



Asymptotic estimates for the widths of classes of functions of high smoothness

Serdyuk A.S., Sokolenko I.V.

We find two-sided estimates for Kolmogorov, Bernstein, linear and projection widths of the classes of convolutions of 2π -periodic functions φ , such that $\|\varphi\|_2 \leq 1$, with fixed generated kernels $\Psi_{\vec{\beta}}$, which have Fourier series of the form $\sum_{k=1}^{\infty} \psi(k) \cos(kt - \beta_k \pi/2)$, where $\psi(k) \geq 0$, $\sum \psi^2(k) < \infty$, $\beta_k \in \mathbb{R}$. It is shown that for rapidly decreasing sequences $\psi(k)$ (in particular, if $\lim_{k \rightarrow \infty} \psi(k+1)/\psi(k) = 0$) the obtained estimates are asymptotic equalities. We establish that asymptotic equalities for widths of this classes are realized by trigonometric Fourier sums.

Key words and phrases: Bernstein width, Kolmogorov width, linear width, projection width, Fourier sum, Weyl-Nagy class, class of the generalized Poisson integrals, $(\psi, \vec{\beta})$ -integral, asymptotic equality.

Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivska str., 01601, Kyiv, Ukraine
E-mail: serdyuk@imath.kiev.ua (Serdyuk A.S.), sokol@imath.kiev.ua (Sokolenko I.V.)

Introduction

Let L_p , $1 \leq p < \infty$, be the space of 2π -periodic functions f summable to the power p on $[-\pi, \pi)$, in which the norm is given by the formula

$$\|f\|_{L_p} = \|f\|_p = \left(\int_{-\pi}^{\pi} |f(t)|^p dt \right)^{1/p},$$

L_{∞} be the space of measurable and essentially bounded 2π -periodic functions f with the norm

$$\|f\|_{L_{\infty}} = \|f\|_{\infty} = \operatorname{ess\,sup}_t |f(t)|,$$

C be the space of continuous 2π -periodic functions f , in which the norm is defined by the equality

$$\|f\|_C = \max_t |f(t)|.$$

YΔK 517.5

2020 *Mathematics Subject Classification*: 41A46, 42A10.

This work was partially supported by the Grant H2020-MSCA-RISE-2019, project number 873071 (SOMPATY: Spectral Optimization: From Mathematics to Physics and Advanced Technology)

Denote by $C_{\bar{\beta},p}^\psi$, $1 \leq p \leq \infty$, the set of all 2π -periodic functions f , representable as convolution

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi(x-t) \Psi_{\bar{\beta}}(t) dt, \quad a_0 \in \mathbb{R}, \quad \varphi \in B_p^0, \quad (1)$$

$$B_p^0 = \{g \in L_p : \|g\|_p \leq 1, g \perp 1\}$$

with a fixed generated kernel $\Psi_{\bar{\beta}} \in L_{p'}$, $1/p + 1/p' = 1$, the Fourier series of which has the form

$$S[\Psi_{\bar{\beta}}](t) = \sum_{k=1}^{\infty} \psi(k) \cos\left(kt - \frac{\beta_k \pi}{2}\right), \quad \beta_k \in \mathbb{R}, \quad \psi(k) \geq 0.$$

A function f in the representation (1) is called $(\psi, \bar{\beta})$ -integral of the function φ and is denoted by $\mathcal{J}_{\bar{\beta}}^\psi \varphi$ ($f = \mathcal{J}_{\bar{\beta}}^\psi \varphi$). If $\psi(k) \neq 0$, $k \in \mathbb{N}$, then the function φ in the representation (1) is called $(\psi, \bar{\beta})$ -derivative of the function f and is denoted by $f_{\bar{\beta}}^\psi$ ($\varphi = f_{\bar{\beta}}^\psi$). The concepts of $(\psi, \bar{\beta})$ -integral and $(\psi, \bar{\beta})$ -derivative was introduced by A.I. Stepanets (see, e.g., [30, 31]). Since $\varphi \in L_p$ and $\Psi_{\bar{\beta}} \in L_{p'}$, the function f of the form (1) is a continuous function, i.e. $C_{\bar{\beta},p}^\psi \subset C$ (see [31, Proposition 3.9.2.]). In the case $\beta_k \equiv \beta$, $\beta \in \mathbb{R}$, the classes $C_{\bar{\beta},p}^\psi$ are denoted by $C_{\beta,p}^\psi$. For $\psi(k) = k^{-r}$, $r > 0$, the classes $C_{\bar{\beta},p}^\psi$ and $C_{\beta,p}^\psi$ are denoted by $W_{\bar{\beta},p}^r$ and $W_{\beta,p}^r$, respectively. The classes $W_{\beta,p}^r$ are the well-known Weyl-Nagy classes (see, e.g., [12, 29–31]). In other words, $W_{\beta,p}^r$, $1 \leq p \leq \infty$, are the classes of 2π -periodic functions f , representable as convolutions of the form

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi(x-t) B_{r,\beta}(t) dt, \quad a_0 \in \mathbb{R}, \quad (2)$$

the Weyl-Nagy kernels $B_{r,\beta}$ of the form

$$B_{r,\beta}(t) = \sum_{k=1}^{\infty} k^{-r} \cos\left(kt - \frac{\beta\pi}{2}\right), \quad r > 0, \quad \beta \in \mathbb{R}, \quad (3)$$

with functions $\varphi \in B_p^0$. The function φ in the formula (2) is called the Weyl-Nagy derivative of the function f and is denoted by f_{β}^r .

If $r \in \mathbb{N}$ and $\beta = r$, then the functions $B_{r,\beta}$ of the form (3) are the well-known Bernoulli kernels and the corresponding classes $W_{\beta,p}^r$ coincide with the well-known classes W_p^r which consist of 2π -periodic functions f with absolutely continuous derivatives $f^{(k)}$ up to $(r-1)$ -th order inclusive and such that $\|f^{(r)}\|_p \leq 1$. In addition, $f^{(r)}(x) = f_r^r(x) = \varphi(x)$ for almost all $x \in \mathbb{R}$, where φ is the function from (2).

For $\psi(k) = e^{-\alpha k^r}$, $\alpha > 0$, $r > 0$, the classes $C_{\bar{\beta},p}^\psi$ and $C_{\beta,p}^\psi$ are denoted by $C_{\bar{\beta},p}^{\alpha,r}$ and $C_{\beta,p}^{\alpha,r}$, respectively. The sets $C_{\beta,p}^{\alpha,r}$ are well-known classes of the generalized Poisson integrals (see, e.g., [30, 31]), i.e. classes of convolutions

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi(x-t) P_{\alpha,r,\beta}(t) dt, \quad a_0 \in \mathbb{R}, \quad \varphi \in B_{p'}^0,$$

with the generalized Poisson kernels

$$P_{\alpha,r,\beta}(t) = \sum_{k=1}^{\infty} e^{-\alpha k^r} \cos\left(kt - \frac{\beta\pi}{2}\right), \quad \alpha > 0, \quad r > 0, \quad \beta \in \mathbb{R}.$$

Let $\mathfrak{N}$ be a some functional class from the space C ($\mathfrak{N} \subset C$). The quantity

$$E_n(\mathfrak{N})_C = \sup_{f \in \mathfrak{N}} E_n(f)_C = \sup_{f \in \mathfrak{N}} \inf_{T_{n-1} \in \mathcal{T}_{2n-1}} \|f - T_{n-1}\|_C$$

is called the best uniform approximation of the class $\mathfrak{N}$ by elements of the subspace $\mathcal{T}_{2n-1}$ of trigonometric polynomials T_{n-1} of the order $n - 1$:

$$T_{n-1}(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{n-1} (\alpha_k \cos kx + \beta_k \sin kx), \quad \alpha_k, \beta_k \in \mathbb{R}.$$

The order estimates for the best approximations $E_n(K)_C$ of classes $K = C_{\bar{\beta},p}^\psi$, $1 \leq p \leq \infty$, (and, hence, classes $W_{\beta,p}^r$, $C_{\beta,p}^{\alpha,r}$ and $C_{\beta,p}^\psi$) depending on rate of decreasing to zero of sequences $\psi(k)$ were obtained, in particular, in the works of V.N. Temlyakov [35], U.Z. Hrabova and A.S. Serdyuk [5], A.S. Serdyuk and T.A. Stepanyuk [25, 26].

If the sequences $\psi(k)$ decrease to zero faster than any geometric progression, then asymptotic equations of the best uniform approximations are even known (see, for example, the authors work [24] and the bibliography available there).

In [24], it was shown that for such classes $C_{\bar{\beta},p}^\psi$ the following asymptotic equations take places

$$E_n(C_{\bar{\beta},p}^\psi)_C \sim \mathcal{E}_n(C_{\bar{\beta},p}^\psi)_C \sim \frac{\|\cos t\|_{p'}}{\pi} \psi(n), \quad 1 \leq p \leq \infty,$$

where

$$\mathcal{E}_n(C_{\bar{\beta},p}^\psi) = \sup_{f \in C_{\bar{\beta},p}^\psi} \|f - S_{n-1}(f)\|_C,$$

$S_{n-1}(f)$ is the partial Fourier sum of order $n-1$ of the function f , $\frac{1}{p} + \frac{1}{p'} = 1$, and $A(n) \sim B(n)$ as $n \rightarrow \infty$ means that $\lim_{n \rightarrow \infty} A(n)/B(n) = 1$.

For $p = \infty$ in the case of $K = W_{\bar{\beta},\infty}^r$, $r > 0$, and in the cases of $K = C_{\bar{\beta},\infty}^{\alpha,r}$, $r \geq 1$, and $K = C_{\bar{\beta},\infty}^\psi$ ($K = C_{\bar{\beta},\infty}^\psi$) for certain restrictions on sequences ψ and $\bar{\beta}$ the exact values of the best uniform approximations are known thanks to the works of J. Favard [6, 7], N.I. Akhiezer and M.G. Krein [1], M.G. Krein [10], B. Sz.-Nagy [12], S.B. Stechkin [29], V.K. Dzyadyk [3, 4], Y.-S. Sun [33], A.V. Bushanskij [2], A. Pinkus [13], A.S. Serdyuk [15–19] etc.

For $p = 2$ and for arbitrary $\bar{\beta} = \beta_k \in \mathbb{R}$, $\sum_{k=1}^{\infty} \psi^2(k) < \infty$ the exact values for the quantity $\mathcal{E}_n(C_{\bar{\beta},2}^\psi)_C$ are also known (see [23]).

In this paper, we establish two-sided estimates of Kolmogorov, Bernstein, linear and projection widths of the classes $C_{\bar{\beta},2}^\psi$ in the space C , which become into asymptotic equations under certain restrictions on the sequence $\psi(k)$ (in particular, if $\lim_{k \rightarrow \infty} \psi(k+1)/\psi(k) = 0$).

Let K be a convex centrally symmetric subset of C and let B be a unit ball of the space C . Let also F_N be an arbitrary N -dimensional subspace of space C , $N \in \mathbb{N}$, and $\mathcal{L}(C, F_N)$ be a set of linear operators from C to F_N . By $\mathcal{P}(C, F_N)$ we denote the subset of projection operators of the set $\mathcal{L}(C, F_N)$, that is, the set of the operators A of linear projection onto the set F_N such that $Af = f$ when $f \in F_N$. The quantities

$$b_N(K, C) = \sup_{F_{N+1}} \sup \{ \varepsilon > 0 : \varepsilon B \cap F_{N+1} \subset K \}, \quad (4)$$

$$d_N(K, C) = \inf_{F_N} \sup_{f \in K} \inf_{u \in F_N} \|f - u\|_C, \quad (5)$$

$$\lambda_N(K, C) = \inf_{F_N} \inf_{A \in \mathcal{L}(C, F_N)} \sup_{f \in K} \|f - Af\|_C, \quad (6)$$

$$\pi_N(K, C) = \inf_{F_N} \inf_{A \in \mathcal{P}(C, F_N)} \sup_{f \in K} \|f - Af\|_C \quad (7)$$

are called Bernstein, Kolmogorov, linear, and projection N -widths of the set K in the space C , respectively.

The results containing order estimates of the widths (4)–(7) in the case of $K = C_{\bar{\beta}, p}^\psi$ (and, in particular, $W_{\bar{\beta}, p}^r$ and $C_{\bar{\beta}, p}^\psi$) can be found, for example, in the works of V.M. Tikhomirov [36], A. Pinkus [13], N.P. Kornejchuk [8], A.K. Kushpel' [11], A.S. Romanyuk [14], V.N. Temlyakov [34, 35] etc.

1 Main results

The main result of this paper is the following statement.

Theorem 1. Let $\bar{\beta} = \{\beta_k\}_{k=1}^\infty$, $\beta_k \in \mathbb{R}$, and $\psi(k) > 0$ satisfies the condition

$$\sum_{k=1}^{\infty} \psi^2(k) < \infty. \quad (8)$$

Then for all $n \in \mathbb{N}$ the following inequalities hold

$$\frac{1}{\sqrt{\pi}} \left(\frac{1}{\psi^2(n)} + 2 \sum_{k=1}^{n-1} \frac{1}{\psi^2(k)} \right)^{-\frac{1}{2}} \leq P_{2n}(C_{\bar{\beta}, 2}^\psi, C) \leq P_{2n-1}(C_{\bar{\beta}, 2}^\psi, C) \leq \frac{1}{\sqrt{\pi}} \left(\sum_{k=n}^{\infty} \psi^2(k) \right)^{\frac{1}{2}}, \quad (9)$$

where P_N is any of the widths b_N, d_N, λ_N or π_N .

If, in addition, $\psi(k)$ satisfies the condition

$$\lim_{n \rightarrow \infty} \max \left\{ \psi(n) \left(\sum_{k=1}^{n-1} \frac{1}{\psi^2(k)} \right)^{\frac{1}{2}}, \frac{1}{\psi(n)} \left(\sum_{k=n+1}^{\infty} \psi^2(k) \right)^{\frac{1}{2}} \right\} = 0, \quad (10)$$

then the following asymptotic equalities hold

$$\left. \frac{P_{2n}(C_{\bar{\beta}, 2}^\psi, C)}{P_{2n-1}(C_{\bar{\beta}, 2}^\psi, C)} \right\} = \psi(n) \left(\frac{1}{\sqrt{\pi}} + \mathcal{O}(1) \max \left\{ \psi(n) \left(\sum_{k=1}^{n-1} \frac{1}{\psi^2(k)} \right)^{\frac{1}{2}}, \frac{1}{\psi(n)} \left(\sum_{k=n+1}^{\infty} \psi^2(k) \right)^{\frac{1}{2}} \right\} \right), \quad (11)$$

where $\mathcal{O}(1)$ are the quantities uniformly bounded in all parameters. The equalities (11) are realized by trigonometric Fourier sums $S_{n-1}(f)$.

Proof. In the work [23], it was proved that if the condition (8) is satisfied, then the following equality

$$\mathcal{E}(C_{\bar{\beta}, 2}^\psi; S_{n-1})_C = \frac{1}{\sqrt{\pi}} \left(\sum_{k=n}^{\infty} \psi^2(k) \right)^{\frac{1}{2}}, \quad \beta_k \in \mathbb{R}, \quad n \in \mathbb{N}, \quad (12)$$

holds.

Since the operator that assigns to each function $f \in C$ its partial Fourier sum is a linear projector, then by virtue (12)

$$\pi_{2n-1} \left(C_{\beta,2}^\psi, C \right) \leq \mathcal{E} \left(C_{\beta,2}^\psi; S_{n-1} \right)_C = \frac{1}{\sqrt{\pi}} \left(\sum_{k=n}^{\infty} \psi^2(k) \right)^{\frac{1}{2}}, \quad \beta_k \in \mathbb{R}, \quad n \in \mathbb{N}. \quad (13)$$

For all $n \in \mathbb{N}$ and $\mathfrak{N} \subset C$

$$P_{2n}(\mathfrak{N}, C) \leq P_{2n-1}(\mathfrak{N}, C),$$

where P_N is any of the widths b_N, d_N, λ_N , and π_N , and, in addition, for all $N \in \mathbb{N}$

$$b_N(\mathfrak{N}, C) \leq d_N(\mathfrak{N}, C) \leq \lambda_N(\mathfrak{N}, C) \leq \pi_N(\mathfrak{N}, C). \quad (14)$$

Therefore on the basis of (13) we obtain an estimate from above for the widths P_N in the formula (9). To obtain a required estimate from below in (9) it suffices to establish that

$$b_{2n} \left(C_{\beta,2}^\psi, C \right) \geq \frac{1}{\sqrt{\pi}} \left(\frac{1}{\psi^2(n)} + 2 \sum_{k=1}^{n-1} \frac{1}{\psi^2(k)} \right)^{-\frac{1}{2}}. \quad (15)$$

In $(2n+1)$ -dimensional space $\mathcal{T}_{2n+1}$ of trigonometric polynomials T_n of order n let us consider a ball of the form

$$B_{2n+1} = \left\{ T_n \in \mathcal{T}_{2n+1} : \|T_n\|_C \leq \frac{1}{\sqrt{\pi}} \left(\frac{1}{\psi^2(n)} + 2 \sum_{k=1}^{n-1} \frac{1}{\psi^2(k)} \right)^{-\frac{1}{2}} \right\} \quad (16)$$

and prove the following embedding

$$B_{2n+1} \subset C_{\beta,2}^\psi. \quad (17)$$

For any trinometric polynomial

$$T_n(x) = \frac{a_0}{2} + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx) \quad (18)$$

from the ball B_{2n+1} its $(\psi, \bar{\beta})$ -derivative has a form

$$\begin{aligned} (T_n)_{\bar{\beta}}^\psi(x) &= \sum_{k=1}^n \left(\frac{a_k}{\psi(k)} \cos \left(kx + \frac{\beta_k \pi}{2} \right) + \frac{b_k}{\psi(k)} \sin \left(kx + \frac{\beta_k \pi}{2} \right) \right) \\ &= \sum_{k=1}^n \left(\frac{a_k \cos \frac{\beta_k \pi}{2}}{\psi(k)} \cos kx - \frac{a_k \sin \frac{\beta_k \pi}{2}}{\psi(k)} \sin kx + \frac{b_k \cos \frac{\beta_k \pi}{2}}{\psi(k)} \sin kx + \frac{b_k \sin \frac{\beta_k \pi}{2}}{\psi(k)} \cos kx \right) \\ &= \sum_{k=1}^n \frac{1}{\psi(k)} \left(\left(a_k \cos \frac{\beta_k \pi}{2} + b_k \sin \frac{\beta_k \pi}{2} \right) \cos kx + \left(-a_k \sin \frac{\beta_k \pi}{2} + b_k \cos \frac{\beta_k \pi}{2} \right) \sin kx \right). \end{aligned}$$

By virtue of Parseval equality, from the above equalities we get

$$\begin{aligned} \|(T_n)_{\bar{\beta}}^\psi\|_2 &= \sqrt{\pi} \left(\sum_{k=1}^n \frac{1}{\psi^2(k)} \left(\left(a_k \cos \frac{\beta_k \pi}{2} + b_k \sin \frac{\beta_k \pi}{2} \right)^2 + \left(-a_k \sin \frac{\beta_k \pi}{2} + b_k \cos \frac{\beta_k \pi}{2} \right)^2 \right) \right)^{\frac{1}{2}} \\ &= \sqrt{\pi} \left(\sum_{k=1}^n \frac{1}{\psi^2(k)} (a_k^2 + b_k^2) \right)^{\frac{1}{2}}. \end{aligned} \quad (19)$$

By Parseval equality for the polynomial T_n of the form (18) we obtaine

$$\frac{a_0^2}{2} + \sum_{k=1}^n (a_k^2 + b_k^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} T_n^2(x) dx.$$

Therefore we have a chain of inequalities

$$a_k^2 + b_k^2 \leq \sum_{k=1}^n (a_k^2 + b_k^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} \|T_n\|_C^2 dx = 2 \|T_n\|_C^2,$$

and, consequently, we obtaine an estimate for $\sqrt{a_k^2 + b_k^2}$ of the following form

$$\sqrt{a_k^2 + b_k^2} \leq \sqrt{2} \|T_n\|_C, \quad k = \overline{1, n}. \quad (20)$$

In the case of $k = n$ this estimate can be improved. To do this, let us consider a trigonometric polynomial

$$\begin{aligned} \tau_n(x) &:= \frac{T_n(x)}{\sqrt{a_n^2 + b_n^2}} = \frac{1}{\sqrt{a_n^2 + b_n^2}} \left(\frac{a_0}{2} + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx) \right) \\ &= \frac{1}{\sqrt{a_n^2 + b_n^2}} \left(\frac{a_0}{2} + \sum_{k=1}^n \sqrt{a_k^2 + b_k^2} \left(\frac{a_k}{\sqrt{a_k^2 + b_k^2}} \cos kx + \frac{b_k}{\sqrt{a_k^2 + b_k^2}} \sin kx \right) \right) \quad (21) \\ &= \frac{\rho_0}{2} + \sum_{k=1}^n \rho_k \cos(kx + \theta_k), \end{aligned}$$

where

$$\rho_0 = \frac{a_0}{\sqrt{a_n^2 + b_n^2}}, \quad \rho_k = \frac{\sqrt{a_k^2 + b_k^2}}{\sqrt{a_n^2 + b_n^2}}, \quad k = \overline{1, n-1}, \quad \rho_n = 1,$$

and θ_k are such that

$$\begin{cases} \cos \theta_k = \frac{a_k}{\sqrt{a_k^2 + b_k^2}}, & k = \overline{1, n}, \\ \sin \theta_k = \frac{-b_k}{\sqrt{a_k^2 + b_k^2}}, & k = \overline{1, n}. \end{cases}$$

As it follows from [9, Statement 2.9.1] for all $p \in [1, \infty]$ the following inequality

$$\|\tau_n(\cdot)\|_p \geq \|\cos n(\cdot)\|_p,$$

holds and, consequently, for $p = \infty$

$$\|\tau_n\|_C \geq 1. \quad (22)$$

From (21) and (22) we get

$$\sqrt{a_n^2 + b_n^2} = \frac{\|T_n\|_C}{\|\tau_n\|_C} \leq \|T_n\|_C. \quad (23)$$

Using the equations (19) and the estimates (20) and (23) we have

$$\begin{aligned} \|(T_n)_{\tilde{\beta}}^{\psi}\|_2 &= \sqrt{\pi} \left(\sum_{k=1}^{n-1} \frac{a_k^2 + b_k^2}{\psi^2(k)} + \frac{a_n^2 + b_n^2}{\psi^2(n)} \right)^{\frac{1}{2}} \\ &\leq \sqrt{\pi} \left(2 \sum_{k=1}^{n-1} \frac{\|T_n\|_C^2}{\psi^2(k)} + \frac{\|T_n\|_C^2}{\psi^2(n)} \right)^{\frac{1}{2}} = \sqrt{\pi} \left(\frac{1}{\psi^2(n)} + 2 \sum_{k=1}^{n-1} \frac{1}{\psi^2(k)} \right)^{\frac{1}{2}} \|T_n\|_C. \end{aligned} \quad (24)$$

Since the polynomials T_n belongs to the ball B_{2n+1} of the form (16), from (24) it follows that

$$\left\| (T_n)_{\beta}^{\psi} \right\|_2 \leq 1.$$

The embedding (17) is proved.

The inequality (15) follows from the definition of the Bernstein width $b_{2n} \left(C_{\beta,2}^{\psi}, C \right)$ and the embedding (17). The relations (13)–(15) prove the inequalities (9). To prove the asymptotic equations (11) under satisfying the condition (10) first of all we note that

$$\left(\sum_{k=n}^{\infty} \psi^2(k) \right)^{\frac{1}{2}} \leq \psi(n) + \left(\sum_{k=n+1}^{\infty} \psi^2(k) \right)^{\frac{1}{2}} \quad (25)$$

and

$$\left(\frac{1}{\psi^2(n)} + 2 \sum_{k=1}^{n-1} \frac{1}{\psi^2(k)} \right)^{\frac{1}{2}} \leq \frac{1}{\psi(n)} + \left(2 \sum_{k=1}^{n-1} \frac{1}{\psi^2(k)} \right)^{\frac{1}{2}}. \quad (26)$$

From (26) we get

$$\begin{aligned} \left(\frac{1}{\psi^2(n)} + 2 \sum_{k=1}^{n-1} \frac{1}{\psi^2(k)} \right)^{-\frac{1}{2}} &\geq \frac{1}{\frac{1}{\psi(n)} + \left(2 \sum_{k=1}^{n-1} \frac{1}{\psi^2(k)} \right)^{\frac{1}{2}}} \\ &= \psi(n) - \left(\frac{1}{\frac{1}{\psi(n)}} - \frac{1}{\frac{1}{\psi(n)} + \left(2 \sum_{k=1}^{n-1} \frac{1}{\psi^2(k)} \right)^{\frac{1}{2}}} \right) = \psi(n) - \frac{\left(2 \sum_{k=1}^{n-1} \frac{1}{\psi^2(k)} \right)^{\frac{1}{2}}}{\frac{1}{\psi(n)} \left(\frac{1}{\psi(n)} + \left(2 \sum_{k=1}^{n-1} \frac{1}{\psi^2(k)} \right)^{\frac{1}{2}} \right)} \\ &= \psi(n) \left(1 - \frac{\left(2 \sum_{k=1}^{n-1} \frac{1}{\psi^2(k)} \right)^{\frac{1}{2}}}{\frac{1}{\psi(n)} + \left(2 \sum_{k=1}^{n-1} \frac{1}{\psi^2(k)} \right)^{\frac{1}{2}}} \right) \geq \psi(n) \left(1 - \psi(n) \left(2 \sum_{k=1}^{n-1} \frac{1}{\psi^2(k)} \right)^{\frac{1}{2}} \right). \end{aligned}$$

So, as it follows from (9) and (25), on the one hand,

$$P_{2n} \left(C_{\beta,2}^{\psi}, C \right) \leq P_{2n-1} \left(C_{\beta,2}^{\psi}, C \right) \leq \psi(n) \left(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi} \psi(n)} \left(\sum_{k=n+1}^{\infty} \psi^2(k) \right)^{\frac{1}{2}} \right), \quad (27)$$

and, on the other hand, by virtue of (9)

$$P_{2n} \left(C_{\beta,2}^{\psi}, C \right) \geq \psi(n) \left(\frac{1}{\sqrt{\pi}} - \sqrt{\frac{2}{\pi}} \psi(n) \left(\sum_{k=1}^{n-1} \frac{1}{\psi^2(k)} \right)^{\frac{1}{2}} \right). \quad (28)$$

The combination of (27) and (28) allows us to write equations

$$P_{2n} \left(C_{\beta,2}^{\psi}, C \right) = \psi(n) \left(\frac{1}{\sqrt{\pi}} + \gamma_n^{(1)} \right), \quad (29)$$

$$P_{2n-1} \left(C_{\beta,2}^{\psi}, C \right) = \psi(n) \left(\frac{1}{\sqrt{\pi}} + \gamma_n^{(2)} \right), \quad (30)$$

in which for $\gamma_n^{(i)}, i = 1, 2$, the following double inequalities

$$-\sqrt{\frac{2}{\pi}}\psi(n) \left(\sum_{k=1}^{n-1} \frac{1}{\psi^2(k)} \right)^{\frac{1}{2}} \leq \gamma_n^{(i)} \leq \frac{1}{\sqrt{\pi}\psi(n)} \left(\sum_{k=n+1}^{\infty} \psi^2(k) \right)^{\frac{1}{2}} \quad (31)$$

hold. If the condition (10) is satisfied, then by virtue of (29)–(31) the asymptotic equations (11) take place. Theorem 1 is proved. $\square$

We note that the condition

$$\lim_{n \rightarrow \infty} \frac{1}{\psi^2(n)} \sum_{k=n+1}^{\infty} \psi^2(k) = 0$$

is satisfied if $\psi(k)$ satisfies the condition

$$\lim_{k \rightarrow \infty} \frac{\psi(k+1)}{\psi(k)} = 0. \quad (32)$$

To make sure of this, let us put

$$\varepsilon_n = \sup_{k \geq n} \frac{\psi(k+1)}{\psi(k)}.$$

By virtue of (32) $\varepsilon_n \downarrow 0$ as $n \rightarrow \infty$. So, we get

$$\begin{aligned} \sum_{k=n+1}^{\infty} \psi^2(k) &= \psi^2(n) \left(\frac{\psi^2(n+1)}{\psi^2(n)} + \frac{\psi^2(n+2)}{\psi^2(n+1)} \frac{\psi^2(n+1)}{\psi^2(n)} + \dots \right) \\ &\leq \psi^2(n) (\varepsilon_n^2 + \varepsilon_n^4 + \dots) = \psi^2(n) \frac{\varepsilon_n^2}{1 - \varepsilon_n^2} = o(\psi^2(n)). \end{aligned}$$

Let us show that for strictly decreasing sequences ψ the fulfillment of condition of the form (32) ensures the truth of the following equality

$$\lim_{n \rightarrow \infty} \psi^2(n) \sum_{k=1}^{n-1} \frac{1}{\psi^2(k)} = 0. \quad (33)$$

To do this, we use Stoltz's theorem, according to which the relation (33) is followed from the following equality

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^{n-1} \frac{1}{\psi^2(k)} - \sum_{k=1}^{n-2} \frac{1}{\psi^2(k)}}{\frac{1}{\psi^2(n)} - \frac{1}{\psi^2(n-1)}} = 0. \quad (34)$$

Since

$$\frac{\sum_{k=1}^{n-1} \frac{1}{\psi^2(k)} - \sum_{k=1}^{n-2} \frac{1}{\psi^2(k)}}{\frac{1}{\psi^2(n)} - \frac{1}{\psi^2(n-1)}} = \frac{\frac{1}{\psi^2(n-1)}}{\frac{1}{\psi^2(n)} - \frac{1}{\psi^2(n-1)}} = \frac{\frac{\psi^2(n)}{\psi^2(n-1)}}{1 - \frac{\psi^2(n)}{\psi^2(n-1)}},$$

then (34) follows from (32). In view of the above, we have the following statement.

Corollary 1. Let $\bar{\beta} = \{\beta_k\}_{k=1}^{\infty}$, $\beta_k \in \mathbb{R}$ and the sequence $\psi(k) > 0$ is strictly decreasing and satisfies the condition of the form (32). Then the asymptotic equalities (11) hold as $n \rightarrow \infty$.

We give the corollaries of Theorem 1 in some important special cases.

Theorem 2. Let $\bar{\beta} = \{\beta_k\}_{k=1}^{\infty}$, $\beta_k \in \mathbb{R}$ and $n \in \mathbb{N}$. Then for all $r \geq \frac{n+1}{2}$ the following inequalities

$$\begin{aligned} \frac{1}{\sqrt{\pi}} n^{-r} \left(1 - \frac{4(1 - \frac{1}{n})^{2r}}{1 + 4(1 - \frac{1}{n})^{2r}} \right)^{\frac{1}{2}} &\leq P_{2n} \left(W_{\bar{\beta}, 2}^r, C \right) \\ &\leq P_{2n-1} \left(W_{\bar{\beta}, 2}^r, C \right) \leq \frac{1}{\sqrt{\pi}} n^{-r} \left(1 + \frac{2 + \frac{1}{n}}{(1 + \frac{1}{n})^r} \right)^{\frac{1}{2}}, \end{aligned} \quad (35)$$

hold, where P_N is any of the widths b_N, d_N, λ_N or π_N .

Proof. Let us put $\psi(k) = k^{-r}$, $r > 1$. Obviously, the condition (8) is satisfied. Since for $2r \geq n+1$, $n \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{k=n+1}^{\infty} \frac{1}{k^{2r}} &< \frac{1}{(n+1)^{2r}} + \int_{n+1}^{\infty} \frac{dt}{t^{2r}} = \frac{1}{(n+1)^{2r}} + \frac{1}{(2r-1)(n+1)^{2r-1}} = \frac{1}{(n+1)^{2r}} \frac{2r+n}{2r-1} \\ &\leq \frac{1}{(n+1)^{2r}} \frac{4r-1}{2r-1} \leq \frac{1}{(n+1)^{2r}} \left(2 + \frac{1}{2r-1} \right) \leq \frac{1}{n^{2r}} \frac{2 + \frac{1}{n}}{(1 + \frac{1}{n})^{2r}}, \end{aligned} \quad (36)$$

then according to the right-hand side of the equality (9) of Theorem 1 we obtain the estimate

$$P_{2n-1} \left(W_{\bar{\beta}, 2}^r, C \right) \leq \frac{1}{\sqrt{\pi}} \left(\sum_{k=n}^{\infty} \frac{1}{k^{2r}} \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{\pi}} n^{-r} \left(1 + \frac{2 + \frac{1}{n}}{(1 + \frac{1}{n})^{2r}} \right)^{\frac{1}{2}}. \quad (37)$$

On the other hand, for $r \geq \frac{n+1}{2}$ and $\psi(k) = k^{-r}$ we have

$$\begin{aligned} \frac{1}{\psi^2(n)} + 2 \sum_{k=1}^{n-1} \frac{1}{\psi^2(k)} &= n^{2r} + 2 \sum_{k=1}^{n-1} k^{2r} \leq n^{2r} + 2 \left((n-1)^{2r} + \int_1^{n-1} t^{2r} dt \right) \\ &= n^{2r} + 2 \left((n-1)^{2r} + \frac{(n-1)^{2r+1}}{2r+1} - \frac{1}{2r+1} \right) \\ &< n^{2r} + 2 \left((n-1)^{2r} + \frac{(n-1)^{2r+1}}{n+2} \right) \\ &< n^{2r} + 4(n-1)^{2r} = n^{2r} \left(1 + 4 \left(1 - \frac{1}{n} \right)^{2r} \right). \end{aligned} \quad (38)$$

By virtue of the left part of the inequality (9) of Theorem 1 and the formula (38) we get the estimate

$$P_{2n} \left(W_{\bar{\beta}, 2}^r, C \right) \geq \frac{1}{\sqrt{\pi}} n^{-r} \left(\frac{1}{1 + 4 \left(1 - \frac{1}{n} \right)^{2r}} \right)^{\frac{1}{2}} = \frac{1}{\sqrt{\pi}} n^{-r} \left(1 - \frac{4 \left(1 - \frac{1}{n} \right)^{2r}}{1 + 4 \left(1 - \frac{1}{n} \right)^{2r}} \right)^{\frac{1}{2}}. \quad (39)$$

Combining the estimates (37) and (39) we obtain (35). Theorem 2 is proved. $\square$

Note that if the condition

$$\lim_{n \rightarrow \infty} \frac{r}{n} = \infty \quad (40)$$

is satisfied, then for $\psi(k) = k^{-r}$ the condition (32) is also satisfied, because

$$\frac{\psi(k+1)}{\psi(k)} = \left(\frac{k}{k+1} \right)^r = \left(1 + \frac{1}{k} \right)^{-r} = \left(\left(1 + \frac{1}{k} \right)^{k+1} \right)^{-\frac{r}{k+1}} \leq e^{-\frac{r}{k+1}} \rightarrow 0, \quad k \rightarrow \infty.$$

Taking the limit as $n \rightarrow \infty$ in the relations (35), we obtain the following statement.

Theorem 3. Let $\bar{\beta} = \{\beta_k\}_{k=1}^\infty, \beta_k \in \mathbb{R}, n \in \mathbb{N}$, and the condition (40) is satisfied. Then the following asymptotic equalities

$$\left. \begin{array}{l} P_{2n}(W_{\bar{\beta},2}^r, C) \\ P_{2n-1}(W_{\bar{\beta},2}^r, C) \end{array} \right\} = n^{-r} \left(\frac{1}{\sqrt{\pi}} + \mathcal{O}(1) \left(1 + \frac{1}{n} \right)^{-r} \right) \quad (41)$$

hold, where P_N is any of the widths b_N, d_N, λ_N or π_N , and $\mathcal{O}(1)$ are the quantities uniformly bounded in all parameters.

Note also that the equalities (41) are easily obtained from the formula (11) and estimates (36) and (38).

Theorem 4. Let $\bar{\beta} = \{\beta_k\}_{k=1}^\infty, \beta_k \in \mathbb{R}, \alpha > 0, r > 1, n \in \mathbb{N}$ and be such that

$$(n-1)^r > \frac{1}{\alpha}, \quad (42)$$

then the following inequalities

$$\begin{aligned} \frac{1}{\sqrt{\pi}} e^{-\alpha n^r} \left(1 - \frac{2\gamma_{\alpha,r,n} e^{-2\alpha r(n-1)^{r-1}}}{1 + 2\gamma_{\alpha,r,n} e^{-2\alpha r(n-1)^{r-1}}} \right)^{\frac{1}{2}} &\leq P_{2n}(C_{\bar{\beta},2}^{\alpha,r}, C) \leq P_{2n-1}(C_{\bar{\beta},2}^{\alpha,r}, C) \\ &\leq \frac{1}{\sqrt{\pi}} e^{-\alpha n^r} \left(1 + e^{-2\alpha r n^{r-1}} \left(1 + \frac{1}{2\alpha r n^{r-1}} \right) \right)^{\frac{1}{2}} \end{aligned} \quad (43)$$

hold, where P_N is any of the widths b_N, d_N, λ_N or π_N and

$$\gamma_{\alpha,r,n} = \left(1 + \frac{1}{\alpha r(n-1)^{r-1}} + e^{-2\alpha(n-1)^r} \max \left\{ e^{4\alpha}, \frac{e^2}{\alpha^{1+1/r}} \right\} \right). \quad (44)$$

Proof. First of all, note that if $\alpha > 0, r > 1, n \in \mathbb{N}$ and satisfy the condition (42), then for a quantity of the form

$$I_{n-1} := \int_1^{n-1} e^{2\alpha t^r} dt, \quad \alpha > 0, r > 1,$$

the following inequality

$$I_{n-1} \leq \frac{e^{2\alpha(n-1)^r}}{\alpha r(n-1)^{r-1}} + \max \left\{ e^{4\alpha}, \frac{e^2}{\alpha^{1+1/r}} \right\} \quad (45)$$

holds. Indeed, integrating by parts we have

$$I_{n-1} = \frac{1}{2\alpha r} \int_1^{n-1} t^{1-r} d e^{2\alpha t^r} = \frac{1}{2\alpha r} \left(\frac{e^{2\alpha(n-1)^r}}{(n-1)^{r-1}} - e^{2\alpha} \right) + \frac{r-1}{2\alpha r} \int_1^{n-1} \frac{e^{2\alpha t^r}}{t^r} dt. \quad (46)$$

For $0 < 2\alpha < 1$, taking into account (42), we obtain

$$\int_1^{n-1} \frac{e^{2\alpha t^r}}{t^r} dt = \int_1^{\alpha^{-1/r}} \frac{e^{2\alpha t^r}}{t^r} dt + \int_{\alpha^{-1/r}}^{n-1} \frac{e^{2\alpha t^r}}{t^r} dt < \frac{e^2}{\alpha^{1/r}} + \alpha \int_{\alpha^{-1/r}}^{n-1} e^{2\alpha t^r} dt \leq \frac{e^2}{\alpha^{1/r}} + \alpha I_{n-1}. \quad (47)$$

From (46) and (47) under condition (42) we get

$$I_{n-1} < \frac{e^{2\alpha(n-1)^r}}{2\alpha r(n-1)^{r-1}} + \frac{e^2}{2\alpha^{1+1/r}} + \frac{1}{2} I_{n-1}$$

and

$$I_{n-1} < \frac{e^{2\alpha(n-1)^r}}{\alpha r(n-1)^{r-1}} + \frac{e^2}{\alpha^{1+1/r}}, \quad 0 < 2\alpha < 1. \quad (48)$$

For $2\alpha \geq 1$ we obtain

$$\begin{aligned} \int_1^{n-1} \frac{e^{2\alpha t^r}}{t^r} dt &= \int_1^{2^{1/r}} \frac{e^{2\alpha t^r}}{t^r} dt + \int_{2^{1/r}}^{n-1} \frac{e^{2\alpha t^r}}{t^r} dt \leq \int_1^{2^{1/r}} e^{2\alpha t^r} dt + \frac{1}{2} \int_{2^{1/r}}^{n-1} e^{2\alpha t^r} dt \\ &= \frac{1}{2} \int_1^{2^{1/r}} e^{2\alpha t^r} dt + \frac{1}{2} I_{n-1} < \frac{2^{1/r}-1}{2} e^{4\alpha} + \frac{1}{2} I_{n-1} < \frac{e^{4\alpha}}{2} + \frac{1}{2} I_{n-1}. \end{aligned} \quad (49)$$

From (46) and (49) under condition (42) we have

$$I_{n-1} < \frac{e^{2\alpha(n-1)^r}}{2\alpha r(n-1)^{r-1}} + \frac{e^{4\alpha}}{4\alpha} + \frac{1}{4\alpha} I_{n-1} < \frac{e^{2\alpha(n-1)^r}}{2\alpha r(n-1)^{r-1}} + \frac{e^{4\alpha}}{2} + \frac{1}{2} I_{n-1}$$

and

$$I_{n-1} < \frac{e^{2\alpha(n-1)^r}}{\alpha r(n-1)^{r-1}} + e^{4\alpha}, \quad 2\alpha \geq 1. \quad (50)$$

The inequality (45) follows from (48) and (50). For $\psi(k) = e^{-\alpha k^r}$, $\alpha > 0$, $r > 1$, under condition (42), taking into account that for $r > 1$ and $n \in \mathbb{N}$ the inequality

$$\left(1 + \frac{1}{n}\right)^r - 1 > \frac{r}{n}$$

holds, we obtain

$$\begin{aligned} \psi^2(n) \sum_{k=1}^{n-1} \frac{1}{\psi^2(k)} &= e^{-2\alpha n^r} \sum_{k=1}^{n-1} e^{2\alpha k^r} \leq e^{-\alpha n^r} \left(e^{2\alpha(n-1)^r} + \int_1^{n-1} e^{2\alpha t^r} dt \right) \\ &\leq e^{-2\alpha n^r} \left(e^{2\alpha(n-1)^r} \left(1 + \frac{1}{\alpha r(n-1)^{r-1}} + e^{-2\alpha(n-1)^r} \max \left\{ e^{4\alpha}, \frac{e^2}{\alpha^{1+1/r}} \right\} \right) \right) \\ &\leq e^{-2\alpha(n-1)^r((1+\frac{1}{n-1})^r-1)} \gamma_{\alpha,r,n} \leq \gamma_{\alpha,r,n} e^{-2\alpha r(n-1)^{r-1}}. \end{aligned} \quad (51)$$

Thus, by virtue of the left part of the inequality (9) of Theorem 1 and (51), we obtain a required estimate from below for widths $P_{2n}(C_{\beta,2}^{\alpha,r}, C)$, $\alpha > 0$, $r > 1$, under condition (42)

$$P_{2n}(C_{\beta,2}^{\alpha,r}, C) \geq \frac{1}{\sqrt{\pi}} \left(e^{2\alpha n^r} + 2 \sum_{k=1}^{n-1} e^{2\alpha k^r} \right)^{-\frac{1}{2}} \geq \frac{1}{\sqrt{\pi}} e^{-\alpha n^r} \left(1 - \frac{2\gamma_{\alpha,r,n} e^{-2\alpha r(n-1)^{r-1}}}{1 + 2\gamma_{\alpha,r,n} e^{-2\alpha r(n-1)^{r-1}}} \right)^{\frac{1}{2}}. \quad (52)$$

As was shown in [30, p. 163–164],

$$\sum_{k=n+1}^{\infty} e^{-\alpha k^r} < e^{-\alpha n^r} \left(1 + \frac{1}{\alpha r n^{r-1}} \right) e^{-\alpha r n^{r-1}}, \quad r > 1, \alpha > 0, n \in \mathbb{N}.$$

Therefore,

$$\frac{1}{\psi^2(n)} \sum_{k=n+1}^{\infty} \psi^2(k) = e^{2\alpha n^r} \sum_{k=n+1}^{\infty} e^{-2\alpha k^r} < \left(1 + \frac{1}{2\alpha r n^{r-1}} \right) e^{-2\alpha r n^{r-1}}. \quad (53)$$

Thus, by virtue of the right part of the inequality (9) of Theorem 1 and the formula (53) we get the estimate

$$P_{2n}(W_{\tilde{\beta},2}^r, C) \leq \frac{1}{\sqrt{\pi}} e^{-\alpha n^r} \left(1 + e^{-2\alpha r n^{r-1}} \left(1 + \frac{1}{2\alpha r n^{r-1}} \right) \right)^{\frac{1}{2}}. \quad (54)$$

Combining the estimates (52) and (54) we obtain (43). Theorem 4 is proved. $\square$

Taking the limit as $n \rightarrow \infty$ in the relations (43), we obtain the following statement.

Theorem 5. *Let $\tilde{\beta} = \{\beta_k\}_{k=1}^{\infty}$, $\beta_k \in \mathbb{R}$, $\alpha > 0$, $r > 1$, $n \in \mathbb{N}$ and the condition (42) is satisfied. Then as $n \rightarrow \infty$ the following asymptotic equalities*

$$\left. \begin{aligned} P_{2n}(C_{\tilde{\beta},2}^{\alpha,r}, C) \\ P_{2n-1}(C_{\tilde{\beta},2}^{\alpha,r}, C) \end{aligned} \right\} = e^{-\alpha n^r} \left(\frac{1}{\sqrt{\pi}} + \mathcal{O}(1) \gamma_{\alpha,r,n} e^{-\alpha r (n-1)^{r-1}} \right) \quad (55)$$

hold, where P_N is any of the widths b_N, d_N, λ_N or π_N and $\gamma_{\alpha,r,n}$ is defined by (44) and $\mathcal{O}(1)$ are the quantities uniformly bounded in all parameters.

Note that the Theorem 5 complements the results of the works [17, 21, 22, 28, 32], which contain exact estimates for the widths of the classes of convolutions with classical or generalized Poisson kernels. As it follows from the proofs of Theorems 4 and 5 the asymptotic equalities for widths in (55) are realized by trigonometric Fourier sums. The asymptotic equalities for deviations of Fourier sums on classes of generalized Poisson integrals $C_{\beta,p}^{\alpha,r}$ in the uniform metric are seen, for example, in [20, 27, 31] and others.

References

- [1] Akhiezer N.I., Krein M.G. *On the best approximation, by trigonometric sums, of differentiable periodic functions.* Dokl. AN SSSR 1937, **15** (3), 107–112. (in Russian)
- [2] Bushanskii A.V. *Best harmonic approximation in the mean of certain functions.* In: Studies in the theory of approximation of functions and their applications. Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1978, 29–37. (in Russian)
- [3] Dzyadyk V.K. *Best approximation on classes of periodic functions defined by kernels which are integrals of absolutely monotone functions.* Izv. Akad. Nauk SSSR Ser. Mat. 1959, **23** (6), 933–950. (in Russian)
- [4] Dzyadyk V.K. *Best approximation in classes of periodic functions defined by integrals of linear combinations of absolutely monotonous kernels.* Mat. Zametki 1974, **16** (5), 691–701.

- [5] Hrabova U.Z., Serdyuk A.S. *Order estimates for the best approximations and approximations by Fourier sums of the classes of (ψ, β) -differential functions*. Ukrainian Math. J. 2014, **65** (9), 1319–1331. doi:10.1007/s11253-014-0861-7
- [6] Favard J. *Sur l'approximation des fonctions périodiques par des polynômes trigonométriques*. C. R. Math. Acad. Sci. Paris 1936, **203**, 1122–1124. (in French)
- [7] Favard J. *Sur les meilleurs procédés d'approximations de certaines classes de fonctions par des polynômes trigonométriques*. Bull. Sci. Math. 1937, **61**, 209–224, 243–256. (in French)
- [8] Korneichuk N.P. *Exact constants in approximation theory*. Encyclopedia of Mathematics and its Applications, 38. Cambridge University Press, Cambridge, 1991.
- [9] Korneichuk N.P., Babenko V.F., Ligun A.A. *Extremal properties of polynomials and splines*. Naukova Dumka, Kiev, 1992. (in Russian)
- [10] Krein M.G. *The theory of best approximation of periodic functions*. Dokl. AN SSSR 1938, **18** (4–5), 245–249. (in Russian)
- [11] Kushpel' A.K. *Estimates for the widths of classes of analytic functions*. Ukrainian Math. J. 1989, **41** (4), 493–496.
- [12] Sz.-Nagy B. *Über gewisse Extremalfragen bei transformierten trigonometrischen Entwicklungen. I. Periodischer Fall*. Ber. Verh. Sächs. Akad. Wiss. Leipzig 1938, **90**, 103–134. (in German)
- [13] Pinkus A. *n-Widths in approximation theory*. In: Ambrosio L. (Ed.) *Ergebnisse der Mathematik und ihrer Grenzgebiete. Folge 3*. Springer-Verlag, Berlin, 1985.
- [14] Romanyuk A.S. *Approximating Characteristics of the Classes of Periodic Functions of Many Variables*. Proceedings of the Institute of Mathematics of the National Academy of Sciences of Ukraine. Kyiv, 2012. (in Russian)
- [15] Serdyuk A.S. *On the best approximation of classes of convolutions of periodic functions by trigonometric polynomials*. Ukrainian Math. J. 1995, **47** (9), 1435–1440. doi:10.1007/BF01057518
- [16] Serdyuk A.S. *Estimates for the widths and best approximations of classes of convolutions of periodic functions*. Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos. 1998, **20**, 286–299. (in Ukrainian)
- [17] Serdyuk A.S. *Widths and best approximations for classes of convolutions of periodic functions*. Ukrainian Math. J. 1999, **51** (5), 748–763. doi:10.1007/BF02591709
- [18] Serdyuk A.S. *On best approximation in classes of convolutions of periodic functions*. Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos. 2002, **35**, 172–194. (in Ukrainian)
- [19] Serdyuk A.S. *Best approximations and widths of classes of convolutions of periodic functions of high smoothness*. Ukrainian Math. J. 2005, **57** (7), 1120–1148. doi:10.1007/s11253-005-0251-2
- [20] Serdyuk A.S. *Approximation of classes of analytic functions by Fourier sums in the uniform metric*. Ukrainian Math. J. 2005, **57** (8), 1275–1296. doi:10.1007/s11253-005-0261-0
- [21] Serdyuk A.S., Bodenchuk V.V. *Exact values of Kolmogorov widths of classes of Poisson integrals*. J. Approx. Theory 2013, **173**, 89–109. doi:10.1016/j.jat.2013.05.002
- [22] Serdyuk A.S., Sokolenko I.V. *Asymptotic behavior of best approximations of classes of Poisson integrals of functions from H_ω* . J. Approx. Theory 2011, **163** (11), 1692–1706. doi:10.1016/j.jat.2011.06.008
- [23] Serdyuk A.S., Sokolenko I.V. *Uniform approximation of classes of $(\psi, \bar{\beta})$ -differentiable functions by linear methods*. Zb. prats Inst. mat. NAN Ukr. Kyiv 2011, **8** (1), 181–189. (in Ukrainian)
- [24] Serdyuk A.S., Sokolenko I.V. *Asymptotic estimates for the best uniform approximations of classes of convolution of periodic functions of high smoothness*. J. Math. Sci. (N.Y.) 2021, **252** (4), 526–540. doi:10.1007/s10958-020-05178-1
- [25] Serdyuk A.S., Stepanyuk T.A. *Estimations of the best approximations for the classes of infinitely differentiable functions in uniform and integral metrics*. Ukrainian Math. J. 2015, **66** (9), 1393–1407. doi:10.1007/s11253-015-1018-z

- [26] Serdyuk A.S., Stepanyuk T.A. *Order estimates for the best approximations and approximations by Fourier sums in the classes of convolutions of periodic functions of low smoothness in the uniform metric*. Ukrainian Math. J. 2015, **66** (12), 1862–1882. doi:10.1007/s11253-015-1056-6
- [27] Serdyuk A.S., Stepanyuk T.A. *Uniform Approximations by Fourier Sums in Classes of Generalized Poisson Integrals*. Anal. Math. 2019, **45** (1), 201–236. doi:10.1007/s10476-018-0310-1
- [28] Shevaldin V.T. *Widths of classes of convolutions with Poisson kernel*. Math. Notes 1992, **51** (6), 611–617. doi:10.1007/BF01263308
- [29] Stechkin S.B. *On the best approximation of certain classes of periodic functions by trigonometric polynomials*. Izv. Akad. Nauk SSSR. Ser. Mat. 1956, **20**, 643–648. (in Russian)
- [30] Stepanets A.I. *Classification and Approximation of Periodic Functions*. In: Hazewinkel M. (Ed.) *Mathematics and Its Applications*, 333. Kluwer Academic Publishers, Dordrecht, 1995.
- [31] Stepanets A.I. *Methods of Approximation Theory*. Utrecht, VSP, 2005.
- [32] Stepanets A.I., Serdyuk A.S. *Lower bounds for the widths of classes of convolutions of periodic functions in the metrics of C and L* . Ukrainian Math. J. 1995, **47** (8), 1271–1282. doi:10.1007/BF01057715
- [33] Sun Y.-S. *On the best approximation of periodic differentiable functions by trigonometric polynomials. II*. Izv. Akad. Nauk SSSR. Ser. Mat. 1961, **25** (1), 143–152. (in Russian)
- [34] Temlyakov V.N. *On estimates for the widths of classes of infinitely differentiable functions*. Mat. Zametki 1990, **47** (5), 155–157. (in Russian)
- [35] Temlyakov V.N. *Approximation of periodic functions*. Comput. Math. Anal. Ser., Nova Science Publishers, Inc. Commack, NY, 1993.
- [36] Tikhomirov V.M. *Some questions in approximation theory*. Izdat. Moskov. Univ., Moscow, 1976. (in Russian)

Received 06.04.2023

Сердюк А.С., Соколенко І.В. Асимптотичні оцінки поперечників класів функцій високої гладкості // Карпатські матем. публ. — 2023. — Т.15, №1. — С. 246–259.

Знайдено двосторонні оцінки колмогоровських, берштейнівських, лінійних та проекційних поперечників класів у просторі згорток 2π -періодичних функцій φ таких, що $\|\varphi\|_2 \leq 1$, із довільними твірними ядрами $\Psi_{\vec{\beta}}$, ряд Фур'є яких має вигляд $\sum_{k=1}^{\infty} \psi(k) \cos(kt - \beta_k \pi/2)$, де $\psi(k) \geq 0$, $\sum \psi^2(k) < \infty$, $\beta_k \in \mathbb{R}$. Показано, що для швидко спадних послідовностей $\psi(k)$ (зокрема, таких, що $\lim_{k \rightarrow \infty} \psi(k+1)/\psi(k) = 0$) одержані оцінки є асимптотичними рівностями. Встановлено, що асимптотичні рівності для поперечників зазначених класів реалізують тригонометричні суми Фур'є.

Ключові слова і фрази: берштейнівський поперечник, колмогоровський поперечник, лінійний поперечник, проекційний поперечник, сума Фур'є, клас Вейля-Надя, клас узагальнених інтегралів Пуассона, $(\psi, \vec{\beta})$ -інтеграл, асимптотична рівність.