ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2025, **17** (1), 152–158
doi:10.15330/cmp.17.1.152-158

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2025, Т.17, №1, С.152–158

Rings with prime units

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A ring R is said to be a UP ring if every invertible element in R is the sum of the identity and a strongly nilpotent element. In this paper, many characterizations of UP rings are given. We prove that for a UP ring R, the following conditions are equivalent: R is weakly exchange, R is exchange, R is clean, and R is nil clean. This is a partial answer to the question, which is asked by A.J. Diesl in [J. Algebra 2013, **383**, 197–211].

Key words and phrases: UP ring, ring extension, exchange ring.

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1 Introduction

Throughout this paper, all rings are associative with identity unless otherwise stated. For a ring R, $a \in R$ is *strongly nilpotent* if every sequence $a = a_0, a_1, \ldots,$ such that $a_{i+1} \in a_i R a_i$ is ultimately zero. Obviously, every strongly nilpotent element is nilpotent. The prime radical $P(R)$ of a ring R, that is, the intersection of all prime ideals, consists of precisely the strongly nilpotent elements. For a ring R, $U(R)$, $J(R)$ and $N(R)$ denote the group of units, the Jacobson radical and the set of all nilpotent elements of R, respectively. Let $\mathbb{Z}$ be the ring of integers. For a positive integer n, $\mathbb{Z}_n$ is the ring of integers modulo n. We write $M_n(R)$ for the ring of all $n \times n$ matrices and $T_n(R)$ for the ring of all $n \times n$ upper triangular matrices over R. Also, we write $R[x]$ for the polynomial ring over a ring R.

The notion of cleanness was first introduced by W.K. Nicholson in [8]. An element a of a ring R is called *clean* if a can be expressed as $e + u$, where e is an idempotent and u is a unit of R. A ring R is clean if all its elements are clean. An element x of a ring R is called *P-clean* if there exist an idempotent $e \in R$ and $p \in P(R)$ such that $x = e + p$. A ring R is called P-clean if each of its elements is P-clean. Evidently, a ring R is P-clean if and only if every element is the sum of an idempotent and a strongly nilpotent that commute by [2]. It is clear that every P-clean ring is clean. But the converse is not true in general.

We call a ring R as UP ring or *a ring with only prime units* if $1 + P(R) = U(R)$. As mentioned above, we prove that if R is a UP ring, then every weakly exchange element of R is P-clean. Hence, every clean element of R is P-clean.

УДК 512.5
2020 *Mathematics Subject Classification:* 13C99, 16D80, 16U80.

We summarize the contents of this paper. In Section 2, we shall characterize UP rings. We prove that a ring R is a UP ring if and only if $T_n(R)$ is UP for all $n \geq 2$. Furthermore, we show that if R is a UP ring, then $R/P(R)$ is reversible and R is directly finite. Finally, in the last section, we focus on weakly exchange properties. For a UP ring R, we prove that the following conditions are equivalent: R is weakly exchange, R is exchange, R is clean, and R is nil clean. Moreover, a ring R is P-clean if and only if it is a weakly exchange UP ring.

2 *UP* rings

For any ring R, it is well known that, $1 + P(R) \subseteq U(R)$. However, this inclusion is strict. So the aim of this paper is to investigate when $U(R)$ is contained in $1 + P(R)$. In this section, we define UP rings and investigate several useful properties of UP rings.

Definition 1. *A ring R is called UP if*

$$1 + P(R) = U(R).$$

We begin with a simple result.

Lemma 1. *For any ring R, we have*

$$U(R) + P(R) = U(R).$$

Proof. It is clear that $U(R) \subseteq U(R) + P(R)$.

To see the converse inclusion, let $x = u + p \in U(R) + P(R)$. Then $x = u(1 + u^{-1}p)$ and so $x \in U(R)$. □

Theorem 1. *For a ring R the following are equivalent:*

(1) *R is a UP ring;*

(2) *$U(R/P(R)) = \{1\}$;*

(3) *$xu - vx \in P(R)$ for any $u, v \in U(R)$ and $x \in R$;*

(4) *$U(R) + U(R) = P(R)$;*

(5) *$R/P(R)$ is a UP ring.*

Proof. (1) $\Rightarrow$ (2) Let $\overline{u}$ be invertible in $R/P(R)$. Then there exists $v \in R$ such that $uv \in 1 + P(R)$. Hence, $uv \in U(R)$, since $1 + P(R) \subseteq U(R)$. Thus, u is right invertible in R. Similarly, u is left invertible in R. So $u \in U(R) = 1 + P(R)$ by hypothesis. Consequently, $\overline{u}$ is the trivial unit in $R/P(R)$.

(2) $\Rightarrow$ (1) It is obvious.

(1) $\Rightarrow$ (3) Let $u, v \in U(R)$ and $x \in R$. Then $u = 1 + p_1$ and $v = 1 + p_2$ for some $p_1, p_2 \in P(R)$. Clearly, $xu - vx \in P(R)$.

(3) $\Rightarrow$ (4) By hypothesis, for $x = 1$, $u - v \in P(R)$ for any $u, v \in U(R)$. This shows that $U(R) + U(R) \subseteq P(R)$. To see the converse inclusion, let $p \in P(R)$. Then we obtain $p = 1 + (-1 + p) \in U(R) + U(R)$.

(4) $\Rightarrow$ (1) Let $u \in U(R)$. By hypothesis, $u + v \in P(R)$ for any $v \in U(R)$. So $u \in 1 + P(R)$.

(2) $\Leftrightarrow$ (5) It is evident, since $P(R/P(R)) = \{0\}$. □

Proposition 1. *The ring* $\bigoplus_{i\in\mathcal{I}} R_i$ *is* UP *if and only if* R_i *are* UP *for all* $i \in \mathcal{I} = \{1, 2, \ldots, n\}$.

Proof. Since $P\big(\bigoplus_{i\in\mathcal{I}} R_i\big) = \bigoplus_{i\in\mathcal{I}} P(R_i)$ and $U\big(\bigoplus_{i\in\mathcal{I}} R_i\big) = \bigoplus_{i\in\mathcal{I}} U(R_i)$, we easily get the result. □

It is well known that $P(R) \subseteq N(R)$ and $P(R) \subseteq J(R)$. But these inclusions might be strict. So we have the following, which is very useful to determine whether a ring is UP.

Proposition 2. *If* R *is a* UP *ring, then*

$$P(R) = J(R) = N(R).$$

Proof. It is clear that $J(R)$ is always in $1 + U(R)$. Since R is UP, we get $1 + U(R) = P(R)$. Hence, $J(R) \subseteq P(R)$. The converse inclusion is clear. Thus, $P(R) = J(R)$.

Further, clearly $P(R) \subseteq N(R)$. Let $n \in N(R)$. Then $1 - n \in U(R)$. By hypothesis there exists $p \in P(R)$ such that $1 - n = 1 + p$. Hence, $n = -p \in P(R)$, as asserted. □

Example 1. *Consider the formal power series ring* $F[[x]]$ *over a field* F. *Then* $J(F[[x]]) =< x >$ *but* $x \notin N(F[[x]])$. *Hence,* $F[[x]]$ *is not a* UP *ring by Proposition 2.*

The converse statement of Proposition 2 is not true in general.

Remark 1. *Consider the ring* $\mathbb{Z}$. *Then*

$$P(\mathbb{Z}) = J(\mathbb{Z}) = N(\mathbb{Z}),$$

but $\mathbb{Z}$ *is not a* UP *ring.*

Recall that an ideal I of a ring R is called *locally nilpotent* if for any $x \in I$, RxR is nilpotent. Further, a ring R is *Boolean* if every element in R is an idempotent.

Proposition 3. *If* $R/J(R)$ *is Boolean and* $J(R)$ *is locally nilpotent, then* R *is a* UP *ring.*

Proof. Assume that $R/J(R)$ is Boolean and $J(R)$ is locally nilpotent. Let $x \in U(R)$. Then $\overline{x} = \overline{1}$ as $R/J(R)$ is Boolean. Hence, $x - 1 \in J(R)$. Since $J(R)$ is locally nilpotent, $R(x-1)R$ is nilpotent and so $x - 1 \in P(R)$. Therefore, $x \in 1 + P(R)$. □

As the following example shows the condition "$R/J(R)$ is Boolean" in Proposition 3 is not superfluous.

Example 2. *Consider the ring* $R = \mathbb{Z}$. *Then* $J(R) = \{0\}$. *Hence,* $J(R)$ *is locally nilpotent. However,* $-1 \in U(R)$ *and* $-1 \notin 1 + P(R)$. *Thus,* R *is not* UP.

Proposition 4. *If* R *is a* UP *ring, then* $R/J(R)$ *is a* UP *ring. The converse holds if* $J(R)$ *is locally nilpotent.*

Proof. If R is UP, then $R/P(R)$ is UP by Theorem 1. Hence, $R/J(R)$ is UP by Proposition 2.

Let $u \in U(R)$. Then $\overline{u} \in U(R/J(R))$. Therefore, $\overline{u} \in 1 + P(R/J(R))$. By Proposition 2, $P(R/J(R)) = J(R/J(R)) = 0$. Hence, $\overline{u} = \overline{1}$. Thus, $u - 1 \in J(R)$. Since $J(R)$ is locally nilpotent, $u - 1 \in P(R)$. Consequently, $U(R) = 1 + P(R)$. □

Theorem 2. *Every subring of a UP ring is UP.*

Proof. Let R be a UP ring and S be a subring of R. To see that S is UP, let $u \in U(S)$. Then $u \in U(R)$. Hence, $u = 1 + p$ for some $p \in P(R)$ by hypothesis. Thus, $1 - u \in P(R) \cap S$. Since $P(R) \cap S \subseteq P(S)$, we get $u \in 1 + P(S)$. So, S is a UP ring. □

Corollary 1. *Every finite subdirect product of UP ring is UP.*

Recall that a ring R is called UU (see [1]), if all units are unipotent.

Theorem 3 ([4, Theorem 6.1])**.** *The following are equivalent:*

(1) *R is a UP ring;*

(2) *R is a 2-primal UU ring;*

(3) *$U(R)$ is a 2-group, $2 \in N(R)$ and R is a 2-primal ring.*

Corollary 2. *If R is a UP ring, then char $(R) = 2^n$ for some $n \geq 0$.*

Proof. By Theorem 3, there exists $n \geq 0$ such that $2^n = 0$ as asserted. □

Proposition 5. *If R is a UP ring, then*

(1) *$R/P(R)$ is reversible;*

(2) *R is directly finite.*

Proof. (1) Let $\overline{ab} = \overline{0}$. Then $\overline{ba} \in N(R/P(R))$, an so $\overline{1} + \overline{ba} \in U(R/P(R))$. By Theorem 1, $\overline{1} + \overline{ba} = \overline{1}$. Hence $\overline{ba} = \overline{0}$, as asserted.

(2) Let $ab = 1$. Then $\overline{ab} = \overline{1}$. By (1), we obtain that $R/P(R)$ is directly finite, and so $\overline{ba} = \overline{1}$. Hence, $ba \in 1 + P(R) \subseteq U(R)$. Further, $(ba)^2 = ba$. Thus, $ba = 1$. □

Recall that a ring R is called *abelian* if all idempotents in R are central.

Corollary 3. *Let R be a UP ring. Then $R/P(R)$ is abelian.*

Proof. Since reversible rings are abelian, the proof is clear by Proposition 5. □

Being R/I is a UP ring does not imply that R is a UP ring. For example, $\mathbb{Z}/2\mathbb{Z}$ is a UP ring, but $\mathbb{Z}$ is not. So we have the following assertion.

Theorem 4. *Let I be a locally nilpotent ideal of R. Then R is a UP ring if and only if R/I is a UP ring.*

Proof. Necessity. Assume that R is UP and $\overline{x} \in U(R/I)$. Then there exists $\overline{y}$ such that $\overline{xy} = \overline{1}$. Hence, $xy - 1 \in I$. By hypothesis, $xy - 1 \in P(R)$ and so $xy \in U(R)$. Thus, x is right invertible. Similarly, x is left invertible. So $x \in U(R)$. Since R is a UP ring, there exists $p \in P(R)$ such that $x = 1 + p$. Hence, $\overline{x} = \overline{1 + p}$. Therefore, $\overline{x} \in 1 + P(R/I)$, because $P(R/I) = P(R)/I$, since $I \subseteq P(R)$.

Sufficiency. To see R is UP, let $u \in U(R)$. Then $\overline{u} \in U(R/I)$. Since R/I is UP and $I \subseteq P(R)$, we get $\overline{u} = \overline{1 + p}$ for some $p \in P(R)$. Hence, $u - 1 - p \in I$. Since $I \subseteq P(R)$, we obtain $u \in 1 + P(R)$, as asserted. □

Corollary 4. *A ring R is UP if and only if $2 \in N(R)$ and $R/2R$ is UP.*

Proposition 6. *Let R be a commutative ring and $\sigma : R \to R$ be an automorphism of R. Then $R[x, \sigma]$ is a UP ring if and only if*

(1) *R is UP;*

(2) *every minimal strongly σ-prime ideal of R is completely prime.*

Proof. It is clear by [9, Proposition 3.2] and Theorem 3. □

Corollary 5. *A commutative ring R is UP if and only if $R[x]$ is UP.*

Proposition 7. *Let R be a ring. Then the following are equivalent:*

(1) *R is UP;*

(2) *$T_n(R)$ is UP for all $n \geq 2$.*

Proof. (1) $\Rightarrow$ (2) Without loss of generality we may consider $n = 2$.

Let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in U(T_2(R))$. Then $a, c \in U(R)$. By hypothesis, $a = 1 + p$ and $c = 1 + q$, where $p, q \in P(R)$. Hence, $A \in I_2 + P(T_2(R))$ since $P(T_2(R)) = \begin{pmatrix} P(R) & R \\ 0 & P(R) \end{pmatrix}$.

(2) $\Rightarrow$ (1) Let $u \in U(R)$. Then $\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \in U(T_2(R))$. Hence, $\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \in 1 + P(T_2(R))$. Thus, $u \in 1 + P(R)$. □

The following shows that Proposition 7 is not true for full matrix rings.

Remark 2. *$M_2(R)$ is not UP for any arbitrary ring R. Choose $U = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in U(M_2(R))$. Then $I_2 - U = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \notin P(M_2(R))$. Hence, $M_2(R)$ is not UP.*

3 Weakly exchange properties

A ring R is *exchange* if for any $a \in R$ there exists an idempotent $e \in R$ such that $e \in aR$ and $1 - e \in (1 - a)R$. Further, in [10] a ring R is called *weakly exchange* if for any $x \in R$ there exists $e^2 = e \in xR$ such that $1 - e \in (1 - x)R$ or $1 - e \in (1 + x)R$. Moreover, an element a in a ring R is called *nil clean* if there exist $e^2 = e \in R$ and $n \in N(R)$ such that $a = e + n$. If also $en = ne$, then an element a is called *strongly nil clean*. A ring R is called nil clean (strongly nil clean) if each of its elements is nil clean (strongly nil clean). It is proved in [6] that every nil clean ring is clean and it is asked is there any nil clean element also clean. So, the following can be seen as a partial answer to this question.

Theorem 5. *Let R be a UP ring. Then the following are equivalent:*

(1) *R is weakly exchange;*

(2) *R is exchange;*

(3) *R is clean;*

(4) *R is nil clean.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) Since R is UP, $2 \in J(R)$. Hence, we get result by [3].

(3) $\Rightarrow$ (4) Let $x \in R$ be clean. So, $x = e + u$ for some $e^2 = e \in R$ and $u \in U(R)$. Then we have $p \in P(R)$ such that $u = 1 + p$. Hence, $x = e + 1 + p = 1 - e + 2e + p$, where $2e + p \in P(R) \subseteq N(R)$. Thus, x is nil clean.

(4) $\Rightarrow$ (3) Let $x \in R$ be nil clean. Then there exist $e^2 = e \in R$ and $n \in N(R)$ such that $x = e + n$. Hence, $x = 1 - e + 2e - 1 + n$. Since R is a UP ring, we get $1 + n = 1 + p$ for some $p \in P(R)$. Thus we have $x = 1 - e + 2e - 1 + p$, which implies that x is clean as $2 \in P(R)$. □

Corollary 6. *Let R be a weakly exchange UP ring. Then $M_n(R)$ is nil clean for all $n \geq 1$.*

Proof. Assume that R is a weakly exchange UP ring. Then R is exchange UU. Hence, it is strongly nil clean by [3]. Further R is 2-primal since it is UP. Thus, $M_n(R)$ is nil clean by [7]. □

Theorem 6. *A ring R is P-clean if and only if*

(1) *R is weakly exchange;*

(2) *R is a UP ring.*

Proof. Necessity. This is obvious.

Sufficiency. Since R is a UP ring, we see that $2 \in R$ is nilpotent. Hence, $2 \in J(R)$. In light of [3, Proposition 2.5], R is an exchange ring. Since R is a UP ring, it is UU (see [5]). Thus, R is strongly nil clean by [4]. To see R is P-clean, let $x \in N(R)$. Since R is UP, $1 + x = 1 + y$ for some $y \in P(R)$. Hence, $x = y \in P(R)$. Thus, R is P-clean. □

Corollary 7. *A ring R is P-clean if and only if it is a clean UP ring.*

Example 3. *Consider the ring $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$. Then we have $P(R) = \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}$ and $R/P(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a Boolean ring, R is a UP ring by [2] and Theorem 6.*

Example 4. *Let $R = \prod_{n=1}^{\infty} \mathbb{Z}_{2^n}$. Then $R/P(R)$ is not Boolean. However, R is exchange. Thus, R is not a UP ring by [2] and Theorem 6.*

Corollary 8. *Let R be a ring. Then the following are equivalent:*

(1) *R is a UP ring;*

(2) *every weakly exchange element of R is P-clean.*

Corollary 9. *A ring R is UP if and only if every unit of R is P-clean.*

Proof. Necessity. It is clear by Corollary 8.

Sufficiency. Let $u \in U(R)$. Then $u = e + p$, where $e^2 = e \in R$ and $p \in P(R)$. Hence, $e = u - p = u(1 - u^{-1}p) \in U(R)$. Thus, $e = 1$ and so $u \in 1 + P(R)$. □

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Received 20.05.2023

Калчі Т.П., Чень Х., Халічоглу С. *Кільця з простими одиницями* // Карпатські матем. публ. — 2025. — Т.17, №1. — С. 152–158.

Кільце R називають UP-кільцем, якщо кожен оборотний елемент у R є сумою одиниці та сильно нільпотентного елемента. У цій статті наведено багато характеристик UP-кілець. Доведено, що для UP-кільця R такі умови є еквівалентними: R є слабко обмінним, R є обмінним, R є чистим і R є ніль-чистим. Це є частковою відповіддю на питання, поставлене А. Дізлом у [J. Algebra 2013, **383**, 197–211].

Ключові слова і фрази: UP-кільце, розширення кілець, обмінне кільце.