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A novel paranormed sequence space derived by Schröder conservative matrix

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In this paper, we offer a new paranormed sequence space $\ell(\tilde{S}, p)$ by utilizing the Schröder conservative matrix and show that this new space is linearly isomorphic to $\ell(p)$. Moreover, we give its Schauder basis and determine some special duals such as α-, β-, γ-duals.

Key words and phrases: Schröder number, paranormed sequence space, conservative matrix, α-, β-, γ-duals.

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1 Preliminaries and motivations

A sequence space is a linear subspace of the set ω of all real valued sequences. We denote by ℓ_∞, c, c_0 and ℓ_p the set of all bounded sequences, the set of all convergent sequences, the set of all convergent to zero sequences and the set of all sequences constituting p-absolutely convergent series, respectively. These are Banach spaces with the following norms

$$\|x\|_{\ell_\infty} = \|x\|_c = \|x\|_{c_0} = \sup_{n\in\mathbb{N}} |x_n|$$

and

$$\|x\|_{\ell_p} = \Big(\sum_n |x_n|^p\Big)^{1/p}.$$

For brevity, here and in what follows, the summation without limits runs from 0 to ∞.

A linear topological space X over the real field $\mathbb{R}$ is referred to as paranormed space if there is a subadditive function $g : X \to \mathbb{R}$ such that $g(\theta) = 0$, $g(-x) = g(x)$ and scalar multiplication is continuous, i.e. $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all $\alpha \in \mathbb{R}$ and $x \in X$. Here, θ is zero vector of X.

Assume from now on that (p_n) is a bounded sequence in the set of real numbers $\mathbb{R}$ with $p_n > 0$, $\sup_{n\in\mathbb{N}} p_n = P$ and $S = \max\{1, P\}$. For any $\xi \in \mathbb{R}$ and $n \in \mathbb{N} = \{1, 2, \dots\}$, from [29] we have

$$|\xi|^{p_n} \le \max\{1, |\xi|^S\}. \tag{1}$$

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Throughout the paper, we assume that $p_n^{-1} + (p_n')^{-1} = 1$ provided $1 < \inf p_n < P < \infty$. Then I.J. Maddox [29,30] (see also [39,41]) defined the linear space $\ell(p)$ as

$$\ell(p) = \Big\{ x = (x_n) \in \omega : \sum_n |x_n|^{p_n} < \infty \Big\}$$

with the complete space paranormed by

$$g_p(x) = \Big(\sum_n |x_n|^{p_n} \Big)^{1/S}.$$

Let (X, g) be a paranormed space. A sequence $\delta_k \in X$ is a Schauder basis for X if and only if for every $x \in X$ there exists a unique sequence of scalars ε_n such that $g\Big(x - \sum_{k=0}^{n} \varepsilon_k \delta_k\Big) \to 0$ as $n \to \infty$. Then we write

$$x = \sum_k \varepsilon_k \delta_k.$$

The multiplier space of X and Y is the set $S(X, Y)$ defined by

$$S(X, Y) = \big\{ x \in \omega : zx \in Y \text{ for all } x \in X \big\}.$$

Adopting this notation, the α-dual, β-dual and γ-dual of a sequence space X are, respectively, defined by

$$X^\alpha = S(X, \ell_1), \quad X^\beta = S(X, cs) \quad \text{and} \quad X^\gamma = S(X, bs).$$

Here, cs and bs correspond to the spaces of sequences with convergent and bounded series, respectively.

Let $T = (t_{mn})$ be an infinite matrix with real entries t_{mn} and T_m be the sequence in the mth row of the matrix T for each $m \in \mathbb{N}$. The T-transform of a sequence $x = (x_n) \in \omega$ is the sequence Tx obtained by the usual matrix product and its entries are stated as

$$(Tx)_m = \sum_n t_{mn} x_n$$

provided that the series is convergent for each $n \in \mathbb{N}$. T is called a matrix mapping from a sequence space X to a sequence space Y if the sequence Tx exists and $Tx \in Y$ for all $x \in X$. The collection of all infinite matrices from X to Y is denoted by (X, Y).

Recall that the set

$$X_T = \{ x \in \omega : Tx \in Y \}$$

is referred to as the domain of the infinite matrix T in the space X. The literature concerned with producing sequence spaces by means of matrix domain of a special limitation method and studying their some algebraic, topological features, and matrix transformations has recently enlarged. In this respect, some important sequence spaces were introduced and matrix mappings on them were studied by several authors. See, for example, [3–5, 8, 10–12, 14, 15, 23, 32, 37, 38] and plenty of references therein. In particular, for the papers concerning with paranormed space, we can refer to [1,2,6,9,13,19,24,28,33–36,40,42,43,47].

Recently, some remarkable numbers such as Catalan, Fibonacci, tribonacci, Bell and some number theoretic functions such as Euler totient and Jordan totient have been considered by introducing a conservative matrix. Also, its matrix domain in several famous sequence spaces of functional analysis, some special duals and some matrix mappings are discussed. The reader can consult [17, 18, 20–22, 25–27, 44–46].

Let us focus on another special number, called as Schröder number S_n and located in a very important position in combinatorics and number theory. These can be defined by the recursive formula

$$S_{n+1} = S_n + \sum_{k=0}^{n} S_k S_{n-k}, \quad n \geq 0,$$

with initial condition $S_0 = 1$. The definition of these numbers can also be given combinatorially as follows: Schröder number S_n counts the number of paths from the southwest corner $(0,0)$ of an $n \times n$ grid to the northeast corner (n,n), using only single steps north, northeast, or east, that do not rise above the southwest–northeast diagonal. The first ten Schröder numbers S_n are

$$1, 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098, 1037718.$$

Also, the Schröder numbers S_n possess the following representation via hypergeometric function as

$$S_n = 2 {}_2F_1\left(-n+1, n+2; 2; -1\right),$$

where the hypergeometric function ${}_2F_1\left(a, b; c; z\right)$ is defined by

$${}_2F_1\left(a, b; c; z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

with the Pochhammer symbol $(x)_n = x(x+1)\cdots(x+n-1)$ for $n \geq 1$, and $(x)_n = 1$ for $n = 0$. For details and applications, we may refer to [7].

Quite recently, the author in [11] introduced a new matrix $\tilde{S} = \left(\tilde{S}_{nk}\right)$, whose entries are Schröder numbers S_n as

$$\tilde{S}_{nk} = \begin{cases} \dfrac{S_k S_{n-k}}{S_{n+1} - S_n}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{if } k > n, \end{cases}$$

and computed its inverse as

$$\tilde{S}^{-1}_{nk} = \begin{cases} (-1)^{n-k} \dfrac{S_{k+1} - S_k}{S_n} P_{n-k}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{if } k > n, \end{cases}$$

where P_n denotes the following determinant

$$\begin{vmatrix} S_1 & S_0 & 0 & \dots & 0 \\ S_2 & S_1 & S_0 & \dots & 0 \\ S_3 & S_2 & S_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_n & S_{n-1} & S_{n-2} & \dots & S_1 \end{vmatrix}$$

for all $n \in \mathbb{N} \setminus \{0\}$ and $P_0 = 1$. It is also shown that this matrix is a conservative matrix.

Following that, in [12], new sequence spaces $\ell_p\left(\tilde{S}\right)$ and $\ell_\infty\left(\tilde{S}\right)$ are defined as

$$\ell_p(\tilde{S}) = \left\{z = (z_n) \in \omega : \sum_{n=0}^{\infty}\left|\frac{1}{S_{n+1}-S_n}\sum_{k=0}^{n}S_kS_{n-k}z_k\right|^p < \infty\right\}$$

and

$$\ell_\infty\left(\tilde{S}\right) = \left\{z = (z_n) \in \omega : \sup_n\left|\frac{1}{S_{n+1}-S_n}\sum_{k=0}^{n}S_kS_{n-k}z_k\right| < \infty\right\}.$$

The sequence t_n represents the $\tilde{S}$-transform of a sequence $z = (z_n)$ as

$$t_n = \tilde{S}_n\left(z\right) = \frac{1}{S_{n+1}-S_n}\sum_{k=0}^{n}S_kS_{n-k}z_k \tag{2}$$

for all $n \in \mathbb{N}$, while the following relation is valid

$$z_k = \sum_{i=0}^{k}\left((-1)^{k-i}\frac{S_{i+1}-S_i}{S_k}P_{k-i}\right)t_i$$

for all $k \in \mathbb{N}$.

In this paper, we extend the normed sequence space $\ell_p(\tilde{S})$ to paranormed space $\ell(\tilde{S}, p)$ and show that the paranormed space $\ell(\tilde{S}, p)$, newly defined here, is linearly isomorphic to $\ell(p)$. Furthermore, its Schauder basis is given and the α-dual, β-dual and γ-dual of this space are determined.

2 The paranormed sequence space $\ell(\tilde{S}, p)$

This section is devoted to introduce a new paranormed space $\ell\left(\tilde{S}, p\right)$ by means of the Schröder numbers, to demonstrate the completeness of the resulting paranormed space, and to present the Schauder basis of $\ell\left(\tilde{S}, p\right)$.

Define the sequence space $\ell\left(\tilde{S}, p\right)$ as

$$\ell\left(\tilde{S}, p\right) = \left\{z = (z_n) \in \omega : \sum_n\left|\frac{1}{S_{n+1}-S_n}\sum_{k=0}^{n}S_kS_{n-k}z_k\right|^{p_n} < \infty\right\}.$$

Note that if $p_n = p$ for all $n \in \mathbb{N}$, the sequence space $\ell\left(\tilde{S}, p\right)$ reduces to the space $\ell_p\left(\tilde{S}\right)$.

We begin to show the completeness of paranormed space $\ell\left(\tilde{S}, p\right)$.

Theorem 1. *With the following paranorm*

$$g_{\tilde{S}}\left(z\right) = \left(\sum_n\left|\frac{1}{S_{n+1}-S_n}\sum_{k=0}^{n}S_kS_{n-k}z_k\right|^{p_n}\right)^{1/S},$$

for all $z = (z_n) \in \ell\left(\tilde{S}, p\right)$, the space $\ell\left(\tilde{S}, p\right)$ is a complete paranormed space.

Proof. For $z = (z_k)$ and $y = (y_k)$ in $\ell\left(\tilde{S}, p\right)$, it follows from [31, p. 30], that

$$\begin{aligned}&\left(\sum_n\left|\frac{1}{S_{n+1}-S_n}\sum_{k=0}^{n}S_kS_{n-k}(z_k+y_k)\right|^{p_n}\right)^{1/S}\\ &\quad\leq\left(\sum_n\left|\frac{1}{S_{n+1}-S_n}\sum_{k=0}^{n}S_kS_{n-k}z_k\right|^{p_n}\right)^{1/S}+\left(\sum_n\left|\frac{1}{S_{n+1}-S_n}\sum_{k=0}^{n}S_kS_{n-k}y_k\right|^{p_n}\right)^{1/S}.\end{aligned} \tag{3}$$

In view of (1) and (3), we see that $\ell\left(\tilde{S}, p\right)$ is linear in accordance with scalar multiplication and the coordinatewise addition. Also, it is clear that $g_{\tilde{S}}\left(\theta\right) = 0$ and $g_{\tilde{S}}\left(-z\right) = g_{\tilde{S}}\left(z\right)$ for all z in $\ell\left(\tilde{S}, p\right)$. We arrive the subadditivity of $g_{\tilde{S}}$ and $g_{\tilde{S}}\left(\xi z\right) \leq \max\left\{1, |\xi|\right\} g_{\tilde{S}}\left(z\right)$ for any $\xi \in \mathbb{R}$ in the light of (1) and (3).

Now, let $\{z^n\}$ be any sequence in $\ell\left(\tilde{S}, p\right)$ such that $g_{\tilde{S}}\left(z^n - z\right) \to 0$ and (ξ_n) be any sequence in $\mathbb{R}$ such that $\xi_n \to \xi$. By aid of the subadditivity of $g_{\tilde{S}}$, one can readily write

$$g_{\tilde{S}}\left(z^n\right) \leq g_{\tilde{S}}\left(z\right) + g_{\tilde{S}}\left(z^n - z\right),$$

from which one can reach the boundedness of $g_{\tilde{S}}\left(z_n\right)$ and the fact

$$\begin{aligned} g_{\tilde{S}}\left(\xi_n z^n - \xi z\right) &= \left(\sum_n \left|\frac{1}{S_{n+1} - S_n} \sum_{k=0}^{n} S_k S_{n-k}\left(\xi_n z_k^n - \xi z_k\right)\right|^{p_n}\right)^{1/S} \\ &\leq |\xi_n - \xi|\, g_{\tilde{S}}\left(z^n\right) + |\xi|\, g_{\tilde{S}}\left(z^n - z\right) \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

which gives the continuity of scalar multiplication. In conclusion, $g_{\tilde{S}}$ is paranorm on $\ell\left(\tilde{S}, p\right)$. For the purpose of the completeness of $\ell\left(\tilde{S}, p\right)$, consider any Cauchy sequence $\left\{u^i\right\}$ in $\ell\left(\tilde{S}, p\right)$ such that $u^i = \left(u_1^i, u_2^i, \dots\right)$ for each $i \in \mathbb{N}$. For a given $\varepsilon > 0$, there exists an integer $n_0\left(\varepsilon\right) \in \mathbb{N}$ such that

$$g_{\tilde{S}}\left(u^i - u^j\right) < \varepsilon \tag{4}$$

for all $i, j \geq n_0\left(\varepsilon\right)$. Using the definition of $g_{\tilde{S}}$, we have

$$\tilde{S}_n\left(u^i - u^j\right) \leq \left(\sum_n \left|\tilde{S}_n\left(u^i\right) - \tilde{S}_n\left(u^j\right)\right|^{p_n}\right)^{1/S} < \varepsilon$$

for each $i, j \geq n_0\left(\varepsilon\right)$, which yields that $\left\{\tilde{S}_n\left(u^1\right), \tilde{S}_n\left(u^2\right), \dots\right\}$ is a Cauchy sequence of real numbers for each fixed $n \in \mathbb{N}$. Since $\mathbb{R}$ is complete, we have $\tilde{S}_n\left(u^i\right) \to \tilde{S}_n\left(u\right)$ as $i \to \infty$ for every fixed $n \in \mathbb{N}$. Taking into consideration these infinitely many limits $\tilde{S}_1\left(u\right), \tilde{S}_2\left(u\right), \dots$, let us define the sequence $\left\{\tilde{S}_1\left(u\right), \tilde{S}_2\left(u\right), \dots\right\}$. It follows from (4), that

$$\sum_{n=1}^{m} \left|\tilde{S}_n\left(u^i\right) - \tilde{S}_n\left(u^j\right)\right|^{p_n} \leq g_{\tilde{S}}\left(u^i - u^j\right)^S < \varepsilon^S \tag{5}$$

for all fixed $m \in \mathbb{N}$ and $i, j \geq n_0\left(\varepsilon\right)$. Taking in (5) the limit as $m \to \infty$ and $j \to \infty$, leads us to $g_{\tilde{S}}\left(u^i - u\right) < \varepsilon$. Now, let $\varepsilon = 1$ in (5) and so $i \geq n_0(1)$. Then, applying Minkowski's inequality, for every fixed $m \in \mathbb{N}$ gives

$$\left(\sum_{n=1}^{m} \left|\tilde{S}_n\left(u\right)\right|^{p_n}\right)^{1/S} \leq g_{\tilde{S}}\left(u^i - u\right) + g_{\tilde{S}}\left(u^i\right) \leq 1 + g_{\tilde{S}}\left(u^i\right),$$

from which we arrive at $u \in \ell\left(\tilde{S}, p\right)$. Since $g_{\tilde{S}}\left(u^i - u\right) \leq \varepsilon$ for all $i \geq n_0\left(\varepsilon\right)$, we have $u^i \to u$ as $i \to \infty$. Consequently, the completeness of $\ell\left(\tilde{S}, p\right)$ is proved.

Remark that we do not have absolute property on $\ell\left(\tilde{S}, p\right)$ due to the fact that we can find a sequence u in $\ell\left(\tilde{S}, p\right)$ such that $g_{\tilde{S}}\left(u\right) \neq g_{\tilde{S}}\left(|u|\right)$, where $|u| = \left(|u_n|\right)$. □

Theorem 2. *The spaces $\ell\left(\tilde{S}, p\right)$ and $\ell\left(p\right)$ are linearly isomorphic.*

Proof. We must provide that there exists a linear transformation between these spaces which is injective, surjective and paranorm preserving.

For any $z \in \ell\left(\tilde{S}, p\right)$, let $L : \ell\left(\tilde{S}, p\right) \to \ell\left(p\right)$ be a transformation such that $L\left(z\right) = \left(\tilde{S}_n\left(z\right)\right)$. The linearity of L is clear by using the fact that any matrix transformation is linear. Also, the transformation L is injective from the fact that if $L\left(z\right) = \theta$, then $z = \theta$. For any sequence $t = \left(t_n\right) \in \ell\left(p\right)$, if the sequence $z = \left(z_n\right)$ is denoted for $n \in \mathbb{N}$ by

$$z_n = \sum_{k=0}^{n}\left(\left(-1\right)^{n-k}\frac{S_{k+1}-S_k}{S_n}P_{n-k}\right)t_k,$$

then we have

$$\begin{aligned}
g_{\tilde{S}}\left(z\right) &= \left(\sum_n\left|\frac{1}{S_{n+1}-S_n}\sum_{k=0}^{n}S_kS_{n-k}z_k\right|^{p_n}\right)^{1/S}\\
&= \left(\sum_n\left|\frac{1}{S_{n+1}-S_n}\sum_{k=0}^{n}S_kS_{n-k}\sum_{i=0}^{k}\left(\left(-1\right)^{k-i}\frac{S_{i+1}-S_i}{S_k}P_{k-i}\right)t_i\right|^{p_n}\right)^{1/S}\\
&= \left(\sum_n\left|\frac{1}{S_{n+1}-S_n}\sum_{i=0}^{n}\left(\sum_{k=0}^{n-i}\left(-1\right)^kS_{n-k-i}P_k\right)\left(S_{i+1}-S_i\right)t_i\right|^{p_n}\right)^{1/S}\\
&= \left(\sum_n\left|t_n\right|^{p_n}\right)^{1/S}\\
&= g_{\tilde{S}}\left(t\right) < \infty,
\end{aligned}$$

from which we get $z \in \ell\left(\tilde{S}, p\right)$. Thus, L is surjective and paranorm preserving, as required. So, the proof is complete. ◻

Now, let us present a Schauder basis of the space $\ell\left(\tilde{S}, p\right)$.

Theorem 3. *Define a sequence $s^{(k)} = \left(s_n^{(k)}\right)$ in $\ell\left(\tilde{S}, p\right)$ as*

$$s_n^{(k)} = \begin{cases}\left(-1\right)^{n-k}\dfrac{S_{k+1}-S_k}{S_n}P_{n-k}, & \text{if } \ 0 \le k \le n,\\ 0, & \text{if } \ k > n,\end{cases}$$

where $n \in \mathbb{N}$ is fixed.

Then $\left\{s^{(0)}, s^{(1)}, \ldots\right\}$ is a Schauder basis for the space $\ell\left(\tilde{S}, p\right)$ and any z in $\ell\left(\tilde{S}, p\right)$, uniquely determined as

$$z = \sum_k t_k s^{(k)},$$

where $t_k = \tilde{S}_k\left(z\right)$.

Proof. As is demonstrated in Theorem 2 the isomorphism $L : \ell\left(\tilde{S}, p\right) \to \ell\left(p\right)$ is surjective. Since the inverse image of the Schauder basis of the space $\ell\left(p\right)$ is a Schauder basis of the space $\ell\left(\tilde{S}, p\right)$, the assertion is clear. ◻

3 The α-, β- and γ-duals of the space $\ell(\tilde{S}, p)$

In this section, we give α-dual, β-dual and γ-dual of $\ell(\tilde{S}, p)$. We need the following lemmas.

Lemma 1 ([16])**.** *Let $1 < p_n \leq P < \infty$ for all $n \in \mathbb{N}$. Then $T = (t_{nk}) \in (\ell_p, \ell_1)$ if and only if*

$$\sup_{N \in \mathcal{F}} \sum_k \left| \sum_{n \in N} t_{nk} K^{-1} \right|^{p_n'} < \infty \tag{6}$$

for some integer $K > 1$. Here $\mathcal{F}$ denotes the family of all finite subsets of $\mathbb{N}$.

Let $0 < p_n \leq 1$ for all $n \in \mathbb{N}$. Then $T = (t_{nk}) \in (\ell_p, \ell_1)$ if and only if

$$\sup_{N \in \mathcal{F}} \sup_k \left| \sum_{n \in N} t_{nk} \right|^{p_n} < \infty.$$

Lemma 2 ([16])**.** *Let $1 < p_n \leq P < \infty$ for all $n \in \mathbb{N}$. Then $T = (t_{nk}) \in (\ell_p, \ell_\infty)$ if and only if*

$$\sup_n \sum_k \left| t_{nk} K^{-1} \right|^{p_n'} < \infty \tag{7}$$

for some integer $K > 1$.

Let $0 < p_n \leq 1$ for all $n \in \mathbb{N}$. Then $T = (t_{nk}) \in (\ell_p, \ell_\infty)$ if and only if

$$\sup_{n,k \in \mathbb{N}} |t_{nk}|^{p_n} < \infty. \tag{8}$$

Lemma 3 ([16])**.** *Let $0 < p_n \leq P < \infty$ for all $n \in \mathbb{N}$. Then $T = (t_{nk}) \in (\ell_p, c)$ if and only if (7) and (8) hold and*

$$\lim_n t_{nk} = c_k, \quad k \in \mathbb{N}, \tag{9}$$

for some c_k.

Theorem 4. *Define the following sets*

$$B_1 = \left\{ b = (b_n) \in \omega : \sup_{N \in \mathcal{F}} \sup_n \left| \sum_{i \in \mathbb{N}} (-1)^{i-n} \frac{S_{n+1} - S_n}{S_i} P_{i-n} b_i \right|^{p_n} < \infty \right\}$$

and

$$B_2 = \bigcup_{K>1} \left\{ b = (b_n) \in \omega : \sup_{N \in \mathcal{F}} \sum_n \left| \sum_{i \in \mathbb{N}} (-1)^{i-n} \frac{S_{n+1} - S_n}{S_i} P_{i-n} b_i K^{-1} \right|^{p_n'} < \infty \right\}.$$

Then the α-dual of the space $\ell\left(\tilde{S}, p\right)$ is as follows

$$\left\{ \ell(\tilde{S}, p) \right\}^\alpha = \begin{cases} B_1, & \text{if } 0 < p_n \leq 1 \text{ for all } n \in \mathbb{N}, \\ B_2, & \text{if } 1 < p_n \leq P < \infty \text{ for all } n \in \mathbb{N}. \end{cases}$$

Proof. It is sufficient to prove only the second case since the first is similar. For any $b = (b_n) \in \omega$, define a matrix $\mathcal{A} = (a_{nk})$ by

$$a_{nk} = \begin{cases} (-1)^{n-k} \dfrac{S_{k+1} - S_k}{S_n} P_{n-k} b_n, & \text{if } 0 \leq k \leq n, \\ 0, & \text{if } k > n. \end{cases}$$

Using the $\tilde{S}$-transform given by (2) yields for all $n \in \mathbb{N}$ that

$$\mathcal{A}_n t = \sum_{k=0}^{n} (-1)^{n-k} \frac{S_{k+1} - S_k}{S_n} P_{n-k} b_n t_k = b_n z_n. \tag{10}$$

Combining (10) and (6) shows that $bz \in \ell_1$ for $z \in \ell\left(\tilde{S}, p\right)$ if and only if $\mathcal{A}t \in \ell_1$ for $t \in \ell_p$. So, $b = (b_n) \in \left\{\ell(\tilde{S}, p)\right\}^{\alpha}$ if and only if $\mathcal{A} \in \left(\ell_p, \ell_1\right)$, which means that $\left\{\ell(\tilde{S}, p)\right\}^{\alpha} = B_2$, as desired. □

By employing Lemma 2, we derive the following theorem.

Theorem 5. *Define the sets B_3, B_4 and B_5 by*

$$B_3 = \bigcup_{K>1} \left\{ b = (b_n) \in \omega : \sup_i \sum_{n=1}^{i} \left| \sum_{i=0}^{n} (-1)^{i-n} \frac{S_{n+1} - S_n}{S_i} P_{i-n} b_n K^{-1} \right|^{p_n'} \right\},$$

$$B_4 = \left\{ b = (b_n) \in \omega : \sup_{i,n \in \mathbb{N}} \left| \sum_{n=0}^{i} (-1)^{i-n} \frac{S_{n+1} - S_n}{S_i} P_{i-n} b_n \right|^{p_n} \right\}$$

and

$$B_5 = \left\{ b = (b_n) \in \omega : \lim_i \sum_{n=0}^{i} (-1)^{i-n} \frac{S_{n+1} - S_n}{S_i} P_{i-n} b_i \text{ exists for } n \in \mathbb{N} \right\}.$$

Then

$$\left\{\ell(\tilde{S}, p)\right\}^{\beta} = B_3 \cup B_4 \cup B_5.$$

Proof. Consider $b = (b_n) \in \omega$ and apply the $\tilde{S}$-transform given by (2) to get

$$\begin{aligned} \sum_{n=1}^{i} b_n z_n &= \sum_{n=1}^{i} b_n \sum_{j=0}^{n} \left((-1)^{n-j} \frac{S_{j+1} - S_j}{S_n} P_{n-j} \right) t_j \\ &= \sum_{n=1}^{i} \left(\sum_{j=0}^{n} (-1)^{n-j} \frac{S_{j+1} - S_j}{S_n} P_{n-j} b_n \right) t_j. \end{aligned} \tag{11}$$

Let $\mathcal{D}$ denotes the matrix given for all $i, n \in \mathbb{N}$ by

$$d_{in} = \begin{cases} \sum_{j=n}^{i} (-1)^{i-j} \frac{S_{j+1} - S_j}{S_i} P_{i-j} b_i, & \text{if } 0 \leq n \leq i, \\ 0, & \text{if } n > i. \end{cases}$$

Using Lemma 3 and equation (11) yields that $bz = (b_n z_n)$ in cs if and only if $\mathcal{D}t$ belongs to c whenever $t = (t_n) \in \ell\,(p)$. From this, we have $b = (b_n) \in \left\{ \ell\left(\tilde{S}, p\right) \right\}^{\beta}$ if and only if $\mathcal{D} \in \left(\ell_p, c\right)$. Thus, from (7) and (9), we reach the desired result. □

Theorem 6.

$$\left\{\ell(\tilde{S}, p)\right\}^{\gamma} = \begin{cases} B_4, & \text{if } 0 < p_n \leq 1 \text{ for all } n \in \mathbb{N}, \\ B_3, & \text{if } 1 < p_n \leq P < \infty \text{ for all } n \in \mathbb{N}. \end{cases}$$

Proof. Proceeding with the arguments as in the proof of Theorem 5, but using Lemma 2 instead of Lemma 3, completes the proof. □

4 Conclusions

For decades, by introducing of a new sequence space via the domain of a special infinite matrix, numerous considerable properties containing special duals and some matrix transformations have been investigated by many scholars. In recent papers, these topics have been studied with respect to number theoretic functions and some special numbers, and consequently, various meaningful and interesting conclusions are revealed.

In this article, as a continuation of the studies [11, 12], we consider a new paranormed space arising from a conservative matrix involving Schröder numbers. Also, we show that this paranormed space is linearly isomorphic to $\ell(p)$. Additionally, we offer a Schauder basis and some special duals of the resulting space.

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Даґлі М.С. *Новий паранормований простір послідовностей, породжений консервативною матрицею Шредера* // Карпатські матем. публ. — 2025. — Т.17, №1. — С. 235–245.

У цій статті ми пропонуємо новий паранормований простір послідовностей $\ell(\tilde{S}, p)$, побудований за допомогою консервативної матриці Шредера, і доводимо, що цей новий простір лінійно ізоморфний до простору $\ell(p)$. Крім того, ми подаємо базис Шаудера для цього простору та визначаємо деякі спеціальні спряжені простори, такі як α-, β-, γ-спряжені.

Ключові слова і фрази: число Шредера, паранормований простір послідовностей, консервативна матриця, α-, β-, γ-спряжені простори.