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$SO(3)$ quasi-monomial polynomial families

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Let H be a subgroup of the affine space group $\mathrm{Aff}(3)$, considered with its natural action on the vector space of three-variable polynomials. The polynomial family $\{B_{m,n,k}(x,y,z)\}$ is called quasi-monomial with respect to H if the group operators in two different bases $\{x^m y^n z^k\}$ and $\{B_{m,n,k}(x,y,z)\}$ have identical matrices. We derive a criterion for quasi-monomiality when the group H is the special orthogonal group $SO(3)$. This criterion is expressed through the exponential generating function of the polynomial family $\{B_{m,n,k}(x,y,z)\}$. It has been proven that Appel's biorthogonal polynomials are quasi-monomials with respect to $SO(3)$ and recurrence relations have been found for them.

Key words and phrases: quasi-monomial polynomial, special orthogonal group, Appel's biorthogonal polynomial, recurrence relation.

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Introduction

Let H be a subgroup of the space affine group $\mathrm{Aff}(3)$, considered with the natural action on the vector space of three-variable polynomials. The polynomial family $B_{m,n,k}(x,y,z)$ is called *quasi-monomial* with respect to H if the group operators in two different bases $\left\{x^m y^n z^k\right\}$ and $\{B_{m,n,k}(x,y,z)\}$ have *identical* matrices. Recently, quasi-monomials have found widespread application in the analysis of 2D and 3D images. For the first time, polynomials with the quasi-monomial property appeared in [1]. The authors proved that the family of polynomials $H_{m,n}(x,y) = H_m(x)H_n(y)$, where $H_n(x)$ are classical Hermitian polynomials, is quasi-monomial with respect to the rotation group of the plane $SO(2)$. Thus, this nice property of Hermitian polynomials allowed for the efficient calculation of the $SO(2)$-invariant image moments. Detailed information about image moment invariants can be found in [2,3].

In the article [4], these ideas were developed, and a complete description of all families of polynomials that are quasi-monomials with respect to the rotation group of the plane $SO(2)$ was obtained. Also in that article, it was proven that the biorthogonal Appell polynomials of two variables are quasi-monomials with respect to $SO(2)$, and the recurrence relations for these polynomials were found. Finding recurrence relations for quasi-monomials is very important because, when computing Appell polynomials numerically with explicit formulas, we may encounter precision loss due to floating-point overflow or underflow. Using recurrence relations allows for efficient and stable calculations, which is crucial for applications.

In the paper [5], a description of quasi-monomials was obtained for the case when the group H is generated by scaling, rotations, and translations of the plane. In [6], biorthogonal

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Appell polynomials of three variables were used to recognize 3D objects.

In this article, we offer a comprehensive description, similar to the 2D case, of families of polynomials that are quasi-monomials with respect to the rotation group of the space $SO(3)$. It is proven that a family of polynomials will be quasi-monomials if and only if the exponential generating function of this family is a function of three variables $ux+vy+wz$, $x^2+y^2+z^2$, and $u^2+v^2+w^2$ only. Additionally, we establish conditions under which the scaling of quasi-monomials preserves the quasi-monomial property. We also prove that the biorthogonal Appell polynomials of three variables are $SO(3)$ quasi-monomials and derive recurrence relations for them.

1 $SO(3)$ quasi-monomials

The 3D rotation group $SO(3)$ (the special orthogonal group) is the group of all rotations about the origin of three-dimensional Euclidean space. It is a three-parameters group with the following matrix realization

$$T_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; T_\beta = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix}, \; T_\gamma = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where the parameters $\alpha, \beta, \gamma \in [0, 2\pi]$ are the *Euler angles*. A rotation $T_{\alpha,\beta,\gamma} = T_\alpha T_\beta T_\gamma$ maps a point (x, y, z) to a new point (x', y', z') as follows:

$$\begin{aligned} x' &= (\cos\alpha\cos\gamma - \sin\alpha\cos\beta\sin\gamma)x + (-\cos\alpha\sin\gamma - \sin\alpha\cos\beta\cos\gamma)y + \sin\alpha\sin\beta z, \\ y' &= (\sin\alpha\cos\gamma + \cos\alpha\cos\beta\sin\gamma)\, x + (\cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma)\, y - \cos\alpha\sin\beta\, z, \\ z' &= \sin\beta\sin\gamma\, x + \sin\beta\cos\gamma\, y + \cos\beta\, z. \end{aligned}$$

From the orthogonality of the group, it follows that the transformations of the group preserve the lengths of vectors, therefore, the subsequent identity

$$\left(x'\right)^2 + \left(y'\right)^2 + \left(x'\right)^2 = x^2 + y^2 + z^2$$

holds true. The monomials $x^m y^n z^k$ are transformed under the rotation as

$$T_{\alpha,\beta,\gamma}\left(x^m y^n z^k\right) = \sum_{\substack{m_1+m_2+m_3=m \\ n_1+n_2+n_3=n \\ k_1+k_2+k_3=k}} C\left(\alpha, \beta, \gamma, m_1, \ldots, k_3\right) x^{m_1+n_1+k_1} y^{m_2+n_2+k_2} z^{m_3+n_3+k_3},$$

where $C(\alpha, \beta, \gamma, m_1, \ldots, k_3)$ is some complicated expression, the explicit form of which does not interest us.

We are interested in polynomials that transform under rotation $T_{\alpha,\beta,\gamma}$ in the same way as the monomials $x^m y^n z^k$ do. This leads us to the following definition.

Definition. *The polynomial family* $\{B_{m,n,k}(x, y, z)\}$ *is called quasi-monomial with respect to* $SO(3)$ *if the following identity*

$$T_{\alpha,\beta,\gamma}\left(B_{m,n,k}(x, y, z)\right) = \sum_{\substack{m_1+m_2+m_3=m \\ n_1+n_2+n_3=n \\ k_1+k_2+k_3=k}} C\left(\alpha, \beta, \gamma, m_1, \ldots, k_3\right) B_{m_1+n_1+k_1, m_2+n_2+k_2, m_3+n_3+k_3}$$

holds for all $m, n, k \in \mathbb{N}$.

Note that the coefficients $C(\alpha, \beta, \gamma, m_1, \ldots, k_3)$ here are the same as the coefficients in the expression for $T_{\alpha,\beta,\gamma}\left(x^m y^n z^k\right)$. The following theorem presents a simple criterion for the quasi-monomiality of a polynomial family in terms of its *exponential* generating function.

Theorem 1. *The polynomial family $\{B_{m,n,k}(x,y,z)\}$ defined by the exponential generating function*

$$G = G(x,y,z,u,v,w) = \sum_{m,n,k=0}^{\infty} B_{m,n,k}(x,y,z)\frac{u^m}{m!}\frac{v^n}{n!}\frac{w^k}{k!}$$

is quasi-polynomial if and only if G is a function of the three variables $ux + vy + wz$, $x^2 + y^2 + z^2$ and $u^2 + v^2 + w^2$ only.

Proof. Necessity. We prove the forward implication first. We start with formulating an auxiliary lemma.

Lemma. *Let $B_{m,n,k}(x,y,z)$ is a quasi-monomial family. Then the following identities hold:*

$$x\frac{\partial B_{m,n,k}(x,y,z)}{\partial y} - y\frac{\partial B_{m,n,k}(x,y,z)}{\partial x} = nB_{m+1,n-1,k}(x,y,z) - mB_{m-1,n+1,k}(x,y,z),$$
$$x\frac{\partial B_{m,n,k}(x,y,z)}{\partial z} - z\frac{\partial B_{m,n,k}(x,y,z)}{\partial x} = kB_{m+1,n,k-1}(x,y,z) - mB_{m-1,n,k+1}(x,y,z),$$
$$y\frac{\partial B_{m,n,k}(x,y,z)}{\partial z} - z\frac{\partial B_{m,n,k}(x,y,z)}{\partial y} = kB_{m,n+1,k-1}(x,y,z) - nB_{m,n-1,k+1}(x,y,z).$$

Proof. Since any rotation $T_{\alpha,\beta,\gamma}$ is the product $T_{\alpha,\beta,\gamma} = T_\alpha T_\beta T_\gamma$, we find that the quasi-monomial satisfies the following identity

$$\begin{aligned} T_\alpha\left(B_{m,n,k}(x,y,z)\right) &= T_{\alpha,\beta,\gamma}\left(B_{m,n,k}(x,y,z)\right)\Big|_{\beta=\gamma=0} \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^j\binom{m}{i}\binom{n}{j}(\cos\alpha)^{m-i+j}(\sin\alpha)^{n-j+i}B_{m+n-i-j,i+j,k}(x,y,z). \end{aligned}$$

After differentiating the relation with respect to α at $\alpha = 0$, we find that the polynomial $B_{m,n,k}(x,y,z)$ satisfies the following differential equation

$$x\frac{\partial B_{m,n,k}(x,y,z)}{\partial y} - y\frac{\partial B_{m,n,k}(x,y,z)}{\partial x} = nB_{m+1,n-1,k}(x,y,z) - mB_{m-1,n+1,k}(x,y,z).$$

In a similar way we get rest two equations. □

We use the Lemma to investigate the derivatives of the generating function G. This implies (similarly to the 2D case, see [4]) that G satisfied the following system of partial linear differential equations

$$\begin{cases} x\dfrac{\partial G}{\partial y} - y\dfrac{\partial G}{\partial x} = v\dfrac{\partial G}{\partial u} - u\dfrac{\partial G}{\partial v}, \\ x\dfrac{\partial G}{\partial z} - z\dfrac{\partial G}{\partial x} = w\dfrac{\partial G}{\partial u} - u\dfrac{\partial G}{\partial w}, \\ y\dfrac{\partial G}{\partial z} - z\dfrac{\partial G}{\partial y} = w\dfrac{\partial G}{\partial v} - v\dfrac{\partial G}{\partial w}. \end{cases}$$

In fact, considering the lemma for the first equation, we obtain:

$$\begin{aligned}
x\frac{\partial G}{\partial y} - y\frac{\partial G}{\partial x} &= \sum_{m,n,k=0}^{\infty} \left(x\frac{\partial B_{m,n,k}(x,y,z)}{\partial y} - y\frac{\partial B_{m,n,k}(x,y,z)}{\partial x} \right) \frac{u^m}{m!}\frac{v^n}{n!}\frac{w^k}{k!} \\
&= \sum_{m,n,k=0}^{\infty} \Big(nB_{m+1,n-1,k}(x,y,z) - mB_{m-1,n+1,k}(x,y,z) \Big) \frac{u^m}{m!}\frac{v^n}{n!}\frac{w^k}{k!} \\
&= v\sum_{m,n,k=0}^{\infty} \Big(B_{m+1,n-1,k}(x,y,z) \Big) \frac{u^m}{m!}\frac{v^{n-1}}{(n-1)!}\frac{w^k}{k!} \\
&\quad - u\sum_{m,n,k=0}^{\infty} \Big(B_{m-1,n+1,k}(x,y,z) \Big) \frac{u^{m-1}}{(m-1)!}\frac{v^n}{n!}\frac{w^k}{k!} = v\frac{\partial G}{\partial u} - u\frac{\partial G}{\partial v}.
\end{aligned}$$

Thus, the generating function G satisfied the partial differential equation

$$x\frac{\partial G}{\partial y} - y\frac{\partial G}{\partial x} = v\frac{\partial G}{\partial u} - u\frac{\partial G}{\partial v}.$$

Two other equations are proved similarly. A system of three differential equation that contains a function of six variables cannot have more than three functionally independent solutions, see [7]. But we can point to three such solutions at once: $x^2+y^2+z^2$, $u^2+v^2+w^2$, $xu+yv+zw$. Thus G is a function of $x^2+y^2+z^2$, $u^2+v^2+w^2$, $xu+yv+zw$, which has to be proved.

Sufficiency. Suppose now that $G = G\left(xu+yv+zw, x^2+y^2+z^2, u^2+v^2+w^2\right)$. Then the following identity

$$T_{\alpha,\beta,\gamma}\big(G\big(xu+yv+zw, x^2+y^2+z^2, u^2+v^2+w^2\big)\big) = G\big(x\bar{u}+y\bar{v}+z\bar{w}, x^2+y^2+z^2, \bar{u}^2+\bar{v}^2+\bar{w}^2\big),$$

holds, where

$$\begin{cases}
\bar{u} = (\cos\alpha\cos\gamma - \sin\alpha\cos\beta\sin\gamma)u + (\sin\alpha\cos\gamma + \cos\alpha\cos\beta\sin\gamma)v + \sin\beta\sin\gamma w, \\
\bar{v} = (-\cos\alpha\sin\gamma - \sin\alpha\cos\beta\cos\gamma)\,u + (\cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma)\,v + \sin\beta\cos\gamma w, \\
\bar{w} = \sin\alpha\sin\beta u - \cos\alpha\sin\beta v + \cos\beta w.
\end{cases}$$

In fact, by direct calculation we obtain:

$$\begin{aligned}
&T_{\alpha,\beta,\gamma}(xu+yv+zw) \\
&\quad = ((\cos\alpha\cos\gamma - \sin\alpha\cos\beta\sin\gamma)x + (-\cos\alpha\sin\gamma - \sin\alpha\cos\beta\cos\gamma)y + \sin\alpha\sin\beta z)u \\
&\qquad + ((\sin\alpha\cos\gamma + \cos\alpha\cos\beta\sin\gamma)x + (\cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma)y - \cos\alpha\sin\beta z)v \\
&\qquad + (\sin\beta\sin\gamma x + \sin\beta\cos\gamma y + \cos\beta z)w = x\bar{u} + y\bar{v} + z\bar{w}.
\end{aligned}$$

Note that

$$(\bar{u},\bar{v},\bar{w}) = T_{-\alpha,-\beta,-\gamma}(u,v,w).$$

Also, the orthogonality of $SO(3)$ implies that $u^2+v^2+w^2 = \bar{u}^2+\bar{v}^2+\bar{w}^2$.

Now, we have already on one side (we dropped the arguments (x,y,z) for simplicity)

$$T_{\alpha,\beta,\gamma}\Big(G\big(xu+yv+zw, x^2+y^2+z^2, u^2+v^2+w^2\big)\Big) = \sum_{m,n,k=0}^{\infty} T_{\alpha,\beta,\gamma}\left(B_{m,n,k}\right)\frac{u^m}{m!}\frac{v^n}{n!}\frac{w^k}{k!},$$

and on another side we get

$$T_{\alpha,\beta,\gamma}(G) = T_{\alpha,\beta,\gamma}\Big(G\big(x\bar{u} + y\bar{v} + z\bar{w}, x^2 + y^2 + z^2, \bar{u}^2 + \bar{v}^2 + \bar{w}^2\big)\Big) = \sum_{m,n,k=0}^{\infty} B_{m,n,k} \frac{\bar{u}^m}{m!} \frac{\bar{v}^n}{n!} \frac{\bar{w}^k}{k!}.$$

Thus

$$\sum_{m,n,k=0}^{\infty} T_{\alpha,\beta,\gamma}\left(B_{m,n,k}\right) \frac{u^m}{m!} \frac{v^n}{n!} \frac{w^k}{k!} = \sum_{m,n,k=0}^{\infty} B_{m,n,k} \frac{\bar{u}^m}{m!} \frac{\bar{v}^n}{n!} \frac{\bar{w}^k}{k!}.$$

We need to find the expression for $T_{\alpha,\beta,\gamma}\left(B_{m,n,k}\right)$ from here. We could expand $\bar{u}^m \bar{v}^n \bar{w}^k$, obtain a series in terms of $u^m v^n w^k$, and then equate the corresponding coefficients. However, we will take a different approach. Let us rewrite this identity for the polynomials $B_{m,n,k}(x,y,z) = x^m y^n z^k$. We can do this because in this case the generating function has the required form $G = e^{xu+yv+zw}$. We have

$$\sum_{m,n,k=0}^{\infty} T_{\alpha,\beta,\gamma}\left(x^m y^n z^k\right) \frac{u^m}{m!} \frac{v^n}{n!} \frac{w^k}{k!} = \sum_{m,n,k=0}^{\infty} x^m y^n z^k \frac{\bar{u}^m}{m!} \frac{\bar{v}^n}{n!} \frac{\bar{w}^k}{k!}.$$

Since, by definition

$$T_{\alpha,\beta,\gamma}(x^m y^n z^k) = \sum_{\substack{m_1+m_2+m_3=m\\ n_1+n_2+n_3=n\\ k_1+k_2+k_3=k}} C\left(\alpha,\beta,\gamma,m_1,\ldots,k_3\right) x^{m_1+n_1+k_1} y^{m_2+n_2+k_2} z^{m_3+n_3+k_3},$$

in the right-hand side, when expanding brackets and returning to the variables u, v, w, we must have

$$\sum_{m,n,k=0}^{\infty} x^m y^n z^k \frac{\bar{u}^m}{m!} \frac{\bar{v}^n}{n!} \frac{\bar{w}^k}{k!}$$
$$= \sum_{m,n,k=0}^{\infty} \left(\sum_{\substack{m_1+m_2+m_3=m\\ n_1+n_2+n_3=n\\ k_1+k_2+k_3=k}} C(\alpha,\beta,\gamma,m_1,\ldots,k_3) x^{m_1+n_1+k_1} y^{m_2+n_2+k_2} z^{m_3+n_3+k_3} \right) \frac{u^m}{m!} \frac{v^n}{n!} \frac{w^k}{k!}.$$

On the other hand, when expanding brackets and returning to the variables u, v, w in

$$\sum_{m,n,k=0}^{\infty} B_{m,n,k} \frac{\bar{u}^m}{m!} \frac{\bar{v}^n}{n!} \frac{\bar{w}^k}{k!},$$

since $\bar{u}, \bar{v}, \bar{w}$ do not depend on x, y, z, we should obtain an expression of the same form

$$\sum_{m,n,k=0}^{\infty} B_{m,n,k} \frac{\bar{u}^m}{m!} \frac{\bar{v}^n}{n!} \frac{\bar{w}^k}{k!}$$
$$= \sum_{m,n,k=0}^{\infty} \left(\sum_{\substack{m_1+m_2+m_3=m\\ n_1+n_2+n_3=n\\ k_1+k_2+k_3=k}} C(\alpha,\beta,\gamma,m_1,\ldots,k_3) B_{m_1+n_1+k_1, m_2+n_2+k_2, m_3+n_3+k_3} \right) \frac{u^m}{m!} \frac{v^n}{n!} \frac{w^k}{k!}.$$

Comparing the coefficients of the powers of u, v, w in these expressions, we can deduce the expression for $T_{\alpha,\beta,\gamma}(B_{m,n,k})$, namely

$$T_{\alpha,\beta,\gamma}(B_{m,n,k}) = \sum_{\substack{m_1+m_2+m_3=m \\ n_1+n_2+n_3=n \\ k_1+k_2+k_3=k}} C(\alpha, \beta, \gamma, m_1, \ldots, k_3) B_{m_1+n_1+k_1, m_2+n_2+k_2, m_3+n_3+k_3}.$$

Thus, the elements of the group $SO(3)$ act on $B_{m,n,k}$ in the same way as they do on the monomials $x^m y^n z^k$. Therefore, the family $\{B_{m,n,k}(x,y,z)\}$ is quasi-monomial as required. □

Example. *The family* $H_{m,n,k}(x,y,z) = H_m(x)H_n(y)H_k(z)$*, where* $H_n(x)$ *is the Hermite polynomial, is a quasi-monomial one. In fact, since*

$$e^{2xu-u^2} = \sum_{m=0}^{\infty} H_m(x)\frac{u^m}{m!}, \quad e^{2yv-v^2} = \sum_{n=0}^{\infty} H_n(y)\frac{v^n}{n!}, \quad e^{2zw-w^2} = \sum_{n=0}^{\infty} H_n(z)\frac{w^n}{n!},$$

the exponential generating function for $H_{m,n,k}(x,y,z)$ *has the form*

$$e^{2xu-u^2} e^{2yv-v^2} e^{2zw-w^2} = e^{2(xu+yv+zw)-(u^2+v^2+w^2)}.$$

Thus, $H_{m,n,k}(x,y,z)$ *is a quasi-monomial family.*

The quasi-monomiality property could potentially be compromised if the polynomials are subject to multiplicative constant scaling, a common practice to maintain the value range within reasonable limits. The subsequent theorem investigates the types of scaling that uphold the property.

Theorem 2. *Suppose that the family* $\{B_{m,n,k}(x,y,z)\}$ *is quasi-monomial. The polynomial family* $\{\widetilde{B}_{m,n,k}(x,y,z)\}$*, where*

$$\widetilde{B}_{m,n,k}(x,y,z) = \alpha_{m,n,k} B_{m,n,n}(x,y,z),$$

is quasi-polynomial if and only if the coefficient $\alpha_{m,n,k}$ *is an arbitrary function of the one variable* $m+n+k$.

Proof. If $\widetilde{B}_{m,n,k}(x,y,z)$ is quasi-monomial, then it satisfies the identities of the Lemma, and we get

$$x\frac{\partial \widetilde{B}_{m,n,k}(x,y,z)}{\partial y} - y\frac{\partial \widetilde{B}_{m,n,k}(x,y,z)}{\partial x} = n\widetilde{B}_{m+1,n-1,k}(x,y,z) - m\widetilde{B}_{m-1,n+1,k}(x,y,z).$$

Then

$$x\frac{\partial B_{m,n,k}(x,y,z)}{\partial y} - y\frac{\partial B_{m,n,k}(x,y,z)}{\partial x} \\ = n\frac{\alpha_{m+1,n-1,k}}{\alpha_{m,n}} B_{m+1,n-1,k}(x,y,z) - m\frac{\alpha_{m-1,n+1,k}}{\alpha_{m,n,k}} B_{m-1,n+1,k}(x,y,z).$$

From this, we can immediately deduce that

$$\frac{\alpha_{m+1,n-1,k}}{\alpha_{m,n,k}} = 1 \qquad \text{and} \qquad \frac{\alpha_{m-1,n+1,k}}{\alpha_{m,n,k}} = 1.$$

Let us do it for each identity, we obtain the system of recurrence equations

$$\begin{cases} \alpha_{m+1,n-1,k} = \alpha_{m,n,k}, \\ \alpha_{m-1,n+1,k} = \alpha_{m,n,k}, \\ \alpha_{m+1,n,k-1} = \alpha_{m,n,k}, \\ \alpha_{m-1,n,k+1} = \alpha_{m,n,k}, \\ \alpha_{m,n-1,k+1} = \alpha_{m,n,k}, \\ \alpha_{m,n+1,k-1} = \alpha_{m,n,k}. \end{cases}$$

It is easily seen that a solution of the system is $\alpha_{m,n,k} = \phi(m+n+k)$, where f is an arbitrary function. In fact, consider an arbitrary index (m,n,k). We can reach the index $(0,0,m+n+k)$ through a series of transformations, as demonstrated below. First, decrease m while increasing n until $m=0$: $\alpha_{0,n+m,k} = \alpha_{m,n,k}$. Then, decrease n while increasing k until $n=0$: $\alpha_{0,0,m+n+k} = \alpha_{0,n+m,k}$. As a result, we obtain

$$\alpha_{m,n,k} = \alpha_{0,n+m,k} = \alpha_{0,0,m+n+k}.$$

We can now define a function $\phi(x) = \alpha_{0,0,x}$. Then, $\alpha_{m,n,k} = \phi(m+n+k)$, showing that any solution of the system is a function of one variable $m+n+k$.

Let us prove the reverse implication. Suppose now that $B_{m,n,k}(x,y,z)$ is a quasi-monomial and that the coefficient $\alpha_{m,n,k}$ is an arbitrary function ϕ of the single variable $m+n+k$, i.e. $\alpha_{m,n,k} = \phi(n+m+k)$. We need to prove that the family

$$\widetilde{B}_{m,n,k}(x,y,z) = \phi(m+n+k)B_{m,n,k}(x,y,z)$$

is quasi-monomial. To do this, let us first check that $\widetilde{B}_{m,n,k}(x,y,z)$ satisfies the identity of the Lemma above. Taking into account the quasi-monomiality of $B_{m,n,k}(x,y,z)$, we have

$$\begin{aligned} x\frac{\partial \widetilde{B}_{m,n,k}(x,y,z)}{\partial y} - y\frac{\partial \widetilde{B}_{m,n,k}(x,y,z)}{\partial x} &= \phi(m{+}n{+}k)\left(x\frac{\partial B_{m,n,k}(x,y,z)}{\partial y} - y\frac{\partial B_{m,n,k}(x,y,z)}{\partial x}\right) \\ &= \phi(m+n+k)\big(nB_{m+1,n-1,k}(x,y,z) - mB_{m-1,n+1,k}(x,y,z)\big) \\ &= n\phi\big((m{+}1){+}(n{-}1)+k\big)B_{m+1,n-1,k}(x,y,z) \\ &\quad - m\phi\big((m{-}1)+(n{+}1)+k\big)B_{m-1,n+1,k}(x,y,z) \\ &= n\widetilde{B}_{m+1,n-1,k}(x,y,z) - m\widetilde{B}_{m-1,n+1,k}(x,y,z). \end{aligned}$$

Similarly, we can verify the other two identities. Theorem 1 now implies that the family $\widetilde{B}_{m,n,k}(x,y,z)$ is a quasi-monomial family. □

In the following section, we will consider two examples of quasi-monomial families of polynomials.

2 Three variable Appel polynomials

Let us consider the two polynomial families $\left\{V^{(s)}_{m,n,k}(x,y,z)\right\}$ and $\left\{U^{(s)}_{m,n,k}(x,y,z)\right\}$ defined by the following exponential generating functions:

$$\frac{1}{\left(1-2(xu+yv+zw)+u^2+v^2+w^2\right)^{\frac{2+s}{2}}} = \sum_{m,n=0}^{\infty} V^{(s)}_{m,n,k}(x,y,z)\frac{u^m}{m!}\frac{v^n}{n!}\frac{w^k}{k!},$$

$$\frac{1}{\left(\left(1-(ux+vy+wz)\right)^2 - (u^2+v^2+w^2)\,(x^2{+}y^2{+}z^2{-}1)\right)^{\frac{s}{2}}} = \sum_{m,n=0}^{\infty} U^{(s)}_{m,n,k}(x,y,z)\frac{u^m}{m!}\frac{v^n}{n!}\frac{w^k}{k!}.$$

The origin of these polynomials can be attributed to the contributions of C. Hermite, F. Didon, J. Kampé de Fériet and P. Appell, see [8–12].

As follows directly from Theorem 1, the polynomial families are quasi-monomials. The explicit formulas (up to constant factor $m!n!k!$) for the polynomials are given in [13, p.42, p.45]. After simplification we get the expressions

$$V^{(s)}_{m,n,k}(x,y,z)=\sum_{i=0}^{\left[\frac{m}{2}\right]}\sum_{j=0}^{\left[\frac{n}{2}\right]}\sum_{t=0}^{\left[\frac{k}{2}\right]}\binom{m}{i}\binom{n}{j}\binom{k}{t}\frac{(1+\frac{s}{2})_{m+n+k-i-j-t}(i-m)_i(j-n)_j(t-k)_t}{2^{2(i+j+t)-(m+n+k)}}x^{m-2i}y^{n-2j}z^{k-2t},$$

$$\begin{aligned}U^{(s)}_{m,n,k}(x,y,z) &= (2s-1)_{m+n+k}\\ &\times\sum_{i=0}^{\left[\frac{m}{2}\right]}\sum_{j=0}^{\left[\frac{n}{2}\right]}\sum_{t=0}^{\left[\frac{k}{2}\right]}\frac{(-1)^{i+j+t}(-m)_{2i}(-n)_{2j}(-k)_{2t}}{2^{2(i+j+t)}i!j!t!(s)_{i+j+t}}x^{m-2i}y^{n-2j}z^{k-2t}\left(1-x^2-y^2-z^2\right)^{i+j+t}.\end{aligned}$$

Here $(x)_n$ is the Pochhammer symbol

$$(x)_n=\begin{cases}1, & n=0,\\ x(x+1)(x+2)\cdots(x+n-1), & n>0.\end{cases}$$

Below are some first such polynomials for the case $s=1$:

$$\begin{aligned}&V_{0,0,0}=1,\ V_{1,0,0}=3x,\ V_{0,1,0}=3y,\ V_{0,0,1}=3z,\ V_{2,0,0}=15x^2-3,\ V_{0,2,0}=15y^2-3,\\ &V_{1,1,0}=15xy,\ V_{1,0,1}=15xz,\ V_{0,1,1}=15xz,\ V_{3,0,0}=105\,x^3-45\,x,\ V_{2,1,0}=105\,x^2y-15\,y;\end{aligned}$$

$$\begin{aligned}&U_{0,0,0}=1,\ U_{1,0,0}=x,\ U_{0,1,0}=y,\ U_{0,0,1}=z,\ U_{2,0,0}=3\,x^2+y^2+z^2-1,\\ &U_{0,0,2}=x^2+y^2+3z^2-1,\ U_{1,1,0}=2xy,\ U_{1,0,1}=2xz,\ U_{0,1,1}=2xz,\\ &U_{3,0,0}=15\,x^3+9\,xy^2+9\,xz^2-9\,x,\ U_{2,1,0}=9\,x^2y+3\,y^3+3\,yz^2-3\,y.\end{aligned}$$

The polynomials are biorthogonal on the sphere B^3 with the weight function

$$w_s(x,y,z)=\left(1-\left(x^2+y^2+z^2\right)\right)^{\frac{s-1}{2}}.$$

In particular, for $s=1$, by using the result [9, p.262] we get

$$\iiint\limits_{x^2+y^2+z^2\le 1}V_{m,n,k}(x,y,z)U_{m',n',k'}(x,y,z)\,dx\,dy\,dz=\frac{4\pi(m+n+k)!\,m!n!k!}{2(m+n+k)+3}\delta_{m,m'}\delta_{n,n'}\delta_{k,k'}.$$

While computing Appell polynomials numerically with explicit formulas, we may encounter precision loss due to floating-point overflow or underflow. This issue is familiar in the context of many 1D orthogonal polynomials, where it can be resolved using three-term recurrence relations. For Appell polynomials, these recurrence relations can be employed to ensure efficient and stable calculations. We will derive explicit recurrence relations for Appell polynomials. Note that these formulas, without proof, were first presented in [6].

In the following two theorems, recurrent relations for Appell polynomials are presented, which were used for their efficient and stable computation.

Theorem 3. *The polynomials $U_{n,m,k} = U_{n,m,k}(x,y,z)$ satisfies the following recurrence relations:*

$$\begin{aligned}
U_{m+1,n,k} = {} & x(2m+n+k+1)U_{m,n,k} + kmxzU_{m,n,k-1} + mnxyU_{m,n-1,k} + 2kmnxyzU_{m,n-1,k-1} \\
& + m\Big((y^2+z^2-1)m + (y^2+2z^2-1)k + (2y^2+z^2-1)n \Big) U_{m-1,n,k} \\
& + mkz\Big((y^2-1)(m+k-1) + (3y^2-1)n \Big) U_{m-1,n,k-1} \\
& + mny\Big((3z^2-1)k + (z^2-1)(m+n-1) \Big) U_{m-1,n-1,k} \\
& - 2kmnyz(m+n+k-2)U_{m-1,n-1,k-1},
\end{aligned}$$

$$\begin{aligned}
U_{m,n+1,k} = {} & y(m+2n+k+1)U_{m,n,k} + knyzU_{m,n,k-1} + mnxyU_{m-1,n,k} + 2kmnxyzU_{m-1,n,k-1} \\
& + n\Big((x^2+z^2-1)n + (x^2+2z^2-1)k + (2x^2+z^2-1)m \Big) U_{m,n-1,k} \\
& + nkz\Big((x^2-1)(k+n-1) + (3x^2-1)m \Big) U_{m,n-1,k-1} \\
& + mnx\Big((3z^2-1)k + (z^2-1)(m+n-1) \Big) U_{m-1,n-1,k} \\
& - 2kmnxz(m+n+k-2)U_{m-1,n-1,k-1},
\end{aligned}$$

$$\begin{aligned}
U_{m,n,k+1} = {} & z(m+n+2k+1)U_{m,n,k} + kmxzU_{m-1,n,k} + knzyU_{m,n-1,k} + 2kmnxyzU_{m-1,n-1,k} \\
& + k\Big((x^2+y^2-1)k + (2x^2+y^2-1)m + (x^2+2y^2-1)n \Big) U_{m,n,k-1} \\
& + mkx\Big((y^2-1)(m+k-1) + (3y^2-1)n \Big) U_{m-1,n,k-1} \\
& + kny\Big((x^2-1)(k+n-1) + (3x^2-1)m \Big) U_{m,n-1,k-1} \\
& - 2kmnyx(m+n+k-2)U_{m-1,n-1,k-1},
\end{aligned}$$

with the initial condition

$$U_{0,0,0} = 1,$$

and $V_{m,n,k} = 0$ if at least one of the indices is negative.

Proof. Direct verification shows that

$$-2f\frac{\partial G}{\partial u} = f_u G,$$

where

$$f = \big(1-(ux+vy+wz)\big)^2 - \Big(u^2+v^2+w^2\Big)\Big(x^2+y^2+z^2-1\Big), \quad f_u = \frac{\partial f}{\partial u}, \quad G = \frac{1}{\sqrt{f}}.$$

Suppose that there exists an identity of the form

$$g_1\frac{\partial G}{\partial u} + g_2\frac{\partial G}{\partial v} + g_3\frac{\partial G}{\partial w} + g_4 G = 0$$

for some polynomials g_1, g_2, g_3, g_4 of x, y, z. Let us multiple it by $-2f$, after cancelation G we get

$$g_1 f_u + g_2 f_v + g_3 f_w - 2f g_4 = 0.$$

Suppose that g_i are polynomials of degree 2. We can solve the polynomial equation and find 4 linearly independent solutions, each of which defines a differential equation. We are omitting the calculations and presenting only the resulting differential identities:

$$x\left(2\,u\frac{\partial G}{\partial u}+v\frac{\partial G}{\partial v}+w\frac{\partial G}{\partial w}+G\right)+uy\frac{\partial G}{\partial v}+\ uz\frac{\partial G}{\partial w}=G\,u+\left(u^2+1\right)\frac{\partial G}{\partial u}+uv\frac{\partial G}{\partial v}+uw\frac{\partial G}{\partial w},$$
$$y\left(u\frac{\partial G}{\partial u}+2\,v\frac{\partial G}{\partial v}+w\frac{\partial G}{\partial w}+G\right)+xv\frac{\partial G}{\partial u}+vz\frac{\partial G}{\partial w}=G\,v+uv\frac{\partial G}{\partial u}+\left(v^2+1\right)\frac{\partial G}{\partial v}+vw\frac{\partial G}{\partial w},$$
$$xw\frac{\partial G}{\partial u}+wy\frac{\partial G}{\partial v}+z\left(u\frac{\partial G}{\partial u}+v\frac{\partial G}{\partial v}+2\,w\frac{\partial G}{\partial w}+G\right)=G\,w+uw\frac{\partial G}{\partial u}+vw\frac{\partial G}{\partial v}+\left(w^2+1\right)\frac{\partial G}{\partial w}.$$

Any such differential identity implies some recurrence relation for $V_{m,n,k}(x,y,z)$ and its derivatives. For instance, if we expand the left side of the first identity, namely

$$x\left(2\,u\frac{\partial G}{\partial u}+v\frac{\partial G}{\partial v}+w\frac{\partial G}{\partial w}+G\right)+uy\frac{\partial G}{\partial v}+\ uz\frac{\partial G}{\partial w},$$

by writing out G and its derivatives as series in terms of u,v,w (similarly to the proof of Theorem 1), we obtain a series with the following general term

$$x(2m+n+k+1)U_{m,n,k}(x,y,z)+m\big(yU_{m-1,n+1,k}(x,y,z)+zU_{m-1,n,k+1}(x,y,z)\big).$$

Similarly, by transforming the left side of that identity, namely

$$G\,u+\left(u^2+1\right)\frac{\partial G}{\partial u}+uv\frac{\partial G}{\partial v}+uw\frac{\partial G}{\partial w}=\frac{\partial G}{\partial u}+\frac{\partial\left(u^2G\right)}{\partial u}+uv\frac{\partial G}{\partial v}+uw\frac{\partial G}{\partial w}-G\,u,$$

we obtain a series with the following general term

$$\begin{aligned}U_{m+1,n,k}+m(m-1)U_{m-1,n,k}+mnV_{m-1,n,k}+mkU_{m-1,n,k}-mU_{m-1,n,k}&\\=m(m+n+k)U_{m-1,n,k}+U_{m+1,n,k}.&\end{aligned}$$

By equating these left and right sides, we obtain the following recurrence equation

$$x(2m+n+k+1)U_{m,n,k}+m(yU_{m-1,n+1,k}+zU_{m-1,n,k+1})=m(m+n+k)U_{m-1,n,k}+U_{m+1,n,k}.$$

Similar reasoning provides us with two more recurrence equations:

$$y(m+2n+k+1)U_{m,n,k}+n(xU_{m+1,n-1,k}+zU_{m,n-1,k+1})=n(m+n+k)U_{m,n-1,k}+U_{m,n+1,k},$$
$$z(m+n+2k+1)U_{m,n,k}+k(xU_{m+1,n,k-1}+yU_{m,n+1,k-1})=k(m+n+k)U_{m,n,k-1}+U_{m,n,k+1}.$$

The obtained recurrence equations are not yet suitable for calculating the values of polynomials, so they need to be transformed into the form

$$U_{m+1,n,k}=\text{an expression of } U \text{ with indices } smaller \text{ than } (m+1,n,k).$$

All subsequent lengthy transformations aim to gradually convert these recurrence equations into the desired form.

By performing the index shifting $m\mapsto m-1$ to two last identities, we get some new identies

$$\begin{aligned}&y(m+2n+k)U_{m-1,n,k}+n(xU_{m,n-1,k}+zU_{m-1,n-1,k+1})\\&\qquad=n(m-1+n+k)U_{m-1,n-1,k}+U_{m-1,n+1,k},\\&z(m+n+2k)U_{m-1,n,k}+k(xU_{m,n,k-1}+yU_{m-1,n+1,k-1})\\&\qquad=k(m-1+n+k)U_{m-1,n,k-1}+U_{m-1,n,k+1},\end{aligned}$$

or

$$\begin{aligned}U_{m-1,n+1,k} &= y(m+2n+k)U_{m-1,n,k} + n(xU_{m,n-1,k} + zU_{m-1,n-1,k+1})\\ &\qquad - n(m-1+n+k)U_{m-1,n-1,k},\\ U_{m-1,n,k+1} &= z(m+n+2k)U_{m-1,n,k} + k(xU_{m,n,k-1} + yU_{m-1,n+1,k-1})\\ &\qquad - k(m-1+n+k)U_{m-1,n,k-1}.\end{aligned}$$

Substitute it into first equation, we get

$$\begin{aligned}U_{m+1,n,k} = {} & x\big(2m+n+k+1\big)U_{m,n,k} + kmxzU_{m,n,k-1} + mnxyU_{m,n-1,k}\\ & + m\Big(\big(y^2+z^2-1\big)m + \big(y^2+2z^2-1\big)k + \big(2y^2+z^2-1\big)n\Big)U_{m-1,n,k}\\ & - m\big(m+n+k-1\big)\big(kzU_{m-1,n,k-1} + nyU_{m-1,n-1,k}\big)\\ & + kmyzU_{m-1,n+1,k-1} + mnyzU_{m-1,n-1,k+1}.\end{aligned}$$

The obtained expression has not yet reached the desired form, as the right-hand side still contains sets of indices $(m-1, n+1, k-1)$, $(m-1, n-1, k+1)$, which are larger than the set $(m+1, n, k)$.

Once more, perform the index shifting for the last two identities. We obtain

$$\begin{aligned}U_{m-1,n+1,k-1} &= y(m+2n+k-1)U_{m-1,n,k-1} + n(xU_{m,n-1,k-1} + zU_{m-1,n-1,k})\\ &\qquad - n(m-2+n+k)U_{m-1,n-1,k-1},\\ U_{m-1,n-1,k+1} &= z(m+n+2k-1)U_{m-1,n-1,k} + k(xU_{m,n-1,k-1} + yU_{m-1,n,k-1})\\ &\qquad - k(m-2+n+k)U_{m-1,n-1,k-1}.\end{aligned}$$

After eliminating $U_{m-1,n+1,k-1}$ and $U_{m-1,n-1,k+1}$, we get

$$\begin{aligned}U_{m+1,n,k} = {} & x\big(2m+n+k+1\big)U_{m,n,k} + kmxzU_{m,n,k-1} + mnxyU_{m,n-1,k} + 2kmnxyzU_{m,n-1,k-1}\\ & + m\Big(\big(y^2+z^2-1\big)m + \big(y^2+2z^2-1\big)k + \big(2y^2+z^2-1\big)n\Big)U_{m-1,n,k}\\ & + mkz\Big(\big(y^2-1\big)\big(m+k-1\big) + \big(3y^2-1\big)n\Big)U_{m-1,n,k-1}\\ & + mny\Big(\big(3z^2-1\big)k + \big(z^2-1\big)\big(m+n-1\big)\Big)U_{m-1,n-1,k}\\ & - 2kmnyz\big(m+n+k-2\big)U_{m-1,n-1,k-1}.\end{aligned}$$

Finally, we have obtained the desired expression for $U_{m+1,n,k}$.

Similar calculations, which we omit, provide us with such recurrence relations for the second and third indices:

$$\begin{aligned}U_{m,n+1,k} = {} & y\big(m+2n+k+1\big)U_{m,n,k} + knyzU_{m,n,k-1} + mnxyU_{m-1,n,k} + 2kmnxyzU_{m-1,n,k-1}\\ & + n\Big(\big(x^2+z^2-1\big)n + \big(x^2+2z^2-1\big)k + \big(2x^2+z^2-1\big)m\Big)U_{m,n-1,k}\\ & + nkz\Big(\big(x^2-1\big)\big(k+n-1\big) + \big(3x^2-1\big)m\Big)U_{m,n-1,k-1}\\ & + mnx\Big(\big(3z^2-1\big)k + \big(z^2-1\big)\big(m+n-1\big)\Big)U_{m-1,n-1,k}\\ & - 2kmnxz\big(m+n+k-2\big)U_{m-1,n-1,k-1},\end{aligned}$$

and

$$\begin{aligned}
U_{m,n,k+1} = & z(m+n+2k+1)U_{m,n,k} + kmxzU_{m-1,n,k} + knzyU_{m,n-1,k} + 2kmnxyzU_{m-1,n-1,k} \\
& + k\Big((x^2+y^2-1)k + (2x^2+y^2-1)m + (x^2+2y^2-1)n\Big)U_{m,n,k-1} \\
& + mkx\Big((y^2-1)(m+k-1) + (3y^2-1)n\Big)U_{m-1,n,k-1} \\
& + kny\Big((x^2-1)(k+n-1) + (3x^2-1)m\Big)U_{m,n-1,k-1} \\
& - 2kmnyx(m+n+k-2)U_{m-1,n-1,k-1}.
\end{aligned}$$

□

The recurrence relations for the polynomials V turned out to be simpler, and we managed to obtain formulas for an arbitrary s. The way of their derivation is similar, and we will only present the final result.

Theorem 4. *The polynomials $V^{(s)}_{n,m,k}(x,y,z)$ satisfies the five-term recurrence relations*

$$\begin{aligned}
(2(1+m+n+k)+s)x\,V_{m,n,k}(x,y,z) &= V_{m+1,n,k}(x,y,z) - n(n-1)V_{m+1,n-2,k}(x,y,z) \\
&\quad - k(k-1)V_{m+1,n,k-2}(x,y,z) + m(m+2n+2k+1+s)V_{m-1,n,k}(x,y,z), \\
(2(1+m+n+k)+s)y\,V_{m,n,k}(x,y,z) &= V_{m,n+1,k}(x,y,z) - m(m-1)V_{m-2,n+1,k}(x,y,z) \\
&\quad - k(k-1)V_{m,n+1,k-2}(x,y,z) + n(n+2m+2k+1+s)V_{m,n-1,k}(x,y,z), \\
(2(1+m+n+k)+s)z\,V_{m,n,k}(x,y,z) &= V_{m,n,k+1}(x,y,z) - m(m-1)V_{m-2,n,k+1}(x,y,z) \\
&\quad - n(n-1)V_{m,n-2,k+1}(x,y,z) + k(k+2m+2n+1+s)V_{m,n,k-1}(x,y,z),
\end{aligned}$$

with the initial conditions

$$V_{0,0,0}(x,y,z) = 1, \quad V_{1,0,0}(x,y,z) = (s+2)\,x, \quad V_{0,1,0}(x,y,z) = (s+2)\,y,$$

$$V_{2,0,0}(x,y,z) = (s+2)\left(x^2 s + 4\,x^2 - 1\right), \quad V_{1,1,0}(x,y,z) = xy\,(s+4)\,(s+2)\,.$$

3 Conclusion

This study provides a comprehensive description of families of polynomials that are quasi-monomials with respect to the rotation group of the space $SO(3)$, extending the results from previous work in 2D case. We have demonstrated that a family of polynomials can be considered quasi-monomials if and only if their exponential generating function depends on three specific variables. Furthermore, we have established the conditions under which the scaling of quasi-monomials maintains their quasi-monomial property.

Significantly, we have proven that biorthogonal Appell polynomials of three variables are $SO(3)$ quasi-monomials, which contributes to the understanding of these polynomials in the context of 3D object recognition. The derivation of recurrence relations for these polynomials addresses the challenge of potential precision loss in numerical calculations, allowing for more efficient and stable computations in practical applications.

This work advances our understanding of quasi-monomials and their properties, opening up new possibilities for further research and applications in various fields, such as image analysis and 3D object recognition. Future studies could explore other transformation groups and additional properties of quasi-monomials, further expanding their applicability.

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Самарук Н.М. $SO(3)$ *квазі-мономіальні сім'ї многочленів* // Карпатські матем. публ. — 2024. — Т.16, №1. — С. 40–52.

Нехай H є підгрупою афінної просторової групи $\mathrm{Aff}(3)$, розглянута зі своєю природною дією на дійсному векторному просторі многочленів від трьох змінних. Сім'ю многочленів $\{B_{m,n,k}(x,y,z)\}$ називають квазі-мономіальною відносно H, якщо оператори групи у двох різних базисах $\{x^m y^n z^k\}$ та $\{B_{m,n,k}(x,y,z)\}$ мають однакові матриці. Ми отримали критерій квазі-мономіальності у випадку, коли група H є спеціальною ортогональною групою $SO(3)$. Цей критерій виражений через експоненціальну генеруючу функцію сім'ї многочленів $\{B_{m,n,k}(x,y,z)\}$. Було також доведено, що біортогональні поліноми Аппеля є квазі-мономіальними відносно $SO(3)$, і для них були знайдені рекурентні співвідношення.

Ключові слова і фрази: квазі-мономіальний многочлен, спеціальна ортогональна група, біортогональний многочлен Аппеля, рекурентне співвідношення.