\|f'\right\|_p$$

holds, where $\frac{1}{p}+\frac{1}{q}=1$, $p>1$.

In [8], S.S. Dragomir et. al. gave the following definition and related Hermite-Hadamard integral inequalities as follow.

Definition 1 ([8]). *A nonnegative function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is said to be P-function if the inequality*

$$f(tx+(1-t)y)\leq f(x)+f(y)$$

holds for all $x,y\in I$ and $t\in(0,1)$.

Theorem 3 ([8]). *Let $f\in P(I)$, $a,b\in I$ with $a<b$ and $f\in L[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right)\leq\frac{2}{b-a}\int_a^b f(x)\,dx\leq 2\left[f(a)+f(b)\right].$$

In [18], S. Numan and İ. İşcan introduced and studied the concept of (s,P)-function and established Hermite-Hadamard integral inequalities for this type of functions. Also, the authors obtained some new Hermite-Hadamard type inequalities for functions whose first derivative in absolute value is (s,P)-function by using Hölder and power-mean integral inequalities. Next, the authors pointed out some applications of the results to give estimates for the approximation error of the integral function in the trapezoidal formula and for some inequalities related to special means of real numbers. S. Numan and İ. İşcan gave the following definition and Hermite-Hadamard integral inequality for the (s,P)-functions.

Definition 2 ([18]). *Let $s\in(0,1]$. A function $f:I\subset\mathbb{R}\to\mathbb{R}$ is called (s,P)-function if*

$$f(tx+(1-t)y)\leq\left(t^s+(1-t)^s\right)\left[f(x)+f(y)\right]$$

for every $x,y\in I$ and $t\in[0,1]$.

Let us denote by $P_s(I)$ the class of all (s, P)-functions on interval I. Clearly, the definition of $(1, P)$-function is coincide with the definition of P-function.

Note that every (s, P)-function is a h-convex function with the function $h(t) = t^s + (1-t)^s$.

Theorem 4 ([18])**.** *Let $s \in (0, 1]$ and $f : [a, b] \to \mathbb{R}$ be a (s, P)-function. If $a < b$ and $f \in L[a, b]$, then the following Hermite-Hadamard type inequalities*

$$2^{s-2} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{2}{s+1}\left[f(a) + f(b)\right]$$

hold.

In recent years, many function classes such as P-function, geometric P-function, n-polynomial P-function, semi P-geometric-arithmetically functions, logarithmic semi P-function, semi harmonically P-function, semi P-geometric-arithmetically function, etc. have been studied by many authors, and integral inequalities belonging to these function classes have been studied in the literature (see [8, 13–17]).

Theorem 5 (Hölder-İşcan integral inequality [12])**.** *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on interval $[a, b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a, b]$, then*

$$\begin{aligned}\int_a^b |f(x)g(x)|\,dx \;\le\; & \frac{1}{b-a}\Bigg\{\left(\int_a^b (b-x)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)\,|g(x)|^q\,dx\right)^{\frac{1}{q}} \\ & + \left(\int_a^b (x-a)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\Bigg\}.\end{aligned}$$

The main purpose of this manuscript is to establish some new Simpson type inequality for (s, P)-functions. The most important feature of this study is that the results obtained with both Hölder and Hölder-İşcan integral inequalities are compared and it is shown that the Hölder-İşcan integral inequality provides a better approximation than the Hölder inequality.

2 Some new Simpson type inequalities for (s, P)-functions

In this section we establish new estimates, that refine Simpson type integral inequality for functions, whose first derivative in absolute value is (s, P)-function. Also, we compare the results, obtained with both Hölder, Hölder-İşcan integral inequalities, and prove that the Hölder-İşcan integral inequality gives a better approximation than the Hölder integral inequality. M. Alomari et. al. [1] used the following lemma.

Lemma 1 ([1])**.** *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I°, where $a, b \in I^\circ$ with $a < b$. Then the following equality*

$$\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx = (b-a)\int_0^1 p(t) f'\left(tb + (1-t)a\right)dt$$

holds, where

$$p(t) = \begin{cases} t - \frac{1}{6}, & t \in \left[0, \frac{1}{2}\right), \\ t - \frac{5}{6}, & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Theorem 6. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a,b \in I^\circ$ with $a < b$. If $|f'|$ is a (s,P)-function on interval $[a,b]$, then the following inequality*

$$\begin{aligned}
&\left|\frac{1}{6}\Big[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\qquad\le 4(b-a)A\left(\left|f'(a)\right|,\left|f'(b)\right|\right)\left[\frac{2\left(5^{s+2}+1\right)+(s-4)6^{s+1}-6.3^{s+1}}{6^{s+2}(s+1)(s+2)}\right]
\end{aligned} \tag{2}$$

holds, where $A(a,b)=\frac{a+b}{2}$ is the arithmetic mean.

Proof. Using Lemma 1 and the following inequality

$$\left|f'(tb+(1-t)a)\right| \le (t^s+(1-t)^s)\left[\left|f'(a)\right|+\left|f'(b)\right|\right],$$

we obtain

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\le (b-a)\left|\int_0^1 p(t)f'(tb+(1-t)a)\,dt\right| \\
&\le (b-a)\int_0^{\frac12}\left|t-\frac16\right|\left|f'(tb+(1-t)a)\right|dt \\
&\quad +(b-a)\int_{\frac12}^1\left|t-\frac56\right|\left|f'(tb+(1-t)a)\right|dt \\
&\le (b-a)\int_0^{\frac12}\left|t-\frac16\right|(t^s+(1-t)^s)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]dt \\
&\quad +(b-a)\int_{\frac12}^1\left|t-\frac56\right|(t^s+(1-t)^s)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]dt \\
&= (b-a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\int_0^{\frac12}\left|t-\frac16\right|(t^s+(1-t)^s)\,dt \\
&\quad +(b-a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\int_{\frac12}^1\left|t-\frac56\right|(t^s+(1-t)^s)\,dt \\
&= (b-a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\int_0^{\frac16}\left(\frac16-t\right)(t^s+(1-t)^s)\,dt \\
&\quad +(b-a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\int_{\frac16}^{\frac12}\left(t-\frac16\right)(t^s+(1-t)^s)\,dt \\
&\quad +(b-a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\int_{\frac12}^{\frac56}\left(\frac56-t\right)(t^s+(1-t)^s)\,dt \\
&\quad +(b-a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\int_{\frac56}^1\left(t-\frac56\right)(t^s+(1-t)^s)\,dt \\
&= (b-a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\left[\frac{5^{s+2}+(s-4)6^{s+1}+1}{6^{s+2}(s+1)(s+2)}+\frac{2^s5^{s+2}-3.6^{s+1}+2^s}{2^s6^{s+2}(s+1)(s+2)}\right] \\
&\quad +(b-a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\left[\frac{2^s5^{s+2}-3.6^{s+2}+2^s}{2^s6^{s+2}(s+1)(s+2)}+\frac{5^{s+2}+(s-4)6^{s+1}+1}{6^{s+2}(s+1)(s+2)}\right] \\
&= 4(b-a)A\left(\left|f'(a)\right|,\left|f'(b)\right|\right)\left[\frac{2\left(5^{s+2}+1\right)+\left[(s-4)2^s-3\right]3^{s+1}}{6^{s+2}(s+1)(s+2)}\right],
\end{aligned}$$

where

$$\int_0^{\frac{1}{6}}\left(\frac{1}{6}-t\right)(t^s+(1-t)^s)\,dt=\frac{5^{s+2}+(s-4)6^{s+1}+1}{6^{s+2}(s+1)(s+2)},$$
$$\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)(t^s+(1-t)^s)\,dt=\frac{2^s5^{s+2}-3.6^{s+1}+2^s}{2^s6^{s+2}(s+1)(s+2)},$$
$$\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)(t^s+(1-t)^s)\,dt=\frac{2^s5^{s+2}-3.6^{s+2}+2^s}{2^s6^{s+2}(s+1)(s+2)},$$
$$\int_{\frac{5}{6}}^{1}\left(t-\frac{5}{6}\right)(t^s+(1-t)^s)\,dt=\frac{5^{s+2}+(s-4)6^{s+1}+1}{6^{s+2}(s+1)(s+2)}.$$

This completes the proof of the theorem. □

Therefore, for $s=1$, we can deduce the following result for P-functions.

Corollary 1. *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|$ is a P-function on interval $[a,b]$, then the following inequality*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{5}{216}(b-a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]$$

holds.

Corollary 2. *In Theorem 6, if $f(a)=f\left(\frac{a+b}{2}\right)=f(b)$, then we get*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq 4(b-a)A\left(\left|f'(a)\right|,\left|f'(b)\right|\right)\left[\frac{2^{s+1}5^{s+2}+\left((s-4)2^s-3\right)6^{s+1}+2^{s+1}}{2^s6^{s+2}(s+1)(s+2)}\right].$$

Corollary 3. *In Corollary 2, by setting $s=1$, we obtain*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{11}{36}(b-a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

Theorem 7. *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$. If $|f'|^q$ is a (s,P)-function on interval $[a,b]$ for some fixed $q>1$, then the following inequality*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq 2(b-a)A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} \tag{3}$$

holds, where $\frac{1}{p}+\frac{1}{q}=1$ and $A(\cdot,\cdot)$ is the arithmetic mean.

Proof. Using Lemma 1, well known Hölder's integral inequality and the following inequality

$$\left|f'(tb+(1-t)a)\right|^q\leq(t^s+(1-t)^s)\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right],$$

which is the properties of (s,P)-function of $|f'|^q$, we get

$$
\begin{aligned}
&\left|\frac{1}{6}\Big[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\qquad\le (b-a)\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left|f'\left(tb+(1-t)a\right)\right|dt+(b-a)\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left|f'\left(tb+(1-t)a\right)\right|dt \\
&\qquad\le (b-a)\bigg(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^p dt\bigg)^{\frac{1}{p}}\bigg(\int_0^{\frac{1}{2}}\left|f'\left(tb+(1-t)a\right)\right|^q dt\bigg)^{\frac{1}{q}} \\
&\qquad\quad+(b-a)\bigg(\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|^p dt\bigg)^{\frac{1}{p}}\bigg(\int_{\frac{1}{2}}^1\left|f'\left(tb+(1-t)a\right)\right|^q dt\bigg)^{\frac{1}{q}} \\
&\qquad= (b-a)\bigg[\int_0^{\frac{1}{6}}\Big(\frac{1}{6}-t\Big)^p dt+\int_{\frac{1}{6}}^{\frac{1}{2}}\Big(t-\frac{1}{6}\Big)^p dt\bigg]^{\frac{1}{p}}\bigg(\int_0^{\frac{1}{2}}\left|f'\left(tb+(1-t)a\right)\right|^q dt\bigg)^{\frac{1}{q}} \\
&\qquad\quad+(b-a)\bigg[\int_{\frac{1}{2}}^{\frac{5}{6}}\Big(\frac{5}{6}-t\Big)^p dt+\int_{\frac{5}{6}}^{1}\Big(t-\frac{5}{6}\Big)^p dt\bigg]^{\frac{1}{p}}\bigg(\int_{\frac{1}{2}}^1\left|f'\left(tb+(1-t)a\right)\right|^q dt\bigg)^{\frac{1}{q}} \\
&\qquad= (b-a)\left[\frac{1}{6^{p+1}(p+1)}+\frac{1}{3^{p+1}(p+1)}\right]^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'\left(tb+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\qquad\quad+(b-a)\left[\frac{1}{3^{p+1}(p+1)}+\frac{1}{6^{p+1}(p+1)}\right]^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'\left(tb+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\qquad= 2(b-a)A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}},
\end{aligned}
$$

where

$$
\begin{aligned}
&\int_0^{\frac{1}{6}}\Big(\frac{1}{6}-t\Big)^p dt=\int_{\frac{5}{6}}^1\Big(t-\frac{5}{6}\Big)^p dt=\frac{1}{6^{p+1}(p+1)}, \\
&\int_{\frac{1}{6}}^{\frac{1}{2}}\Big(t-\frac{1}{6}\Big)^p dt=\int_{\frac{1}{2}}^{\frac{5}{6}}\Big(\frac{5}{6}-t\Big)^p dt=\frac{1}{3^{p+1}(p+1)}, \\
&\int_0^{\frac{1}{2}}\left|f'\left(tb+(1-t)a\right)\right|^q dt=\int_{\frac{1}{2}}^1\left|f'\left(tb+(1-t)a\right)\right|^q dt=\frac{\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]}{s+1}.
\end{aligned}
$$

This completes the proof of the theorem. □

Therefore, for $s=1$, we can write the following result for P-functions.

Corollary 4. *Let $f:I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|$ is a P-function on interval $[a,b]$, then the following inequality*

$$
\begin{aligned}
&\left|\frac{1}{6}\Big[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\qquad\le 2(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)
\end{aligned}
$$

holds, where $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 5. *In Theorem 7, if we take $f(a)=f\left(\frac{a+b}{2}\right)=f(b)$, then we get*

$$\begin{aligned}\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq 2(b-a)\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right).\end{aligned} \tag{4}$$

Corollary 6. *Under the conditions of Corollary 5, if we take $s=1$ in 4, then we get the following inequality*

$$\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq 2(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right).$$

Theorem 8. *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$. If $|f'|^q$ is a (s,P)-function on interval $[a,b]$ for some fixed $q\geq 1$, then the following inequality holds*

$$\begin{aligned}\left|\frac{1}{6}\Big[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq (b-a)\left(\frac{5}{36}\right)^{1-\frac{1}{q}} A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\left(\frac{(2^s+1)\left(5^{s+2}+1\right)+(s-7)6^{s+1}}{6^{s+2}(s+1)(s+2)}\right)^{\frac{1}{q}}.\end{aligned} \tag{5}$$

Proof. From Lemma 1, well known power-mean integral inequality and the property of the (s,P)-function of $|f'|^q$, we obtain

$$\begin{aligned}&\left|\frac{1}{6}\Big[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq (b-a)\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left|f'(tb+(1-t)a)\right|dt+(b-a)\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left|f'(tb+(1-t)a)\right|dt \\
&\leq (b-a)\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad+(b-a)\left(\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
&=(b-a)\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left(t^s+(1-t)^s\right)\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&\quad+(b-a)\left(\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left(t^s+(1-t)^s\right)\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&=(b-a)\left(\frac{5}{36}\right)^{1-\frac{1}{q}} A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\left(\frac{2\left(5^{s+2}+1\right)+\left[(s-4)2^s-3\right]3^{s+1}}{6^{s+2}(s+1)(s+2)}\right)^{\frac{1}{q}},\end{aligned}$$

where

$$\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|dt=\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|dt=\frac{5}{72},$$

$$\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left(t^s+(1-t)^s\right)dt=\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left(t^s+(1-t)^s\right)dt=\frac{2(5^{s+2}+1)+[(s-4)2^s-3]3^{s+1}}{6^{s+2}(s+1)(s+2)}.$$

This completes the proof of the theorem. □

Corollary 7. *Under the assumption of Theorem 8, if we take $q=1$ in the inequality* (5)*, then we get the following inequality*

$$\begin{aligned}&\left|\frac{1}{6}\Big[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\right|\\&\qquad\leq (b-a)A\left(\left|f'(a)\right|,\left|f'(b)\right|\right)\left(\frac{2\left(5^{s+2}+1\right)+(s-4)6^{s+1}-6.3^{s+1}}{6^{s+2}(s+1)(s+2)}\right).\end{aligned}$$

This inequality coincides with the inequality (2).

Theorem 9. *Let $f:I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$. If $|f'|^q$ is a (s,P)-function on interval $[a,b]$ for some fixed $q>1$, then the following inequality*

$$\begin{aligned}\left|\frac{1}{6}\Big[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\right|&\leq 4(b-a)A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\\&\times\Bigg[\left(\frac{2^{p+2}+3p+5}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{s2^s+1}{2^s(s+1)(s+2)}\right)^{\frac{1}{q}}\\&+\left(\frac{(3p+4)\,2^{p+1}+1}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{2^{s+1}-1}{2^s(s+1)(s+2)}\right)^{\frac{1}{q}}\Bigg]\end{aligned}\tag{6}$$

holds, where $\frac{1}{p}+\frac{1}{q}=1$ and $A(\cdot,\cdot)$ is the arithmetic mean.

Proof. Using Lemma 1, Hölder-İşcan integral inequality and the following inequality

$$\left|f'(tb+(1-t)a)\right|^q\leq\left(t^s+(1-t)^s\right)\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right],$$

we have

$$\begin{aligned}&\left|\frac{1}{6}\Big[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\right|\\&\quad\leq(b-a)\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left|f'(tb+(1-t)a)\right|dt+(b-a)\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left|f'(tb+(1-t)a)\right|dt\\&\quad\leq 2(b-a)\left(\int_0^{\frac{1}{2}}\Big(\frac{1}{2}-t\Big)\left|t-\frac{1}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\Big(\frac{1}{2}-t\Big)\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\\&\qquad+2(b-a)\left(\int_0^{\frac{1}{2}}t\left|t-\frac{1}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}t\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\\&\qquad+2(b-a)\left(\int_{\frac{1}{2}}^1(1-t)\left|t-\frac{5}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1(1-t)\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\\&\qquad+2(b-a)\left(\int_{\frac{1}{2}}^1\Big(t-\frac{1}{2}\Big)\left|t-\frac{5}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\Big(t-\frac{1}{2}\Big)\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\end{aligned}$$

$$
\begin{aligned}
&= 2(b-a)A^{\frac{1}{q}}\left(\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right)\left(\frac{2^{p+2}+3p+5}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{s2^{s}+1}{2^{s}(s+1)(s+2)}\right)^{\frac{1}{q}} \\
&\quad +2(b-a)A^{\frac{1}{q}}\left(\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right)\left(\frac{(3p+4)\,2^{p+1}+1}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{2^{s+1}-1}{2^{s}(s+1)(s+2)}\right)^{\frac{1}{q}} \\
&\quad +2(b-a)A^{\frac{1}{q}}\left(\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right)\left(\frac{(3p+4)\,2^{p+1}+1}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{2^{s+1}-1}{2^{s}(s+1)(s+2)}\right)^{\frac{1}{q}} \\
&\quad +2(b-a)A^{\frac{1}{q}}\left(\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right)\left(\frac{2^{p+2}+3p+5}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{s2^{s}+1}{2^{s}(s+1)(s+2)}\right)^{\frac{1}{q}} \\
&= 4(b-a)A^{\frac{1}{q}}\left(\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right)\left[\left(\frac{2^{p+2}+3p+5}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{s2^{s}+1}{2^{s}(s+1)(s+2)}\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\frac{(3p+4)\,2^{p+1}+1}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{2^{s+1}-1}{2^{s}(s+1)(s+2)}\right)^{\frac{1}{q}}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
&\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)\left|t-\frac{1}{6}\right|^{p}dt=\int_{\frac{1}{2}}^{1}\left(t-\frac{1}{2}\right)\left|t-\frac{5}{6}\right|^{p}dt=\frac{2^{p+2}+3p+5}{6^{p+2}(p+1)(p+2)}, \\
&\int_0^{\frac{1}{2}}t\left|t-\frac{1}{6}\right|^{p}dt=\int_{\frac{1}{2}}^{1}(1-t)\left|t-\frac{5}{6}\right|^{p}dt=\frac{(3p+4)\,2^{p+1}+1}{6^{p+2}(p+1)(p+2)}, \\
&\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)\left|f'(tb+(1-t)a)\right|^{q}dt=\frac{s2^{s}+1}{2^{s+1}(s+1)(s+2)}\left[\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}\right], \\
&\int_0^{\frac{1}{2}}t\left|f'(tb+(1-t)a)\right|^{q}dt=\frac{2^{s+1}-1}{2^{s+1}(s+1)(s+2)}\left[\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}\right], \\
&\int_{\frac{1}{2}}^{1}(1-t)\left|f'(tb+(1-t)a)\right|^{q}dt=\frac{2^{s+1}-1}{2^{s+1}(s+1)(s+2)}\left[\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}\right], \\
&\int_{\frac{1}{2}}^{1}\left(t-\frac{1}{2}\right)\left|f'(tb+(1-t)a)\right|^{q}dt=\frac{s2^{s}+1}{2^{s+1}(s+1)(s+2)}\left[\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}\right].
\end{aligned}
$$

This completes the proof of the theorem. □

Remark 1. *The inequality* (6) *gives better result than the inequality* (3)*. Let us show that*

$$
\begin{aligned}
&2\left[\left(\frac{2^{p+2}+3p+5}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{s2^{s}+1}{2^{s}(s+1)(s+2)}\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\frac{(3p+4)\,2^{p+1}+1}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{2^{s+1}-1}{2^{s}(s+1)(s+2)}\right)^{\frac{1}{q}}\right] \\
&\qquad \leq 2^{\frac{1}{q}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}.
\end{aligned}
$$

Using the well known classic inequality $x^{\frac{1}{p}}y^{\frac{1}{q}}+z^{\frac{1}{p}}w^{\frac{1}{q}}\leq(x+z)^{\frac{1}{p}}(y+w)^{\frac{1}{q}}$*,* $x,y,z,w\in(0,\infty)$*,* *by sample calculations we get*

$$2\left[\left(\frac{2^{p+2}+3p+5}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{s2^s+1}{2^s(s+1)(s+2)}\right)^{\frac{1}{q}}+\left(\frac{(3p+4)\,2^{p+1}+1}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{2^{s+1}-1}{2^s(s+1)(s+2)}\right)^{\frac{1}{q}}\right]$$

$$\leq 2\left(\frac{2^{p+2}+3p+5}{6^{p+2}(p+1)(p+2)}+\frac{(3p+4)\,2^{p+1}+1}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}+\left(\frac{s2^s+1}{2^s(s+1)(s+2)}+\frac{2^{s+1}-1}{2^s(s+1)(s+2)}\right)^{\frac{1}{q}}$$

$$\leq 2\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\frac{1}{2}\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}=2\,(2)^{-\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}$$

$$=2^{\frac{1}{q}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}},$$

which completes the proof of remark.

3 Applications for special means for (s, P)-functions

Throughout this section, for shortness, the following notations will be used for special means of two nonnegative numbers a, b with $b > a$.

1. The arithmetic mean

$$A := A(a,b) = \frac{a+b}{2}, \quad a,b \geq 0.$$

2. The geometric mean

$$G := G(a,b) = \sqrt{ab}, \quad a,b \geq 0.$$

3. The harmonic mean

$$H := H(a,b) = \frac{2ab}{a+b}, \quad a,b > 0.$$

4. The logarithmic mean

$$L := L(a,b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & a \neq b, \\ a, & a = b, \end{cases} \quad a,b > 0.$$

5. The p-logarithmic mean

$$L_p := L_p(a,b) = \begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & a \neq b,\ p \in \mathbb{R}\backslash\{-1,0\}, \\ a, & a = b, \end{cases} \quad a,b > 0.$$

6.The identric mean

$$I := I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, \quad a,b > 0.$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships

$$H \leq G \leq L \leq I \leq A$$

are known in the literature.

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 1. *Let $a, b \in [0, \infty)$ with $a < b$, $s \in (0, 1]$ and $n \geq 2$. Then the following inequality*

$$\left|\frac{1}{3}\left(A\left(a^n, b^n\right) + 2A^n\right) - L_n^n\right| \leq 4n(b-a)A\left(a^{n-1}, b^{n-1}\right)\left[\frac{2\left(5^{s+2}+1\right) + (s-4)6^{s+1} - 6.3^{s+1}}{6^{s+2}(s+1)(s+2)}\right]$$

holds.

Proof. The assertion follows from the inequality (2) for the function $f(x) = x^n$, $x \in [0, \infty)$. $\square$

Proposition 2. *Let $a, b \in (0, \infty)$ with $a < b$ and $s \in (0, 1]$. Then the following inequaly*

$$\left|\frac{1}{3}\left(H^{-1}(a, b) + 2A^{-1}\right) - L^{-1}\right| \leq 4n(b-a)H^{-1}\left(a^2, b^2\right)\left[\frac{2\left(5^{s+2}+1\right) + (s-4)6^{s+1} - 6.3^{s+1}}{6^{s+2}(s+1)(s+2)}\right]$$

holds.

Proof. The assertion follows from the inequality (2) for the function $f(x) = x^{-1}$, $x \in (0, \infty)$. $\square$

4 Conclusion

In this paper, some new Simpson type inequalities are obtained for functions whose first derivative in absolute value is (s, P)-function using Hölder, power-mean and Hölder-İşcan inequalities. The authors can obtain new types of integral inequalities for (s, P)-functions using different identities. Then the authors compare the obtained results with both Hölder and Hölder-İşcan integral inequalities and show that Hölder-İşcan integral inequality provides a better approximation than Hölder inequality.

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Кадакал М., Ішкан І., Кадакал Х. *Деякі нові інтегральні нерівності типу Сімпсона для (s,P)-функцій* // Карпатські матем. публ. — 2025. — Т.17, №1. — С. 331–342.

У цій статті встановлено деякі нові нерівності типу Сімпсона для функцій, абсолютне значення першої похідної яких є (s,P)-функцією, із використанням нерівностей Гельдера, Гельдера-Ішкана та нерівності для середніх степеневих. Далі порівнюються отримані результати як з інтегральною нерівністю Гельдера, так і з нерівністю Гельдера-Ішкана, і доводиться, що остання дає кращу апроксимацію, ніж нерівність Гельдера. Також наведено деякі застосування до спеціальних середніх дійсних чисел.

Ключові слова і фрази: (s,P)-функція, нерівність типу Сімпсона, нерівність Ерміта-Адамара, нерівність Гельдера-Ішкана.