Topological transitivity of translation operators in a non-separable Hilbert space

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We consider a Hilbert space of entire analytic functions on a non-separable Hilbert space, associated with a non-separable Fock space. We show that under some conditions operators, like the differentiation operators and translation operators, are topologically transitive in this space.

Key words and phrases: topologically transitive operator, hypercyclic operator, function space.

Introduction

Linear dynamics is popular and new branch of mathematics. Linear dynamics is closely related to hypercyclic and topologically transitive operators [12]. We introduce the definition of topologically transitive operator.

Definition 1. Let $X$ be a Hausdorff locally convex space. A continuous operator $T: X \to X$ is called topologically transitive if for each pair $U, V$ of non-empty open subsets of $X$ there is some $n \in \mathbb{N}$ with $T^n(U) \cap V \neq \emptyset$.

If the underlying space is a separable Baire space, transitivity is equivalent to hypercyclicality.

Definition 2. A continuous linear operator $T: X \to X$ acting on a separable Fréchet space $X$ is called hypercyclic if there is a vector $x \in X$ for which the orbit under $T$, $\text{Orb}(T, x) = \{x, Tx, T^2x, \ldots\}$ is dense in $X$.

Every such vector $x$ is called a hypercyclic vector of $T$. Orbits can be quite complicated sets, even for very simple nonlinear mappings. However, there are some orbits which are especially simple and play a central role in the study of the dynamical system. We can suppose that $\text{Orb}(T, x)$ is an at most countable set.

The assumption of separability for the existence of a hypercyclic operator is necessary since if $X$ is not separable, then $X$ has no countable dense subset and hence, $\text{Orb}(T, x)$ cannot be dense in $X$ in according to the operator $T$ and vector $x$ chosen. Non-separable spaces do not support hypercyclic operators.

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The weighted unilateral backward shift or also known as the Rolewicz operator was one of the first hypercyclic operators found in 1960’s. It was shown that no linear operator is hypercyclic on any finite-dimensional space [21]. S. Rolewicz posed the problem: does an infinite-dimensional space support a hypercyclic operator? Results of S. Ansari [1] and L. Bernal-Gonzalez [5] state that every infinite-dimensional separable Banach space supports a hypercyclic operator.

An approach to obtain something similar to hypercyclic in a non-separable Hilbert and Banach spaces was given by K. Chan [10] and A. Montes and C. Romero [17], respectively. Working on non-separable Banach spaces it is clear that the definition of hypercyclic has to be replaced by the topological transitivity. Since every separable Banach space admits hypercyclic and hence topological transitive operators it is natural to ask whether every non-separable Banach space admits a topological transitive operator? However there exist several examples of Banach spaces which do not admit topologically transitive operator. For Hilbert spaces the latter question is true as shown in [4] by T. Bermudez and N.J. Kalton. Some non-separable spaces such as the space $\ell^2$, see [6].

A lot of results on hypercyclicity has been based on the Kitai-Gethner-Shapiro theorem (so-called Hypercyclicity Criterion). In [4] T. Bermudez and N.J. Kalton pointed that by the same way it is possible to prove the similar Criterion holds for topologically transitive operators. Here we present this theorem with a proof that is quite analogous to the proof of the hypercyclicity criterion (see, e.g., [3, p. 5]).

**Theorem 1** (Topologically transitive criterion). Let $T$ be a bounded linear operator on a complex Banach space $X$ (not necessarily separable). Suppose that there exists a strictly increasing sequence $\{n_k\}$ of positive integers for which there are:

(i) a dense subset $Z_0 \subset X$ such that $T^{n_k}(x) \to 0$ for every $x \in Z_0$ as $k \to \infty$;

(ii) a dense subset $Y_0 \subset X$ and a sequence of mappings (not necessary linear and not necessary continuous) $S_k : Y_0 \to X$ such that $S_k(y) \to 0$ for every $y \in Y_0$ and $T^{n_k} \circ S_k(y) \to y$ for every $y \in Y_0$ as $k \to \infty$.

Then $T$ is topologically transitive.

**Proof.** Let $U$ and $V$ be open and nonempty subsets of $X$. Pick $x \in U \cap Z_0$, $y \in V \cap Y_0$ and $\varepsilon > 0$ so that $B(x, \varepsilon) \subset U$ and $B(y, 2\varepsilon) \subset V$. By (i) and (ii) there exists $n_k$, arbitrarily large, satisfying

$$T^{n_k}(x) \in B(0, \varepsilon),$$

(that is, the fact that $T^{n_k}$ tends point-wise to zero on a dense subset of $X$ insures that $T^{n_k}(x) \cap B(0, \varepsilon)$ is nonempty for all sufficiently large $k$), also $S_k(y) \in B(0, \varepsilon)$.

So $x + S_k(y) \in U$ and

$$T^{n_k}(x + S_k y) = T^{n_k}(x) + T^{n_k}S_k(y) \to 0 + y \in V.$$

Hence, $T^{n_k}(x + S_k y) \in V$ for sufficiently large $k$. This means $T^{n_k}(U) \cap V \neq \emptyset$. Thus, $T$ is topologically transitive.
In 1929, G.D. Birkhoff proved that the translation operator \( f(x) \mapsto f(x + a) \) is hypercyclic on the Fréchet space of entire functions on the complex plane \( \mathbb{C} \) for any nonzero complex number \( a \) (see [7]). This result was generalized by many authors for composition operators in spaces of analytic functions in finite-dimensional spaces [8, 9, 13], and in infinite-dimensional Banach spaces [2, 11, 16, 18–20, 22].

In this paper, we generalize results of [18] for non-separable Hilbert spaces.

1 Non-separable function Hilbert space

In this section, we consider a general construction of a non-separable function Hilbert space as the dual space to the \( \ell_2 \)-sum of symmetric Hilbertian tensor products (see [14, 15, 18] for the separable case). Let \( E \) be a complex non-separable Hilbert space with an orthonormal basis \((e_i)_{i \in I}\), where \( I \) is an uncountable set. \( E \) is endowed with the scalar product \( (x|y)_E \) and the norm \( \|x\|_E = (x|x)_E^{1/2}, x, y \in E \). Clearly, for every \( n \in \mathbb{N} \) the \( n \)-th tensor power \( \otimes^n E \) is defined to be complex linear span of elements

\[
\{x_1 \otimes \cdots \otimes x_n: x_1, \ldots, x_n \in E\}.
\]

It is well-known that it is possible to define a norm \( \| \cdot \|_{\otimes^n E} \) on the vector space \( \otimes^n E \) such that the corresponding completion \( \otimes^n E \) is a Hilbert space. More exactly, the scalar product on \( \otimes^n E \) is defined by the equality

\[
\langle x_1 \otimes \cdots \otimes x_n | y_1 \otimes \cdots \otimes y_n \rangle_{\otimes^n E} = (x_1 | y_1)_E \cdots (x_n | y_n)_E
\]

for all \( x_i, y_i \in E, i = 1, \ldots, n \). Let \([i]\) denotes a multi-index \((i_1, \ldots, i_n), i_k \in I\). We denote by \( \mathcal{M} \) the set of all multi-indexes \([i]\). Since the system

\[
\{e_{i_1} \otimes \cdots \otimes e_{i_n} \in \otimes^n E : [i] \in \mathcal{M}\}
\]

forms an orthonormal basis in \( \otimes^n E \), every such vector \( v \in \otimes^n E \) can be represented by the Fourier series expansion

\[
v = \sum_{[i] \in \mathcal{M}} \alpha_{[i]} e_{i_1} \otimes \cdots \otimes e_{i_n},
\]

and we put

\[
\|v\|_{\otimes^n E} = \langle v | v \rangle_{\otimes^n E}^{1/2} = \left( \sum_{[i] \in \mathcal{M}} |\alpha_{[i]}|^2 \right)^{1/2}.
\]

It is clear that the above norm, generated by the scalar product, is a cross-norm on \( \otimes^n E \), that is,

\[
\|x_1 \otimes \cdots \otimes x_n\|_{\otimes^n E} = \|x_1\|_E \cdots \|x_n\|_E.
\]

We denote by \( \otimes^n E \) the \( n \)-fold symmetric algebraic tensor product of space \( E \). Every element from \( \otimes^n E \) can be defined by formula

\[
x_1 \otimes_s \cdots \otimes_s x_n := \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)},
\]

where \( x_1, \ldots, x_n \in E \) and \( S_n \) is the group of permutations on the set \( \{1, \ldots, n\} \).
We will use the following notation $e_i^\otimes k = e_i \otimes \cdots \otimes e_i$ for any $k, i \in \mathbb{N}$. We denote by $(k)$ an arbitrary multi-index $(k_1, \ldots, k_n) \in \mathbb{Z}_+^n$, and $|(k)| = \sum_i k_i! = \prod_i k_i!$.

The vectors
\[
\left\{ e_i^\otimes (k) := e_i^{\otimes k_1} \otimes \cdots \otimes e_i^{\otimes k_n} : [i] \in \mathcal{M}, (k) \in \mathbb{Z}_+^n, \quad |(k)| = n \right\}
\]
form an orthogonal basis in the closure $\otimes_n^\eta E$ of $\otimes_n^\eta E$ in $\otimes_n^\eta E$ and
\[
\left\| e_i^\otimes (k) \right\|_{\otimes_n^\eta E} = \sqrt{\frac{(k)!}{n!}}, \quad n = |(k)|.
\]

Hermitian dual of a Hilbert space $E$ we can define by the relation
\[
E^* = \left\{ y^* := \langle \cdot, y \rangle_E : y \in E \right\}.
\]

Note that the classical symmetric Fock space $\mathcal{F}$ is the Hilbert direct sum of $\otimes_{s}\eta E$, $n = 0, 1, \ldots$, where $\otimes_0^\eta E = \mathbb{C}$.

We say that a Hilbert space $\mathcal{F}_\eta$ with an arbitrary Hilbert norm $\| \cdot \|_\eta$ is a (generalized) symmetric Fock space over a given Hilbert space $E$ if vectors
\[
1, e_i^{(k)} = e_i^{\otimes k_1} \otimes \cdots \otimes e_i^{\otimes k_n}, \quad k \in \mathbb{N}, \quad k_1 + \cdots + k_n = n, \quad i_1 < \cdots < i_n,
\]
form an orthogonal basis in $\mathcal{F}_\eta$. Thus $\mathcal{F}_\eta$ can be represented by the Hilbert direct sum of symmetric tensor powers
\[
\mathcal{F}_\eta = \mathbb{C} \oplus E \oplus \otimes_2^\eta E \oplus \cdots \oplus \otimes_n^\eta E \oplus \cdots.
\]

Evidently, the norm $\| \cdot \|_\eta$ is completely defined by its value on the basis vectors. Hence, setting $\| e_i^\otimes (k) \|_\eta$ by arbitrary positive numbers, we can get various symmetric Fock spaces over $E$. Let $\langle \cdot, \cdot \rangle_\eta$ be the scalar product in $\mathcal{F}_\eta$.

Put $c_{[i]}^{(k)} := \| e_i^\otimes (k) \|_\eta^{-2}$ and $c_0 = 1$. Let us consider a power series
\[
\eta(x) = \sum_{k_1 + \cdots + k_n = 0}^{\infty} \sum_{i_1 < \cdots < i_n} c_{i_1 \cdots i_n}^{k_1 \cdots k_n} x_{i_1}^{k_1} \cdots x_{i_n}^{k_n} e_i^{\otimes k_n} = \sum_{|(k)| = 0}^{\infty} \sum_{[i] \in \mathcal{M}} c_{[i]}^{(k)} x_{[i]}^{\otimes (k)}
\]
for any $x = \sum_{i=1}^{\infty} x_i e_i \in E$.

It is known (see [18]), that there are constants $c > 0$ and $M > 0$ such that for all multi-indexes $[i] \in \mathcal{M}$, $(k) \in \mathbb{Z}_+^n$ and $n = k_1 + \cdots + k_n$ inequalities
\[
0 < c_{[i]}^{(k)} = c_{i_1 \cdots i_n}^{k_1 \cdots k_n} \leq c M^{2n} \frac{(k_1 + \cdots + k_n)!}{k_1! \cdots k_n!} = c M^{2n} \frac{n!}{k_1! \cdots k_n!}
\]
hold. Then there exists an open subset $U \subset E$, $U \ni 0$, such that for every $\phi \in \mathcal{F}_\eta$ the map $f_\phi(x) = \langle \eta(x), \phi \rangle_\eta$ is an analytic function on $U$. The function $\langle \eta(x), e_i^\otimes (k) \rangle_\eta$ is an $n$-homogeneous polynomial and
\[
\langle \eta(x), e_i^\otimes (k) \rangle_\eta = x_{i_1}^{k_1} \cdots x_{i_n}^{k_n}.
\]

Let us denote by $\mathcal{H}_\eta(E)$ the Hilbert space of analytic function $f_\phi = \langle \eta(\cdot), \phi \rangle_\eta$ that is Hermitian dual to $\mathcal{F}_\eta(E)$. 
2 Topologically transitive composition operator on Hilbert space

In this section, we consider the case when \( \mathcal{H}_\eta(E) = \mathcal{F}_\eta^a(E) \) consists with entire functions on \( E \).

Let us consider translation operator \( T_a : \mathcal{H}_\eta(E) \rightarrow \mathcal{H}_\eta(E) \), defined as

\[
T_a : f(x) \mapsto f(x+a), \quad x, a \in E, \quad a \neq 0.
\]

Since \( \mathcal{H}_\eta(E) \) is not separable space, \( T_a \) is not hypercyclic. Our purpose is to find conditions when \( T_a \) is topologically transitive.

Let us define a differentiation operator \( D_a \) on \( \mathcal{H}_\eta(E) \) by

\[
D_a(f(x)) = \sum_{i=1}^{\infty} a_i \frac{\partial}{\partial x_i} f(x),
\]

where \( a = (a_i) \in \ell_2, x \in E \). If we write

\[
f = \sum_{n=0}^{\infty} f_n = (f^0, f^1, \ldots, f^k, \ldots),
\]

then \( D_a \) can be considered as a weighted backward shift.

Let us denote a weight of the differentiation operator \( D_a \) by \( \gamma_{k_i} = \frac{k_j \sqrt{c^{(k_k)}_{[i]}}}{\sqrt{c_{[i]}}} \). Thus,

\[
D_a f = \sum_{i} \sum_{k_{[i]}} a_{k_{[i]}} \cdot \gamma_{k_{[i]}} \left( f \right) \mathcal{E}^{(k)}_{[i]} \mathcal{E}^{(k')}_{[i]},
\]

where coefficients \( c^{(k)}_{[i]} \) are defined in (1), \( n \in \mathbb{N}, [i] \in \mathcal{M}, (k) \in \mathbb{Z}_+^n, n = |(k)|, (k') = (k_1, k_2, \ldots, k_{j-1}, \ldots, k_n), j = 1, \ldots, n, \)

\[
\mathcal{E}^{(k)}_{[i]}(x) = \left( x | e^{(k)}_{[i]} \right) = \left( x | e^{(k)}_{[i]} \right) \sqrt{c^{(k)}_{[i]}} = \sqrt{c^{(k)}_{[i]}} x^{(k)}_{[i]} = \sqrt{c^{(k)}_{[i]}} c_{k_1} \ldots c_{k_n} x_{1} \ldots x_{n}.
\]

Let us suppose, in sequel, that the set \( \left\{ \frac{k_j \sqrt{c^{(k_k)}_{[i]}}}{\sqrt{c_{[i]}}} \right\} \) is bounded. Then the operator \( D_a \) is continuous on \( \mathcal{H}_\eta(E) \) [18]. Moreover, we will suppose that for every function of exponential type \( \Psi = \sum_{n=0}^{\infty} c_n t^n \) and linear functional \( \psi \in E^* \) the element \( \Psi(\psi) \) belongs to \( \mathcal{H}_\eta(E) \). From [18] it follows that this condition holds, for example, if \( \eta = \sum_{n=0}^{\infty} \frac{c_n}{(n)!^2} \).

We denote

\[
\hat{\Psi}(D_a)f(x) = \sum_{n=0}^{\infty} c_n D_a^n f(x),
\]

where \( D_a^n \) is the \( n \)th power of \( D_a \) and so \( D_a^n f(x) \) is the \( n \)th Fréchet derivative of \( f \) at the point \( x \in E \) towards \( a, a \in E \), and \( c_n \) are coefficients of function of exponential type \( \Psi \).
Theorem 2. Let $E$ be a Hilbert space and $\Psi = \sum_{n=0}^{\infty} c_n t^n$ be a nonconstant function of exponential type. If $D_a$ is continuous, then the operator

$$\hat{\Psi}(D_a): \mathcal{H}_\eta(E) \to \mathcal{H}_\eta(E),$$

$$f(x) \mapsto \sum_{n=0}^{\infty} c_n D_a^n f(x)$$

is topologically transitive.

The following lemmas were proved in [18] for a separable Hilbert spaces $E$. However they are obviously true for the non-separable case as well.

Lemma 1. $B = \{ e_\psi : \psi \in E^*, e_\psi \in \mathcal{H}_\eta \}$ is a linearly independent subset of $\mathcal{H}_\eta$.

Lemma 2. Let $U$ be a non-empty open subset of a ball in $E^*$ with radius $\delta$ and center at 0. Suppose that $e_\psi \in \mathcal{H}_\eta(E)$ for every $\psi \in U$. Then $S = \text{span} \{ e_\psi : \psi \in U \}$ is dense in $\mathcal{H}_\eta(E)$.

Proof. Let $a$ be a fixed element in $E$. Consider the function $p : E^* \to \mathbb{C}$ defined by

$$p(\psi) = \sum_{n=1}^{\infty} c_n \psi^n(a),$$

where $n \geq 0$.

Since $\hat{\Psi}$ is a function of exponential type, there exists $R > 0$ so that $|c_n| \leq \frac{R^n}{n!}$, $n \geq 1$.

Given $\psi \in E^*$ with $\|\psi\| \leq 1$, we have

$$|c_n| \cdot |\psi^n(a)| \leq \frac{R^n}{n!} \|\psi\|^n \|a\|^n \leq \left( \frac{e\|a\| R}{n} \right)^n,$$

and so

$$\left( |c_n| \cdot |\psi^n(a)| \right)^\frac{1}{n} \to 0, \quad n \to \infty.$$

It is clear that $p : E^* \to \mathbb{C}$ is a continuous and nonconstant function. So the sets

$$U : = \{ \psi \in E^* : \|p(\psi)\| < 1 \}, \quad V : = \{ \psi \in E^* : \|p(\psi)\| > 1 \},$$

where $\|p(\psi)\| = \left| \sum_{n=1}^{\infty} c_n \psi^n(a) \right|$, are both open and non-empty. Hence, according to Lemma 2,

$$X_0 = \text{span} \{ e_\psi : \psi \in U \}, \quad Y_0 = \text{span} \{ e_\psi : \psi \in V \} \quad (3)$$

are both dense subspaces of $\mathcal{H}_\eta$. Next, notice that if $T = \hat{\Psi}(D_a)$, given $\psi \in E^*$ we get

$$T(e_\psi) = \sum_{n=0}^{\infty} c_n D_a^n(e_\psi) = \sum_{n=0}^{\infty} c_n \psi^n(a)e_\psi = p(\psi)e_\psi.$$

By (3),

$$T^n \to 0 \text{ for } n \to \infty \text{ pointwise on } X_0.$$  

Also, by Lemma 1 there exists a linear map $S : Y_0 \to Y_0$ determined by

$$S(e_\psi) = [p(\psi)]^{-1} e_\psi, \quad \psi \in E^*, \quad (4)$$

which by (3) and (4) satisfies

$$\begin{cases} 
S^n \to 0 & \text{for } n \to \infty \text{ pointwise on } Y_0, \\
TS = \text{id}_{Y_0} & \text{on } Y_0.
\end{cases}$$

By Theorem 1, $T = \hat{\Psi}(D_a)$ is topologically transitive. \qed
We note that $\Psi(D_a) = e^{Da}$.

**Corollary 1.** Let $E$ be a Hilbert space and

$$T_a: \mathcal{H}_\eta(E) \to \mathcal{H}_\eta(E),$$

where $\mathcal{H}_\eta(E)$ as in Theorem 2, and

$$T_a: f(x) \mapsto f(x + a).$$

Then

$$T_a = \sum_{n=0}^{\infty} \frac{1}{n!} D_a^n$$

and so $T_a$ is topologically transitive.

**References**


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У статті розглянуто гільбертові простори цілих аналітичних функцій на несепарабельному гільбертовому просторі, асоційованому з деяким несепарабельним простором Фока. Показано, що за деяких умов такі оператори, як оператор диференціювання та оператор зсуву, будуть топологічно транзитивними в цьому просторі.

Ключові слова і фрази: топологічно транзитивний оператор, гіперциклічний оператор, функційний простір.