ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2025, **17** (1), 343–363
doi:10.15330/cmp.17.1.343-363

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2025, Т.17, №1, С.343–363

Some fixed point results on controlled rectangular partial metric spaces with a graph structure

Alamgir N.[1], Kiran Q.[2], Ali Y.[3], Aydi H.[4,5]

In this work, we define a controlled rectangular partial metric space, which is a generalization of a controlled rectangular metric space and a controlled metric space. Furthermore, we derive some fixed point results with suitable conditions in this setting. As an application, we give a fixed point theorem for graphic contractions on a controlled rectangular partial metric space via a graph structure.

Key words and phrases: controlled rectangular partial metric space, fixed point, graph.

[1] SNS, National University of Sciences and Technology, Sector H-12, Islamabad, Pakistan
[2] SEECS, National University of Sciences and Technology, Sector H-12, Islamabad, Pakistan
[3] CEME, National University of Sciences and Technology, Islamabad, Pakistan
[4] Institut Supérieur d'Informatique et des Techniques de Communication, Université de Sousse, 4000, H. Sousse, Tunisia
[5] Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa
E-mail: nayab@sns.nust.edu.pk (Alamgir N.), quanita.kiran@seecs.edu.pk (Kiran Q.), yali@ceme.nust.edu.pk (Ali Y.), hassen.aydi@isima.rnu.tn (Aydi H.)

1 Introduction

In the recent years, fixed point theory appears in many fields of mathematics by its convenient application to resolve problems of nonlinear analysis, graph theory and differential equations. For more details, see [3, 8, 10, 16, 19, 29]. A large number of researchers have exerted in finding new metric spaces, besides to variant contraction mappings. I.A. Bakhtin [11] and S. Czerwik [12] presented the famous b-metric space generalization. A convenient application of the b-metric space opened avenues for researchers to apply fixed point theorems for resolving problems in variety of situations. Resultantly, many generalizations of the metric space have emerged. Namely, T. Kamran et. al. [15] in 2017 generalized the concept of a b-metric space by replacing in the triangle inequality the constant s by a function $\theta(u,v)$ as follows

$$d_\varsigma(u,w) \leq \varsigma(u,w)[d_\varsigma(u,v) + d_\varsigma(v,w)]$$

for all $u, v, w \in U$. Note that each b-metric space is an extended b-metric space, but the converse is not true in general. Since then, many researchers have successfully generalized it in different directions, see [5, 7, 9, 17, 23, 27]. Lately, N. Mlaiki et. al. [18] developed the notion of a controlled metric space by generalizing the extended b-metric with the following triangle inequality

$$d_\varsigma(u,w) \leq \varsigma(u,v)d_\varsigma(u,v) + \varsigma(v,w)d_\varsigma(v,w)$$

for all $u, v, w \in U$. Since then, a good amount of very valuable papers have been published, see [2, 6, 13, 21, 22, 28]. N. Alamgir et. al. [4] introduced a controlled rectangular metric space as a generalization of a controlled metric space as follows.

УДК 515.126.4, 517.988
2020 *Mathematics Subject Classification:* 46T99, 47H10, 54H25.

Definition 1. *A mapping $d_\varsigma : U \times U \to [0,\infty)$ on a nonempty set U is called an extended rectangular b-metric, if for all $u, v \in U$, and all distinct $\mu_1, \mu_2 \in U \setminus \{u, v\}$, the following axioms are satisfies:*

(i) $d_\varsigma(u,v) = 0$ *if and only if* $u = v$,

(ii) $d_\varsigma(u,v) = d_\varsigma(v,u)$,

(iii) $d_\varsigma(u,v) \leq \varsigma(u,\mu_1)d_\varsigma(u,\mu_1) + \varsigma(\mu_1,\mu_2)d_\varsigma(\mu_1,\mu_2) + \varsigma(\mu_2,v)d_\varsigma(\mu_2,v)$,

where $\varsigma : U \times U \to [1,\infty)$ is a map. Then (U, d_ς) is a controlled rectangular metric space.

Example 1. *Let $U = [0,\infty)$. Define a mapping $d_\varsigma : U \times U \to [0,\infty)$ by*

$$d_\varsigma(u,v) = \begin{cases} 0, & \text{if } u = v, \\ \frac{1}{u}, & \text{if } u \geq 1 \text{ and } v \in [0,1), \\ \frac{1}{v}, & \text{if } v \geq 1 \text{ and } u \in [0,1), \\ 1, & \text{otherwise.} \end{cases}$$

Then (U, d_ς) is a controlled rectangular metric space, where $\varsigma : U \times U \to [1,\infty)$ is a mapping defined by

$$\varsigma(u,v) = \begin{cases} u, & \text{if } u, v \geq 1, \\ 1, & \text{otherwise.} \end{cases}$$

In this paper, we introduce the notion of a controlled rectangular partial metric space, which generalizes a controlled rectangular metric space and a controlled metric space. Further, we derive several fixed point results on such spaces via a graph structure [25].

2 Some fixed point results on controlled rectangular partial metric spaces

In this section, we introduce a controlled rectangular partial metric space and prove some related fixed point results.

Definition 2. *A mapping $d_\varsigma : U \times U \to [0,\infty)$ on a nonempty set U with a real valued function $\varsigma : U \times U \to [1,\infty)$ is called a controlled rectangular partial metric, if for all $u, v \in U$ and all distinct $\mu_1, \mu_2 \in U \setminus \{u, v\}$, the following axioms are satisfied:*

(i) $d_\varsigma(u,v) \geq 0$,

(ii) $u = v$ *if and only if* $d_\varsigma(u,u) = d_\varsigma(u,v) = d_\varsigma(v,v)$,

(iii) $d_\varsigma(u,v) = d_\varsigma(v,u)$,

(iv) $d_\varsigma(u,u) \leq d_\varsigma(u,v)$,

(v) $$d_\varsigma(u,v) \leq \varsigma(u,\mu_1)d_\varsigma(u,\mu_1) + \varsigma(\mu_1,\mu_2)d_\varsigma(\mu_1,\mu_2) + \varsigma(\mu_2,v)d_\varsigma(\mu_2,v) - d_\varsigma(\mu_1,\mu_1) - d_\varsigma(\mu_2,\mu_2).$$

The pair (U, d_ς) is a controlled rectangular partial metric space.

Remark 1. *A controlled rectangular partial metric space is different from a controlled partial metric space [25]. Also, every controlled rectangular metric space is a controlled rectangular partial metric space with a zero self-distance, but the converse statement is not true. For explanation, see the following example.*

Example 2. *Let $U = [0,3]$ and $\sigma \geq 3$ be a constant. Define a mapping $d_\varsigma : U \times U \to [0,\infty)$ by*

$$d_\varsigma(u,v) = \begin{cases} \sigma, & \text{if } u = v, \\ \frac{1}{u} + \frac{1}{\sigma}, & \text{if } u \geq 1 \text{ and } v \in [0,1), \\ \frac{1}{v} + \frac{1}{\sigma}, & \text{if } v \geq 1 \text{ and } u \in [0,1), \\ 1 + \frac{1}{\sigma}, & \text{otherwise.} \end{cases}$$

Then (U, d_ς) is a controlled rectangular partial metric space, where $\varsigma : U \times U \to [1,\infty)$ is a function defined by

$$\varsigma(u,v) = \begin{cases} u, & \text{if } u, v \geq 1, \\ 1, & \text{otherwise.} \end{cases}$$

Note that (U, d_ς) is not a controlled partial metric space, because if we take $u = 3$, $v = 0.5$, $w = 4$ and $\sigma = 6$, then $d_\varsigma(u,w) = 1 + \frac{1}{6} = 1.16666$, $d_\varsigma(u,v) = \frac{1}{3} + \frac{1}{6} = 0.49999$, $d_\varsigma(v,w) = \frac{1}{4} + \frac{1}{6} = 0.41666$ and $\varsigma(u,v) = 1$, $\varsigma(v,w) = 1$, so the triangle inequality does not hold. Indeed,

$$d_\varsigma(u,w) = 1.16666 > \varsigma(u,v)d_\varsigma(u,v) + \varsigma(v,w)d_\varsigma(v,w) = 0.49999 + 0.41666 = 0.91665.$$

Clearly, (U, d_ς) is not also a controlled rectangular metric space because the self distance is not zero.

Next, we will define convergent and Cauchy sequences in the setting of controlled rectangular partial metric spaces.

Definition 3. *Let $\{u_n\}$ be a sequence in the controlled rectangular partial metric space (U, d_ς) and $u \in U$. Then*

(i) $\{u_n\}$ converges to u, if $\lim_{n\to\infty} d_\varsigma(u_n, u) = d_\varsigma(u,u)$, in such a case, we write $\lim_{n\to\infty} u_n = u$,

(ii) $\{u_n\}$ is called Cauchy sequence, if $\lim_{m,n\to\infty} d_\varsigma(u_m, u_n)$ exists and is finite,

(iii) (U, d_ς) is called complete, if every Cauchy sequence in M converges to u, so that

$$\lim_{m,n\to\infty} d_\varsigma(u_m, u_n) = d_\varsigma(u,u).$$

Remark 2. *It should be noted that*

(i) a controlled rectangular partial metric space is not continuous in general,

(ii) the limit of a convergent sequence in a controlled rectangular partial metric space need not be unique.

The following example confirms the second statement.

Example 3. *Let $U = [0, \infty)$. Define a map $d_\varsigma : U \times U \to [0, \infty)$ by $d_\varsigma(u, v) = \max\{u, v\} + \sigma$, where $\sigma > 0$ is an arbitrary constant. Also, define a function $\varsigma : U \times U \to [1, \infty)$ by*

$$\varsigma(u,v) = \begin{cases} uv, & \text{if } u, v \in [1, \infty), \\ 1, & \text{otherwise.} \end{cases}$$

Then (U, d_ς) is a complete controlled rectangular partial metric space. Now, consider the sequence $u_n = \{\frac{1}{n}\}$ for all $n \in \mathbb{N}$. Then for all $v \geq 1$, we have

$$\lim_{n \to \infty} d_\varsigma(u_n, v) = d_\varsigma(v, v).$$

Hence, $\{u_n\}$ has more than one limit in U.

Theorem 1. *Let $T : U \to U$ be a mapping on complete controlled rectangular partial metric space (U, d_ς). Suppose that the following axioms hold:*

(i) for all $u, v \in U$ we have

$$d_\varsigma(Tu, Tv) \leq \lambda d_\varsigma(u, v), \tag{1}$$

where $\lambda \in [0, 1)$,

(ii) for each $u_0 \in U$ we have $\lim_{n,m \to \infty} \varsigma(u_n, u_m)\lambda < 1$, where $u_n = T^n u_0$, $n = 1, 2, \ldots$.

Then T has a unique fixed point $\mu \in U$ and $d_\varsigma(\mu, \mu) = 0$.

Proof. First, we will show that if μ is a fixed point of T, then $d_\varsigma(\mu, \mu) = 0$. Let us suppose that $d_\varsigma(\mu, \mu) > 0$, then from inequality (1) it follows that

$$d_\varsigma(\mu, \mu) = d_\varsigma(T\mu, T\mu) \leq \lambda d_\varsigma(\mu, \mu) < d_\varsigma(\mu, \mu).$$

It is a contradiction. Hence, $d_\varsigma(\mu, \mu) = 0$.

For the existence of a fixed point, let us take an arbitrary element $u_0 \in U$, and define an iterative sequence $\{u_n\}$ over u_0 as follows

$$u_1 = Tu_0, \quad u_2 = Tu_1 = T(Tu_0) = T^2 u_0, \ldots, \quad u_n = T^n u_0, \ldots.$$

Now, by recursively applying inequality (1), we obtain

$$\begin{aligned} d_\varsigma(u_n, u_{n+1}) = d_\varsigma(T^n u_0, T^{n+1} u_0) &\leq \lambda d_\varsigma(T^{n-1} u_0, T^n u_0) \\ &\leq \lambda^2 d_\varsigma(T^{n-2} u_0, T^{n-1} u_0) \leq \cdots \leq \lambda^n d_\varsigma(u_0, u_1). \end{aligned}$$

So, we have

$$d_\varsigma(u_n, u_{n+1}) \leq \lambda^n d_\varsigma(u_0, u_1). \tag{2}$$

By taking limit in the above inequality as $n \to \infty$, we get

$$\lim_{n \to \infty} d_\varsigma(u_n, u_{n+1}) = 0.$$

Next, we show that $\{u_n\}$ is a Cauchy sequence. For this end, we consider the following two cases.

Case 1. Let κ be odd, that is, $\kappa = 2m+1$ with $m \geq 1$. Then from condition *(iii)* of Definition 2 and inequality (2) for $n+\kappa > n$, we have

$$\begin{aligned}
d_\varsigma(u_n, u_{n+2m+1}) &\leq \varsigma(u_n, u_{n+1}) d_\varsigma(u_n, u_{n+1}) + \varsigma(u_{n+1}, u_{n+2}) d_\varsigma(u_{n+1}, u_{n+2}) \\
&\quad + \varsigma(u_{n+2}, u_{n+2m+1}) d_\varsigma(u_{n+2}, u_{n+2m+1}) - d_\varsigma(u_{n+1}, u_{n+1}) - d_\varsigma(u_{n+2}, u_{n+2}) \\
&\leq \left[\varsigma(u_n, u_{n+1})\lambda^n + \varsigma(u_{n+1}, u_{n+2})\lambda^{n+1}\right] d_\varsigma(u_0, u_1) \\
&\quad + \varsigma(u_{n+2}, u_{n+2m+1}) d_\varsigma(u_{n+2}, u_{n+2m+1}) - d_\varsigma(u_{n+1}, u_{n+1}) - d_\varsigma(u_{n+2}, u_{n+2}) \\
&\leq \left[\varsigma(u_n, u_{n+1})\lambda^n + \varsigma(u_{n+1}, u_{n+2})\lambda^{n+1}\right] d_\varsigma(u_0, u_1) - d_\varsigma(u_{n+1}, u_{n+1}) \\
&\quad - d_\varsigma(u_{n+2}, u_{n+2}) + \varsigma(u_{n+2}, u_{n+2m+1})\varsigma(u_{n+2}, u_{n+3}) d_\varsigma(u_{n+2}, u_{n+3}) \\
&\quad + \varsigma(u_{n+2}, u_{n+2m+1})\varsigma(u_{n+3}, u_{n+4}) d_\varsigma(u_{n+3}, u_{n+4}) \\
&\quad + \varsigma(u_{n+2}, u_{n+2m+1})\varsigma(u_{n+4}, u_{n+2m+1}) d_\varsigma(u_{n+4}, u_{n+2m+1}) \\
&\quad - \varsigma(u_{n+2}, u_{n+2m+1}) d_\varsigma(u_{n+3}, u_{n+3}) - \varsigma(u_{n+2}, u_{n+2m+1}) d_\varsigma(u_{n+4}, u_{n+4}) \\
&\leq \left[\varsigma(u_n, u_{n+1})\lambda^n + \varsigma(u_{n+1}, u_{n+2})\lambda^{n+1}\right] d_\varsigma(u_0, u_1) \\
&\quad + \varsigma(u_{n+2}, u_{n+2m+1})\varsigma(u_{n+2}, u_{n+3})\lambda^{n+2} d_\varsigma(u_0, u_1) \\
&\quad + \varsigma(u_{n+2}, u_{n+2m+1})\varsigma(u_{n+3}, u_{n+4})\lambda^{n+3} d_\varsigma(u_0, u_1) + \ldots \\
&\quad + d_\varsigma(u_0, u_1)\varsigma(u_{n+2}, u_{n+2m+1}) \ldots \varsigma(u_{n+2m-2}, u_{n+2n-1})\lambda^{n+2m-2} \\
&\quad + \varsigma(u_{n+2}, u_{n+2m+1}) \ldots \varsigma(u_{n+2m-1}, u_{n+2m})\lambda^{n+2m-1} d_\varsigma(u_0, u_1) \\
&\quad + \varsigma(u_{n+2}, u_{n+2m+1}) \ldots \varsigma(u_{n+2m}, u_{n+2m+1})\lambda^{n+2m} d_\varsigma(u_0, u_1) \\
&\quad - \left[\lambda^{n+1} - \lambda^{n+2}\right] d_\varsigma(u_0, u_0) - \varsigma(u_{n+2}, u_{n+2m+1}) d_\varsigma(u_0, u_0)\lambda^{n+3} \\
&\quad - \varsigma(u_{n+2}, u_{n+2m+1}) d_\varsigma(u_0, u_0)\lambda^{n+4} - \ldots \\
&\quad - d_\varsigma(u_0, u_0)\varsigma(u_{n+2}, u_{n+2m+1}) \ldots \varsigma(u_{n+2m-2}, u_{n+2m+1})\lambda^{n+2m-1} \\
&\quad - d_\varsigma(u_0, u_0)\varsigma(u_{n+2}, u_{n+2m+1}) \ldots \varsigma(u_{n+2m-2}, u_{n+2m+1})\lambda^{n+2m} \\
&\leq d_\varsigma(u_0, u_1)\left[\varsigma(u_n, u_{n+1})\lambda^n + \varsigma(u_{n+2}, u_{n+3})\lambda^{n+1}\right] \\
&\quad + \sum_{i=1}^{m-1} \prod_{j=1}^{i} \varsigma(u_{n+2j}, u_{n+2m+1})\varsigma(u_{n+2i}, u_{n+2i+1})\lambda^{n+2i} d_\varsigma(u_0, u_1) \\
&\quad + \sum_{i=1}^{m-1} \prod_{j=1}^{i} \varsigma(u_{n+2j+1}, u_{n+2m+1})\varsigma(u_{n+2i+1}, u_{n+2i+2})\lambda^{n+2i+1} d_\varsigma(u_0, u_1) \\
&\quad + \prod_{i=1}^{m} \varsigma(u_{n+2i}, u_{n+2m+1})\lambda^{n+2m} d_\varsigma(u_0, u_1) - \left[\lambda^{n+1} - \lambda^{n+2}\right] d_\varsigma(u_0, u_0) \\
&\quad - \sum_{i=1}^{m-1} \prod_{j=1}^{i} \varsigma(u_{n+2j}, u_{n+2m+1})\lambda^{n+2i+1} d_\varsigma(u_0, u_0) \\
&\quad - \sum_{i=1}^{m-1} \prod_{j=1}^{i} \varsigma(u_{n+2j}, u_{n+2m+1})\lambda^{n+2i+2} d_\varsigma(u_0, u_0).
\end{aligned}$$

We get

$$\sum_{i=1}^{m-1} \prod_{j=1}^{i} \varsigma(u_{n+2j}, u_{n+2m+1})\varsigma(u_{n+2i}, u_{n+2i+1})\lambda^{n+2i} \leq \sum_{i=1}^{m-1} \prod_{j=1}^{i} \varsigma(u_{2j}, u_{n+2m+1})\varsigma(u_{2i}, u_{2i+1})\lambda^{2i}$$

and

$$\sum_{i=1}^{m-1}\prod_{j=1}^{i}\varsigma(u_{n+2j+1},u_{n+2m+1})\varsigma(u_{n+2i+1},u_{n+2i+2})\lambda^{n+2i+1} \le \sum_{i=1}^{m-1}\prod_{j=1}^{i}\varsigma(u_{2j+1},u_{n+2m+1})\varsigma(u_{2i+1},u_{n+2i+2})\lambda^{2i+1}.$$

Therefore, we obtain

$$\begin{aligned} d_\varsigma(u_n,u_{n+2m+1}) \le{}& d_\varsigma(u_0,u_1)\left[\varsigma(u_n,u_{n+1})\lambda^n+\varsigma(u_{n+2},u_{n+3})\lambda^{n+1}\right] \\ &+\sum_{i=1}^{m-1}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m+1})\varsigma(u_{2i},u_{2i+1})\lambda^{2i}d_\varsigma(u_0,u_1) \\ &+\sum_{i=1}^{m-1}\prod_{j=1}^{i}\varsigma(u_{2j+1},u_{n+2m+1})\varsigma(u_{2i+1},u_{2i+2})\lambda^{2i+1}d_\varsigma(u_0,u_1) \\ &+\prod_{i=1}^{m-1}\varsigma(u_{n+2i},u_{n+2m+1})\lambda^{n+2m}d_\varsigma(u_0,u_1)-\left[\lambda^{n+1}-\lambda^{n+2}\right]d_\varsigma(u_0,u_0) \\ &-\sum_{i=1}^{m-1}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m+1})\lambda^{2i+1}d_\varsigma(u_0,u_0) \\ &-\sum_{i=1}^{m-1}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m+1})\lambda^{2i+2}d_\varsigma(u_0,u_0). \end{aligned} \tag{3}$$

Since $\lim_{n,m\to\infty}\varsigma(u_n,u_m)\lambda<1$, the series

$$\sum_{i=1}^{\infty}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m+1})\varsigma(u_{2i},u_{2i+1})\lambda^{2i}, \qquad \sum_{i=1}^{\infty}\prod_{j=1}^{i}\varsigma(u_{2j+1},u_{n+2m+1})\varsigma(u_{2i+1},u_{2i+2})\lambda^{2i+1},$$

$$\sum_{i=1}^{\infty}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m+1}\lambda^{2i+1}, \qquad \sum_{i=1}^{\infty}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m+1}\lambda^{2i+2}$$

converge by the ratio test. Let

$$S=\sum_{i=1}^{\infty}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m+1})\varsigma(u_{2i},u_{2i+1})\lambda^{2i}, \quad S'=\sum_{i=1}^{\infty}\prod_{j=1}^{i}\varsigma(u_{2j+1},u_{n+2m+1})\varsigma(u_{2i+1},u_{2i+2})\lambda^{2i+1},$$

$$S''=\sum_{i=1}^{\infty}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m+1})\lambda^{2i+1}, \quad S'''=\sum_{i=1}^{\infty}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m+1})\lambda^{2i+2},$$

$$S_n=\sum_{i=1}^{n}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m+1})\varsigma(u_{2i},u_{2i+1})\lambda^{2i}, \quad S'_n=\sum_{i=1}^{n}\prod_{j=1}^{i}\varsigma(u_{2j+1},u_{n+2m+1})\varsigma(u_{2i+1},u_{2i+2})\lambda^{2i+1},$$

$$S''_n=\sum_{i=1}^{n}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m+1})\lambda^{2i+1}, \quad S'''_n=\sum_{i=1}^{n}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m+1})\lambda^{2i+2}.$$

Then inequality (3) takes the following form

$$\begin{aligned} d_\varsigma(u_n, u_{n+2m+1}) \leq\ & d_\varsigma(u_0, u_1)\big[\varsigma(u_n, u_{n+1})\lambda^n + \varsigma(u_{n+2}, u_{n+3})\lambda^{n+1}\big] \\ & + d_\varsigma(u_0, u_1)\,[S_{m-1} - S_{n+1}] + d_\varsigma(u_0, u_1)\big[S'_{m-1} - S'_{n+1}\big] d_\varsigma(u_0, u_1) \\ & + \prod_{i=1}^{m-1} \varsigma(u_{n+2j}, u_{n+2m+1})\lambda^{n+2m} d_\varsigma(u_0, u_1) - \big[\lambda^{n+1} - \lambda^{n+2}\big] d_\varsigma(u_0, u_0) \\ & - d_\varsigma(u_0, u_0)\big[S''_{m-1} - S''_{n+1}\big] - d_\varsigma(u_0, u_0)\big[S'''_{m-1} - S'''_{n+1}\big]. \end{aligned} \tag{4}$$

By taking limit in the above inequality as $n \to \infty$, we get $\lim_{n\to\infty} d_\varsigma(u_n, u_{n+2m+1}) = 0$.

Case 2. Let κ be even, that is, $\kappa = 2m$ with $m \geq 1$. Then from condition *(iii)* of Definition 2 and inequality (2) for $n + \kappa > n$, we have

$$\begin{aligned} d_\varsigma(u_n, u_{n+2m}) \leq\ & \varsigma(u_n, u_{n+1}) d_\varsigma(u_n, u_{n+1}) + \varsigma(u_{n+1}, u_{n+2}) d_\varsigma(u_{n+1}, u_{n+2}) \\ & + \varsigma(u_{n+2}, u_{n+2m}) d_\varsigma(u_{n+2}, u_{n+2m}) - d_\varsigma(u_{n+1}, u_{n+1}) - d_\varsigma(u_{n+2}, u_{n+2}) \\ \leq\ & \big[\varsigma(u_n, u_{n+1})\lambda^n + \varsigma(u_{n+1}, u_{n+2})\lambda^{n+1}\big] d_\varsigma(u_0, u_1) \\ & + \varsigma(u_{n+2}, u_{n+2m}) d_\varsigma(u_{n+2}, u_{n+2m}) - d_\varsigma(u_{n+1}, u_{n+1}) - d_\varsigma(u_{n+2}, u_{n+2}) \\ \leq\ & \big[\varsigma(u_n, u_{n+1})\lambda^n + \varsigma(u_{n+1}, u_{n+2})\lambda^{n+1}\big] d_\varsigma(u_0, u_1) - d_\varsigma(u_{n+1}, u_{n+1}) \\ & - d_\varsigma(u_{n+2}, u_{n+2}) + \varsigma(u_{n+2}, u_{n+2m})\varsigma(u_{n+2}, u_{n+3}) d_\varsigma(u_{n+2}, u_{n+3}) \\ & + \varsigma(u_{n+2}, u_{n+2m})\varsigma(u_{n+3}, u_{n+4}) d_\varsigma(u_{n+3}, u_{n+4}) \\ & + \varsigma(u_{n+2}, u_{n+2m})\varsigma(u_{n+4}, u_{n+2m}) d_\varsigma(u_{n+4}, u_{n+2m}) \\ & - \varsigma(u_{n+2}, u_{n+2m}) d_\varsigma(u_{n+3}, u_{n+3}) - \varsigma(u_{n+2}, u_{n+2m}) d_\varsigma(u_{n+4}, u_{n+4}) \\ \leq\ & \big[\varsigma(u_n, u_{n+1})\lambda^n + \varsigma(u_{n+1}, u_{n+2})\lambda^{n+1}\big] d_\varsigma(u_0, u_1) \\ & + \varsigma(u_{n+2}, u_{n+2m})\big(\varsigma(u_{n+2}, u_{n+3})\lambda^{n+2} + \varsigma(u_{n+3}, u_{n+4})\lambda^{n+3}\big) d_\varsigma(u_0, u_1) \\ & + \varsigma(u_{n+2}, u_{n+2m}) \ldots \varsigma(u_{n+2m-2}, u_{n+2m})\varsigma(u_{n+2m-2}, u_{n+2m-1})\lambda^{n+2m-2} d_\varsigma(u_0, u_1) \\ & + \varsigma(u_{n+2}, u_{n+2m}) \ldots \varsigma(u_{n+2m-1}, u_{n+2m})\varsigma(u_{n+2m-1}, u_{n+2m})\lambda^{n+2m-1} d_\varsigma(u_0, u_1) \\ & + \varsigma(u_{n+2}, u_{n+2m}) \ldots \varsigma(u_{n+2m-2}, u_{n+2m})\varsigma(u_{n+2m}, u_{n+2m})\lambda^{n+2m} d_\varsigma(u_0, u_0) \\ & - \big[\lambda^{n+1} - \lambda^{n+2}\big] d_\varsigma(u_0, u_0) - \varsigma(u_{n+2}, u_{n+2m}) d_\varsigma(u_0, u_0)\lambda^{n+3} \\ & - \varsigma(u_{n+2}, u_{n+2m}) d_\varsigma(u_0, u_0)\lambda^{n+4} - \ldots \\ & - \varsigma(u_{n+2}, u_{n+2m}) \ldots \varsigma(u_{n+2m-2}, u_{n+2m})\big(\lambda^{n+2m-1} - \lambda^{n+2m}\big) d_\varsigma(u_0, u_0) \\ \leq\ & d_\varsigma(u_0, u_1)\big[\varsigma(u_n, u_{n+1})\lambda^n + \varsigma(u_{n+2}, u_{n+3})\lambda^{n+1}\big] \\ & + \sum_{i=1}^{m-1} \prod_{j=1}^{i} \varsigma(u_{n+2j}, u_{n+2m})\varsigma(u_{n+2i}, u_{n+2i+1})\lambda^{n+2i} d_\varsigma(u_0, u_1) \\ & + \sum_{i=1}^{m-1} \prod_{j=1}^{i} \varsigma(u_{n+2j}, u_{n+2m})\varsigma(u_{n+2i+1}, u_{n+2i+2})\lambda^{n+2i+1} d_\varsigma(u_0, u_1) \\ & + \prod_{i=1}^{m} \varsigma(u_{n+2i}, u_{n+2m})\lambda^{n+2m} d_\varsigma(u_0, u_0) - \big[\lambda^{n+1} - \lambda^{n+2}\big] d_\varsigma(u_0, u_0) \\ & - \sum_{i=1}^{m-1} \prod_{j=1}^{i} \varsigma(u_{n+2j}, u_{n+2m})\big(\lambda^{n+2i+1} + \lambda^{n+2i+2}\big) d_\varsigma(u_0, u_0). \end{aligned}$$

We have

$$\sum_{i=1}^{m-1}\prod_{j=1}^{i}\varsigma(u_{n+2j},u_{n+2m})\varsigma(u_{n+2i},u_{n+2i+1})\lambda^{n+2i} \leq \sum_{i=1}^{m-1}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m})\varsigma(u_{2i},u_{2i+1})\lambda^{2i},$$

$$\sum_{i=1}^{m-1}\prod_{j=1}^{i}\varsigma(u_{n+2j},u_{n+2m})\varsigma(u_{n+2i+1},u_{n+2i+2})\lambda^{n+2i+1} \leq \sum_{i=1}^{m-1}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m})\varsigma(u_{2i+1},u_{2i+2})\lambda^{2i+1}.$$

Therefore, we obtain

$$\begin{aligned} d_\varsigma(u_n,u_{n+2m}) \leq\ & d_\varsigma(u_0,u_1)\left[\varsigma(u_n,u_{n+1})\lambda^n+\varsigma(u_{n+2},u_{n+3})\lambda^{n+1}\right] \\ &+\sum_{i=1}^{m-1}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m})\varsigma(u_{2i},u_{2i+1})\lambda^{2i}d_\varsigma(u_0,u_1) \\ &+\sum_{i=1}^{m-1}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m})\varsigma(u_{2i+1},u_{2i+2})\lambda^{2i+1}d_\varsigma(u_0,u_1) \\ &+\prod_{i=1}^{m}\varsigma(u_{n+2i},u_{n+2m})\lambda^{n+2m}d_\varsigma(u_0,u_0)-\left[\lambda^{n+1}-\lambda^{n+2}\right]d_\varsigma(u_0,u_0) \\ &-\sum_{i=1}^{m-1}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m})\left(\lambda^{2i+1}+\lambda^{2i+2}\right)d_\varsigma(u_0,u_0). \end{aligned}$$

Since $\lim_{n,m\to\infty}\varsigma(u_n,u_m)\lambda<1$, the series

$$\sum_{i=1}^{\infty}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m})\varsigma(u_{2i},u_{2i+1})\lambda^{2i}, \qquad \sum_{i=1}^{\infty}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m})\varsigma(u_{2i+1},u_{2i+2})\lambda^{2i+1},$$

$$\sum_{i=1}^{\infty}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m})\lambda^{2i+1}, \qquad \sum_{i=1}^{\infty}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m})\lambda^{2i+2},$$

converge by the ratio test. Let

$$S=\sum_{i=1}^{\infty}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m})\varsigma(u_{2i},u_{2i+1}))\lambda^{2i}, \qquad S'=\sum_{i=1}^{\infty}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m})\varsigma(u_{2i+1},u_{2i+2})\lambda^{2i+1},$$

$$S''=\sum_{i=1}^{\infty}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m})\lambda^{2i+1}, \qquad S'''=\sum_{i=1}^{\infty}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m})\lambda^{2i+2},$$

$$S_n=\sum_{i=1}^{n}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m})\varsigma(u_{2i},u_{2i+1}))\lambda^{2i}, \qquad S'_n=\sum_{i=1}^{n}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m})\varsigma(u_{2i+1},u_{2i+2})\lambda^{2i+1},$$

$$S''_n=\sum_{i=1}^{n}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m})\lambda^{2i+1}, \qquad S'''_n=\sum_{i=1}^{n}\prod_{j=1}^{i}\varsigma(u_{2j},u_{n+2m})\lambda^{2i+2}.$$

Then inequality (3) takes the following form

$$\begin{aligned} d_\varsigma(u_n,u_{n+2m}) \leq\ & d_\varsigma(u_0,u_1)\left[\varsigma(u_n,u_{n+1})\lambda^n+\varsigma(u_{n+2},u_{n+3})\lambda^{n+1}\right] \\ &+d_\varsigma(u_0,u_1)\left[S_{m-1}-S_{n+1}\right]+d_\varsigma(u_0,u_1)\left[S'_{m-1}-S'_{n+1}\right]d_\varsigma(u_0,u_1) \\ &+\prod_{i=1}^{m}\varsigma(u_{n+2j},u_{n+2m})\lambda^{n+2m}d_\varsigma(u_0,u_0)-\left[\lambda^{n+1}-\lambda^{n+2}\right]d_\varsigma(u_0,u_0) \\ &-d_\varsigma(u_0,u_0)\left[S''_{m-1}-S''_{n+1}\right]-d_\varsigma(u_0,u_0)\left[S'''_{m-1}-S'''_{n+1}\right]. \end{aligned}$$

By taking limit in the above inequality as $n \to \infty$, we get $\lim_{n\to\infty} d_\varsigma(u_n, u_{n+2m}) = 0$.

Hence, in both cases

$$\lim_{n\to\infty} d_\varsigma(u_n, u_{n+2m}) = 0.$$

This shows that $\{u_n\}$ is a Cauchy sequence. As U is complete, so there exists $\mu \in U$ such that

$$\lim_{n\to\infty} d_\varsigma(u_n, \mu) = \lim_{n,m\to\infty} d_\varsigma(u_n, u_m) = d_\varsigma(\mu, \mu) = 0. \tag{5}$$

Now we show that μ is a fixed point of T. From axiom *(v)* of Definition 2, we have

$$\begin{aligned} d_\varsigma(\mu, T\mu) &\leq \varsigma(\mu, u_n)d_\varsigma(\mu, u_n) + \varsigma(u_n, u_{n+1})d_\varsigma(u_n, u_{n+1}) \\ &\quad + \varsigma(u_{n+1}, T\mu)d_\varsigma(u_{n+1}, T\mu) - d_\varsigma(u_n, u_n) - d_\varsigma(u_{n+1}, u_{n+1}) \\ &\leq \varsigma(\mu, u_n)d_\varsigma(\mu, u_n) + \varsigma(u_n, u_{n+1})d_\varsigma(Tu_{n-1}, Tu_n) + \varsigma(u_{n+1}, T\mu)d_\varsigma(Tu_n, T\mu) \\ &\leq \varsigma(\mu, u_n)d_\varsigma(\mu, u_n) + \varsigma(u_n, u_{n+1})\lambda d_\varsigma(u_{n-1}, u_n) + \varsigma(u_{n+1}, \mu)\lambda d_\varsigma(u_n, \mu). \end{aligned}$$

Since $\lim_{n,m\to\infty} \varsigma(u_n, u_m)\lambda < 1$ for any $u_n \in U$, by taking limit as $n \to \infty$ and using (5), we get

$$\lim_{n\to\infty} d_\varsigma(\mu, T\mu) = 0.$$

Therefore, $T\mu = \mu$. Hence μ is a fixed point of T.

For uniqueness, let ν be another fixed point of T and $\mu \neq \nu$. From inequality (1), we obtain

$$d_\varsigma(\mu, \nu) = d_\varsigma(T\mu, T\nu) \leq \lambda d_\varsigma(\mu, \nu) < d_\varsigma(\mu, \nu).$$

It is a contradiction. Hence, $d_\varsigma(\mu, \nu) = 0$, that is, $\mu = \nu$ and μ is the unique fixed point of T. □

Example 4. *Let $U = [0, 2]$. Define a mapping $d_\varsigma : U \times U \to [0, \infty)$ by*

$$d_\varsigma(u, v) = \begin{cases} (u - v)^2 + \max\{u^2, v^2\}, & \text{if } u \neq v, \\ u, & \text{if } u = v \neq 0, \\ 0, & \text{if } u = v = 0. \end{cases}$$

Also, define the function $\varsigma : U \times U \to [1, \infty)$ by $\varsigma(u, v) = 3u + 2v + 2$ for all $u, v \in U$. Then (U, d_ς) is a controlled rectangular partial metric space.

Define a mapping $T : U \to U$ by

$$T(u) = \begin{cases} \frac{u}{2}, & \text{if } u \in (0, 1), \\ 0, & \text{if } u = 0, \\ 1, & \text{if } u = 2. \end{cases}$$

Then inequality (1) *holds for all $u, v \in U$.*

Also, note that for each $u \in U$ we have

$$\lim_{n,m\to\infty} \varsigma(u_n, u_m)\lambda < 1,$$

where $\lambda \in [0, 1)$. Hence, all the axioms of Theorem 1 are satisfied, and 0 *is the unique fixed point of T.*

Theorem 2. *Let $T : U \to U$ be a mapping on a complete controlled rectangular partial metric space (U, d_ς), which satisfies the following axioms:*

(i) for all $u, v \in U$ we have

$$d_\varsigma(Tu, Tv) \leq \lambda[d_\varsigma(u, Tu) + d_\varsigma(v, Tv)], \tag{6}$$

where $\lambda \in [0, \frac{1}{2})$,

(ii) for each $u_0 \in U$ we have $\lim_{n,m\to\infty} \varsigma(u_n, u_m)\lambda < 1$, where $u_n = T^n u_0$, $n = 1, 2, \ldots$, and $\mu = \frac{\lambda}{1-\lambda} \in [0, 1)$,

(iii) for each $u \in U$ limits $\lim_{n\to\infty} \varsigma(u, u_n)$ and $\lim_{n\to\infty} \varsigma(u_n, u)$ exist and are finite.

Then T has a unique fixed point $\mu \in U$ and $d_\varsigma(\mu, \mu) = 0$.

Proof. First, we will show that if μ is a fixed point of T, then $d_\varsigma(\mu, \mu) = 0$. Let us suppose that $d_\varsigma(\mu, \mu) > 0$. Then from inequality (6) it follows that

$$\begin{aligned} d_\varsigma(\mu, \mu) = d_\varsigma(T\mu, T\mu) &\leq \lambda[d_\varsigma(\mu, T\mu) + d_\varsigma(\mu, T\mu)] \\ &= \lambda[d_\varsigma(\mu, \mu) + d_\varsigma(\mu, \mu)] = 2\lambda d_\varsigma(\mu, \mu) < d_\varsigma(\mu, \mu). \end{aligned}$$

It is a contradiction. Hence, $d_\varsigma(\mu, \mu) = 0$.

For the existence of a fixed point, consider $u_n = T^n u_0$. Let us take an arbitrary element $u_0 \in U$ and choose $u_1 = Tu_0$ and $u_2 = Tu_1$. Then from inequality (6) we obtain

$$d_\varsigma(u_1, u_2) = d_\varsigma(Tu_0, Tu_1) \leq \lambda[d_\varsigma(u_0, Tu_0) + d_\varsigma(u_1, Tu_1)] = \lambda[d_\varsigma(u_0, u_1) + d_\varsigma(u_1, u_2)].$$

This implies that

$$d_\varsigma(u_1, u_2) \leq \frac{\lambda}{1-\lambda} d_\varsigma(u_0, u_1),$$

where $\omega = \frac{\lambda}{1-\lambda} < 1$ for $\lambda < \frac{1}{2}$. By recursively applying inequality (6), we obtain

$$d_\varsigma(u_n, u_{n+1}) \leq \omega^n d_\varsigma(u_0, u_1). \tag{7}$$

Thus, by taking limit in the above inequality, we get

$$\lim_{n\to\infty} d_\varsigma(u_n, u_{n+1}) = 0. \tag{8}$$

Now, we will show that $\{u_n\}$ is a Cauchy sequence. By following the same procedure as in the proof of Theorem 1 and using inequality (7), we conclude that $\{u_n\}$ is a Cauchy sequence. As U is complete, so there exists $\mu \in U$ such that

$$\lim_{n\to\infty} d_\varsigma(u_n, \mu) = \lim_{n,m\to\infty} d_\varsigma(u_n, u_m) = d_\varsigma(\mu, \mu) = 0. \tag{9}$$

Next we show that μ is a fixed point of T. From condition *(iii)* of Definition 2, for any $n \in \mathbb{N}$ we have

$$\begin{aligned} d_\varsigma(\mu, T\mu) &\leq \varsigma(\mu, u_n) d_\varsigma(\mu, u_n) + \varsigma(u_n, u_{n+1}) d_\varsigma(u_n, u_{n+1}) \\ &\qquad + \varsigma(u_{n+1}, T\mu) d_\varsigma(u_{n+1}, T\mu) - d_\varsigma(u_n, u_n) - d_\varsigma(u_{n+1}, u_{n+1}) \\ &\leq \varsigma(\mu, u_n) d_\varsigma(\mu, u_n) + \varsigma(u_n, u_{n+1}) d_\varsigma(u_n, u_{n+1}) + \varsigma(u_{n+1}, T\mu) d_\varsigma(Tu_n, T\mu) \\ &\leq \varsigma(\mu, u_n) d_\varsigma(\mu, u_n) + \varsigma(u_n, u_{n+1}) d_\varsigma(u_n, u_{n+1}) \\ &\qquad + \varsigma(u_{n+1}, T\mu) \lambda[d_\varsigma(u_n, Tu_n) + d_\varsigma(\mu, T\mu)]. \end{aligned}$$

This implies that

$$\begin{aligned} [1-\lambda\varsigma(u_{n+1},T\mu)]d_\varsigma(\mu,T\mu) \leq\ & \varsigma(\mu,u_n)d_\varsigma(\mu,u_n) \\ & + [\varsigma(u_n,u_{n+1})+\lambda\varsigma(u_{n+1},T\mu)]d_\varsigma(u_n,u_{n+1}). \end{aligned} \tag{10}$$

Since for each $u_0 \in U$ we have $\lim_{n\to\infty} \varsigma(u_n,u_m)\lambda < 1$, and for each $u \in U$ limits $\lim_{n\to\infty} \varsigma(u_n,u)$ and $\lim_{n\to\infty} \varsigma(u,u_n)$ exist and are finite, by taking limit in (10) as $n \to \infty$ and using equations (8) and (9), we obtain $d_\varsigma(\mu,T\mu) = 0$. This implies that $T\mu = \mu$.

For the uniqueness, let ν be another fixed point of T and $\mu \neq \nu$. From inequality (6), we obtain

$$d_\varsigma(\mu,\nu) = d_\varsigma(T\mu,T\nu) \leq \lambda[d_\varsigma(\mu,T\mu)+d_\varsigma(\nu,T\nu)] = \lambda[d_\varsigma(\mu,\mu)+d_\varsigma(\nu,\nu)].$$

As $d_\varsigma(\mu,\mu) = 0$ and $d_\varsigma(\nu,\nu) = 0$, hence from above inequality, we obtain $d_\varsigma(\mu,\nu) = 0$, that is, $\mu = \nu$ and μ is the unique fixed point of T. □

Theorem 3. *Let $T : U \to U$ be a mapping on a complete controlled rectangular partial metric space (U, d_ς), which satisfies the following axioms:*

(i) for all $u, v \in U$, we have

$$d_\varsigma(Tu,Tv) \leq \lambda \max\{d_\varsigma(u,v), d_\varsigma(u,Tu), d_\varsigma(v,Tv)\}, \tag{11}$$

where $\lambda \in [0, \frac{1}{2})$,

(ii) for each $u_0 \in U$ we have $\lim_{n,m\to\infty} \varsigma(u_n,u_m)\lambda < 1$ and $\lim_{n\to\infty} \varsigma(u_n,y)\lambda < 1$ for each $y \in U$, where $u_n = T^n u_0$, $n = 1,2,\ldots$,

(iii) for each $u \in U$ limits $\lim_{n\to\infty} \varsigma(u_n,u)$ and $\lim_{n\to\infty} \varsigma(u,u_n)$ exist and are finite.

Then T has a unique fixed point $\mu \in U$ and $d_\varsigma(\mu,\mu) = 0$.

Proof. First, let us suppose that if μ is a fixed point of T, then $d_\varsigma(\mu,\mu) = 0$. Suppose that $d_\varsigma(\mu,\mu) > 0$. Then from inequality (11) it follows that

$$d_\varsigma(\mu,\mu) = d_\varsigma(T\mu,T\mu) \leq \lambda \max\{d_\varsigma(\mu,\mu), d_\varsigma(\mu,T\mu), d_\varsigma(\mu,T\mu)\} = \lambda d_\varsigma(\mu,\mu) < d_\varsigma(\mu,\mu).$$

It is a contradiction. Hence, $d_\varsigma(\mu,\mu) = 0$.

For the existence of a fixed point, let us take an arbitrary element $u_0 \in U$, and define an iterative sequence $u_n = T^n u_0$ for each $n \geq 0$. From (11), we obtain

$$\begin{aligned} d_\varsigma(u_n,u_{n+1}) = d_\varsigma(Tu_{n-1},Tu_n) &\leq \lambda \max\{d_\varsigma(u_{n-1},u_n), d_\varsigma(u_{n-1},Tu_{n-1}), d_\varsigma(u_n,Tu_n)\} \\ &= \lambda \max\{d_\varsigma(u_{n-1},u_n), d_\varsigma(u_{n-1},u_n), d_\varsigma(u_n,u_{n+1})\} \\ &= \lambda \max\{d_\varsigma(u_{n-1},u_n), d_\varsigma(u_n,u_{n+1})\}. \end{aligned}$$

If $\max\{d_\varsigma(u_{n-1},u_n), d_\varsigma(u_n,u_{n+1})\} = d_\varsigma(u_n,u_{n+1})$, then from above inequality, we obtain

$$d_\varsigma(u_n,u_{n+1}) \leq \lambda d_\varsigma(u_n,u_{n+1}) < d_\varsigma(u_n,u_{n+1}).$$

It is a contradiction. Hence, $\max\{d_\varsigma(u_{n-1},u_n), d_\varsigma(u_n,u_{n+1})\} = d_\varsigma(u_{n-1},u_n)$, and from above inequality we get $d_\varsigma(u_n,u_{n+1}) \leq \lambda d_\varsigma(u_n,u_{n+1}) < d_\varsigma(u_{n-1},u_n)$.

By repeating the same procedure recursively, we obtain

$$d_\varsigma(u_n, u_{n+1}) \leq \lambda^n d_\varsigma(u_0, u_1) \tag{12}$$

for all $n \geq 0$. Thus, by taking limit in the above inequality, we have

$$\lim_{n\to\infty} d_\varsigma(u_n, u_{n+1}) = 0. \tag{13}$$

Now, we will show that $\{u_n\}$ is a Cauchy sequence. By following the same procedure as in the proof of Theorem 1 and using inequality (12), we conclude that $\{u_n\}$ is a Cauchy sequence. Since U is complete, there exists $\mu \in U$ such that

$$\lim_{n\to\infty} d_\varsigma(u_n, \mu) = \lim_{n,m\to\infty} d_\varsigma(u_n, u_m) = d_\varsigma(\mu, \mu) = 0. \tag{14}$$

Next, we show that μ is a fixed point of T. From condition *(iii)* of Definition 2, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} d_\varsigma(\mu, T\mu) &\leq \varsigma(\mu, u_n) d_\varsigma(\mu, u_n) + \varsigma(u_n, u_{n+1}) d_\varsigma(u_n, u_{n+1}) \\ &\quad + \varsigma(u_{n+1}, T\mu) d_\varsigma(u_{n+1}, T\mu) - d_\varsigma(u_n, u_n) - d_\varsigma(u_{n+1}, u_{n+1}) \\ &\leq \varsigma(\mu, u_n) d_\varsigma(\mu, u_n) + \varsigma(u_n, u_{n+1}) d_\varsigma(u_n, u_{n+1}) + \varsigma(u_{n+1}, T\mu) d_\varsigma(Tu_n, T\mu) \\ &\leq \varsigma(\mu, u_n) d_\varsigma(\mu, u_n) + \varsigma(u_n, u_{n+1}) d_\varsigma(u_n, u_{n+1}) \\ &\quad + \varsigma(u_{n+1}, T\mu) \lambda \max\{d_\varsigma(u_n, \mu), d_\varsigma(u_n, Tu_n), d_\varsigma(\mu, T\mu)\} \\ &= \varsigma(\mu, u_n) d_\varsigma(\mu, u_n) + \varsigma(u_n, u_{n+1}) d_\varsigma(u_n, u_{n+1}) \\ &\quad + \varsigma(u_{n+1}, T\mu) \lambda \max\{d_\varsigma(u_n, \mu), d_\varsigma(u_n, u_{n+1}), d_\varsigma(\mu, T\mu)\}. \end{aligned}$$

Assume that $T\mu \neq \mu$. Then $d_\varsigma(\mu, T\mu) > 0$. Due to equations (13) and (14), there exists an integer n_0 so that for all $n \geq n_0$, one can write $\max\{d_\varsigma(u_n, \mu), d_\varsigma(u_n, u_{n+1})\} < d_\varsigma(\mu, T\mu)$. That is, for all $n \geq n_0$, we have $\max\{d_\varsigma(u_n, \mu), d_\varsigma(u_n, u_{n+1}), d_\varsigma(\mu, T\mu)\} = d_\varsigma(\mu, T\mu)$. Thus,

$$d_\varsigma(\mu, T\mu) \leq \varsigma(\mu, u_n) d_\varsigma(\mu, u_n) + \varsigma(u_n, u_{n+1}) d_\varsigma(u_n, u_{n+1}) + \varsigma(u_{n+1}, T\mu) \lambda d_\varsigma(\mu, T\mu).$$

This implies that

$$[1 - \varsigma(u_{n+1}, T\mu)\lambda] d_\varsigma(\mu, T\mu) \leq \varsigma(\mu, u_n) d_\varsigma(\mu, u_n) + \varsigma(u_n, u_{n+1}) d_\varsigma(u_n, u_{n+1}). \tag{15}$$

Since for each $u_0 \in U$ we have $\lim_{n\to\infty} \varsigma(u_n, u_m)\lambda < 1$, $\lim_{n\to\infty} [1 - \varsigma(u_{n+1}, T\mu)\lambda] < 0$ and for each $u \in U$ the limit $\lim_{n\to\infty} \varsigma(u, u_n)$ exists and is finite, the right-hand side of inequality (15) goes to 0 as $n \to \infty$. Necessarily, we obtain that $d_\varsigma(\mu, T\mu) = 0$. This implies that $T\mu = \mu$. It is a contradiction. That is, μ is a fixed point of T.

For the uniqueness, let ν be another fixed point of T. From inequality (11), we obtain

$$\begin{aligned} d_\varsigma(\mu, \nu) = d_\varsigma(T\mu, T\nu) &\leq \lambda \max\{d_\varsigma(\mu, \nu), d_\varsigma(\mu, T\mu), d_\varsigma(\nu, T\nu)\} \\ &= \lambda \max\{d_\varsigma(\mu, \nu), d_\varsigma(\mu, \mu), d_\varsigma(\nu, \nu)\}. \end{aligned}$$

As $d_\varsigma(\mu, \mu) = 0$ and $d_\varsigma(\nu, \nu) = 0$, hence from above inequality, we obtain $d_\varsigma(\mu, \nu) = 0$, that is, $\mu = \nu$ and μ is the unique fixed point of T. □

3 Fixed point results for graphic contractions in controlled rectangular partial metric spaces

In this section, we present some fixed point results for graphic contractions in controlled rectangular partial metric spaces. J. Jachymski [14] established some strong results, which combine fixed point theory with graph theory. Afterwards, many researchers have developed this approach for different metric spaces with several contractions, see [1, 20, 24–26]. In this section, we prove some fixed point results for graphic G_η-contractive maps.

For more understanding, we review some basic concepts of graph theory. Let (U, d_ς) be a complete controlled rectangular partial metric space. We denote by Δ the diagonal of the Cartesian product $U \times U$. A graph $\mathfrak{G}$ is denoted by the pair $(\mathfrak{V}, \mathfrak{E})$, where $\mathfrak{V}$ is a set of vertices coinciding with U and $\mathfrak{E}$ is the set of its edges, so that $\Delta \subset \mathfrak{E}$. Throughout this paper, we suppose that the graph $\mathfrak{G}$ has no parallel edges, that is, $\mathfrak{G}$ has no edges that are incident to the same two vertices.

Definition 4 ([14]). *Let us take two vertices a and b in a graph $\mathfrak{G}$. A path in $\mathfrak{G}$ from a to b of length k, $k \in \mathbb{N} \cup \{0\}$, is a sequence $\{t\}_{t=0}^{k}$ of $k+1$ distinct vertices such that $t_0 = a$, $t_k = b$ and $(t_i, t_{i+1}) \in \mathfrak{E}(\mathfrak{G})$ for $i = 1, 2, \ldots, k$.*

Let (U, d_ς) be a complete controlled rectangular partial metric space. The graph $\mathfrak{G}$ can be converted to a directed graph by assigning to each edge the distance given by the controlled partial metric between its vertices. We denote by $\mathfrak{G}^{-1}$ the graph obtained from $\mathfrak{G}$ by reversing the direction of edges in $\mathfrak{G}$. Hence,

$$\mathfrak{E}(\mathfrak{G}^{-1}) = \{(u, \mathfrak{V}) \in U \times U : (\mathfrak{V}, u) \in \mathfrak{E}(\mathfrak{G})\} \quad \text{and} \quad \mathfrak{V}(\mathfrak{G}^{-1}) = \mathfrak{V}(\mathfrak{G}).$$

Let us take an undirected graph $\tilde{\mathfrak{G}}$ by ignoring the direction of all edges, that is

$$\mathfrak{E}(\tilde{\mathfrak{G}}) = \mathfrak{E}(\mathfrak{G}) \cup \mathfrak{E}(\mathfrak{G}^{-1}).$$

Definition 5. *A subgraph of a graph $\mathfrak{G}$ is itself graph, that consists the subsets of graph edges and its correlated vertices.*

Definition 6. (i) *A graph $\mathfrak{G}$ is said to be connected, if there is a path between any two vertices of $\mathfrak{G}$, and is called weakly connected, if $\tilde{\mathfrak{G}}$ is connected.*

(ii) *A graph $\mathfrak{G}$ is said to be simple, if it has no multiple edges nor loops.*

Let us take a vertex u of graph $\mathfrak{G}$. Then the subgraph $\mathfrak{G}_u$ consisting all the edges and vertices, that are contained in some path in $\mathfrak{G}$ starting from u, is called the component of $\mathfrak{G}$ containing u. In this case, $\mathfrak{V}(\mathfrak{G}_u) = [u]_{\mathfrak{G}}$, where $[u]_{\mathfrak{G}}$ denotes the equivalence class of relation R (we say that $u \, R \, \mathfrak{V}$, if there is a path from u to $\mathfrak{V}$ in $\mathfrak{G}$). In order to apply condition *(iii)* of Definition 2 to the vertices of the graph $\mathfrak{G}$, we consider a graph of length bigger than 2, which means that between any two vertices we can find a path through at least two other vertices. We denote by U_T the following set

$$U_T = \{u \in U : (u, Tu) \in \mathfrak{E}(\mathfrak{G}) \text{ or } (Tu, u) \in \mathfrak{E}(\mathfrak{G})\}.$$

Next, we will define graphic $\mathfrak{G}_\eta$-contractions.

Definition 7. *A mapping $T : U \to U$ on a complete controlled rectangular partial metric space (U, d_ς) endowed with a graph $\mathfrak{G}$ is called a graphic $\mathfrak{G}_\eta$-contraction, if it satisfies the following conditions:*

(i) for all $u, \mathfrak{V} \in U$ we have that $(u, \mathfrak{V}) \in \mathfrak{E}(\mathfrak{G})$ implies $(Tu, T\mathfrak{V}) \in \mathfrak{E}(\mathfrak{G})$,

(ii) there exists a function $\eta : [0, \infty) \to [0, \infty)$ such that $d_\varsigma(Tu, T^2u) \le \eta(d_\varsigma(u, Tu)$ for all $u \in U_T$, where η is an increasing function and $\{\eta^n(t)\}_{n\in\mathbb{N}}$ converges to 0 for all $t > 0$.

Definition 8. *A mapping $T : U \to U$ on a controlled rectangular partial metric space (U, d_ς) endowed with a graph $\mathfrak{G}$ is called orbitally $\mathfrak{G}$-continuous, if for all $u, \mathfrak{V} \in U$ and any positive sequence $\{t_n\}_{n\in\mathbb{N}}$ it satisfies the following condition:*

$$T^{t_n}u \to \mathfrak{V},\ \ (T^{t_n}u, T^{t_{n+1}}u) \in \mathfrak{E}(\mathfrak{G})\ \text{ implies }\ T(T^{t_n}u) \to T\mathfrak{V}\ \text{ as }\ n \to \infty.$$

Lemma 1. *Let $T : U \to U$ be a graphic $\mathfrak{G}_\eta$-contraction on complete controlled rectangular partial metric space (U, d_ς) endowed with a graph $\mathfrak{G}$. If $u \in U_T$, then there exists $\gamma(u) \ge 0$ such that*

$$d_\varsigma(T^nu, T^{n+1}u) \le \eta(\gamma(u)),$$

for all $n \in \mathbb{N}$, where $\gamma(u) = d_\varsigma(u, Tu)$.

Proof. Let $u \in U_T$, then $(u, Tu) \in \mathfrak{E}(\mathfrak{G})$ or $(Tu, u) \in \mathfrak{E}(\mathfrak{G})$. Let us suppose $(u, Tu) \in \mathfrak{E}(\mathfrak{G})$. Then by induction we obtain $(T^nu, T^{n+1}u) \in \mathfrak{E}(\mathfrak{G})$ for all $n \in \mathbb{N}$. Thus, we get

$$d_\varsigma(T^nu, T^{n+1}u) \le \eta(d_\varsigma(T^{n-1}u, T^nu)) \le \eta^2(d_\varsigma(T^{n-2}u, T^{n-1}u)) \le \cdots \le \eta^n(d_\varsigma(u, Tu)).$$

Since $\gamma(u) = d_\varsigma(u, Tu)$, we obtain $d_\varsigma(T^nu, T^{n+1}u) \le \eta^n(\gamma(u))$. □

Theorem 4. *Let $T : U \to U$ be orbitally $\mathfrak{G}$-continuous graphic $\mathfrak{G}_\eta$-contraction on complete controlled rectangular partial metric space (U, d_ς) endowed with a graph $\mathfrak{G}$. Suppose that the following axioms hold:*

(i) for every sequence $\{u_n\}_{n\in\mathbb{N}}$ in U, if $u_n \to u$ and $(u_n, u_{n+1}) \in \mathfrak{E}(\mathfrak{G})$, then there exists a subsequence $\{u_{k_n}\}_{n\in\mathbb{N}}$ with $(u_{k_n}, u) \in \mathfrak{E}(\mathfrak{G})$ for $n \in \mathbb{N}$,

(ii) for each $i \ge 1$ and $u \in U$ we have $\lim_{n\to\infty} \varsigma(T^iu, T^nu) < 1$,

(iii) a function ς satisfies the inequality

$$\lim_{n\to\infty} \frac{\varsigma(\mu, T^{k_n}u)}{1 - \varsigma(T^{k_{n+1}}u, T\mu)} \le 0. \tag{16}$$

Then the restriction $T\big|_{[u]_{\tilde{\mathfrak{G}}}}$ of T to $[u]_{\tilde{\mathfrak{G}}}$ has a unique fixed point.

Proof. Let $u \in U_T$. Then from Lemma 1 we have

$$d_\varsigma(T^nu, T^{n+1}u) \le \eta^n(\gamma(u)), \tag{17}$$

where $\gamma(u) = d_\varsigma(u, Tu) \ge 0$. By taking limit in (17) as $n \to \infty$ and using property of η, we obtain

$$\lim_{n\to\infty} d_\varsigma(T^nu, T^{n+1}u) = 0. \tag{18}$$

Next, we will show that $\{T^n u\}$ is a Cauchy sequence. For this end, we consider the following two cases.

Case 1. Let κ be odd, that is $\kappa = 2m+1$ with $m \geq 1$. Then from condition *(iii)* of Definition 2 and inequality (17) for $n+\kappa > n$, we have

$$
\begin{aligned}
&d_\varsigma(T^n u, T^{n+2m+1}u)\\
&\quad\leq \varsigma(T^n u, T^{n+1}u)d_\varsigma(T^n u, T^{n+1}u) + \varsigma(T^{n+1}u, T^{n+2}u)d_\varsigma(T^{n+1}u, T^{n+2}u)\\
&\qquad + \varsigma(T^{n+2}u, T^{n+2m+1}u)d_\varsigma(T^{n+2}u, T^{n+2m+1}u)\\
&\qquad - d_\varsigma(T^{n+1}u, T^{n+1}u) - d_\varsigma(T^{n+2}u, T^{n+2}u)\\
&\quad\leq \varsigma(T^n u, T^{n+1}u)\eta^n(\gamma(u)) + \varsigma(T^{n+1}u, T^{n+2}u)\eta^{n+1}(\gamma(u))\\
&\qquad + \varsigma(T^{n+2}u, T^{n+2m+1}u)d_\varsigma(T^{n+2}u, T^{n+2m+1}u)\\
&\qquad - d_\varsigma(T^{n+1}u, T^{n+1}u) - d_\varsigma(T^{n+2}u, T^{n+2}u)\\
&\quad\leq \varsigma(T^n u, T^{n+1}u)\eta^n(\gamma(u)) + \varsigma(T^{n+1}u, T^{n+2}u)\eta^{n+1}(\gamma(u))\\
&\qquad - d_\varsigma(T^{n+1}u, T^{n+1}u) - d_\varsigma(T^{n+2}u, T^{n+2}u)\\
&\qquad + \varsigma(T^{n+2}u, T^{n+2m+1}u)\varsigma(T^{n+2}u, T^{n+3}u)d_\varsigma(T^{n+2}u, T^{n+3}u)\\
&\qquad + \varsigma(T^{n+2}u, T^{n+2m+1}u)\varsigma(T^{n+3}u, T^{n+4}u)d_\varsigma(T^{n+3}u, T^{n+4}u)\\
&\qquad + \varsigma(T^{n+2}u, T^{n+2m+1}u)\varsigma(T^{n+4}u, T^{n+2m+1}u)d_\varsigma(T^{n+4}u, T^{n+2m+1}u)\\
&\qquad - \varsigma(T^{n+2}u, T^{n+2m+1}u)d_\varsigma(T^{n+3}u, T^{n+3}u)\\
&\qquad - \varsigma(T^{n+2}u, T^{n+2m+1}u)d_\varsigma(T^{n+4}u, T^{n+4}u)\\
&\quad\leq \varsigma(T^n u, T^{n+1}u)\eta^n(\gamma(u)) + \varsigma(T^{n+1}u, T^{n+2}u)\eta^{n+1}(\gamma(u))\\
&\qquad + \varsigma(T^{n+2}u, T^{n+2m+1}u)\varsigma(T^{n+2}u, T^{n+3}u)\eta^{n+2}(\gamma(u))\\
&\qquad + \varsigma(T^{n+2}u, T^{n+2m+1}u)\varsigma(T^{n+3}u, T^{n+4}u)\eta^{n+3}(\gamma(u)) + \ldots\\
&\qquad + \varsigma(T^{n+2}u, T^{n+2m+1}u)\ldots\varsigma(T^{n+2m-2}u, T^{n+2m+1}u)\eta^{n+2m-2}(\gamma(u))\\
&\qquad + \varsigma(T^{n+2}u, T^{n+2m+1}u)\ldots\varsigma(T^{n+2m-1}u, T^{n+2m}u)\eta^{n+2m-1}(\gamma(u))\\
&\qquad + \varsigma(T^{n+2}u, T^{n+2m+1}u)\ldots\varsigma(T^{n+2m}u, T^{n+2m+1}u)\eta^{n+2m}(\gamma(u))\\
&\qquad - d_\varsigma(T^{n+1}u, T^{n+1}u) - d_\varsigma(T^{n+2}u, T^{n+2}u)\\
&\qquad - \varsigma(T^{n+2}u, T^{n+2m+1}u)d_\varsigma(T^{n+3}u, T^{n+3}u)\\
&\qquad - \varsigma(T^{n+2}u, T^{n+2m+1}u)d_\varsigma(T^{n+4}u, T^{n+4}u) - \ldots\\
&\qquad - \varsigma(T^{n+2}u, T^{n+2m+1}u)\ldots\varsigma(T^{n+2m-2}u, T^{n+2m+1}u)d_\varsigma(T^{n+2m-1}u, T^{n+2m-1}u)\\
&\qquad - \varsigma(T^{n+2}u, T^{n+2m+1}u)\ldots\varsigma(T^{n+2m-2}u, T^{n+2m+1}u)d_\varsigma(T^{n+2m}u, T^{n+2m}u)\\
&\quad\leq \varsigma(T^n u, T^{n+1}u)\eta^n(\gamma(u)) + \varsigma(T^{n+1}u, T^{n+2}u)\eta^{n+1}(\gamma(u))\\
&\qquad + \sum_{i=1}^{m-1}\prod_{j=1}^{i}\varsigma(T^{n+2j}u, T^{n+2m+1}u)\varsigma(T^{n+2i}u, T^{n+2i+1}u)\eta^{n+2i}(\gamma(u))\\
&\qquad + \sum_{i=1}^{m-1}\prod_{j=1}^{i}\varsigma(T^{n+2j+1}u, T^{n+2m+1}u)\varsigma(T^{n+2i+1}u, T^{n+2i+2}u)\eta^{n+2i+1}(\gamma(u))\\
&\qquad + \prod_{i=1}^{m}\varsigma(T^{n+2i}u, T^{n+2m+1}u)\eta^{n+2m}(\gamma(u)).
\end{aligned}
\tag{19}
$$

By using the property of function η and axiom *(ii)*, we conclude that the following two series

$$\sum_{i=1}^{\infty}\prod_{j=1}^{i}\varsigma(T^{n+2j}u, T^{n+2m+1}u)\varsigma(T^{n+2i}u, T^{n+2i+1}u)\eta^{n+2i}(\gamma(u)),$$

and

$$\sum_{i=1}^{\infty}\prod_{j=1}^{i}\varsigma(T^{n+2j+1}u, T^{n+2m+1}u)\varsigma(T^{n+2i+1}u, T^{n+2i+2}u)\eta^{n+2i+1}(\gamma(u)),$$

converge by the ratio test.

Let

$$\begin{aligned}
S &= \sum_{i=1}^{\infty}\prod_{j=1}^{i}\varsigma(T^{n+2j}u, T^{n+2m+1}u)\varsigma(T^{n+2i}u, T^{n+2i+1}u)\eta^{n+2i}(\gamma(u)),\\
S' &= \sum_{i=1}^{\infty}\prod_{j=1}^{i}\varsigma(T^{n+2j+1}u, T^{n+2m+1}u)\varsigma(T^{n+2i+1}u, T^{n+2i+2}u)\eta^{n+2i+1}(\gamma(u)),\\
S_n &= \sum_{i=1}^{n}\prod_{j=1}^{i}\varsigma(T^{n+2j}u, T^{n+2m+1}u)\varsigma(T^{n+2i}u, T^{n+2i+1}u)\eta^{n+2i}(\gamma(u)),\\
S'_n &= \sum_{i=1}^{n}\prod_{j=1}^{i}\varsigma(T^{n+2j+1}u, T^{n+2m+1}u)\varsigma(T^{n+2i+1}u, T^{n+2i+2}u)\eta^{n+2i+1}(\gamma(u)).
\end{aligned}$$

Then inequality (19) takes the following form

$$\begin{aligned}
d_{\varsigma}(T^n u, T^{n+2m+1}u) \leq\ & \varsigma(T^n u, T^{n+1}u)\eta^n(\gamma(u))\\
& + \varsigma(T^{n+1}u, T^{n+2}u)\eta^{n+1}(\gamma(u))\\
& + [S_{m-1} - S_{n+1}] + [S'_{m-1} - S'_{n+1}]\\
& + \prod_{i=1}^{m-1}\varsigma(T^{n+2j}u, T^{n+2m+1}u))\eta^{n+2m}(\gamma(u)).
\end{aligned}$$

By taking limit in the above inequality as $n \to \infty$, we get

$$\lim_{n\to\infty} d_{\varsigma}(T^n u, T^{n+2m+1}u) = 0.$$

Case 2. Let κ be even, that is $\kappa = 2m$ with $m \geq 1$. Then from condition *(iii)* of Definition 2 and inequality (17) for $n + \kappa > n$, we have

$$
\begin{aligned}
&d_\varsigma(T^n u, T^{n+2m}u)\\
&\quad\le \varsigma(T^n u, T^{n+1}u)d_\varsigma(T^n u, T^{n+1}u) + \varsigma(T^{n+1}u, T^{n+2}u)d_\varsigma(T^{n+1}u, T^{n+2}u)\\
&\qquad + \varsigma(T^{n+2}u, T^{n+2m}u)d_\varsigma(T^{n+2}u, T^{n+2m}u)\\
&\qquad - d_\varsigma(T^{n+1}u, T^{n+1}u) - d_\varsigma(T^{n+2}u, T^{n+2}u)\\
&\quad\le \varsigma(T^n u, T^{n+1}u)\eta^n(\gamma(u)) + \varsigma(T^{n+1}u, T^{n+2}u)\eta^{n+1}(\gamma(u))\\
&\qquad + \varsigma(T^{n+2}u, T^{n+2m}u)d_\varsigma(T^{n+2}u, T^{n+2m}u)\\
&\qquad - d_\varsigma(T^{n+1}u, T^{n+1}u) - d_\varsigma(T^{n+2}u, T^{n+2}u)\\
&\quad\le \varsigma(T^n u, T^{n+1}u)\eta^n(\gamma(u)) + \varsigma(T^{n+1}u, T^{n+2}u)\eta^{n+1}(\gamma(u))\\
&\qquad - d_\varsigma(T^{n+1}u, T^{n+1}u) - d_\varsigma(T^{n+2}u, T^{n+2}u)\\
&\qquad + \varsigma(T^{n+2}u, T^{n+2m}u)\varsigma(T^{n+2}u, T^{n+3}u)d_\varsigma(T^{n+2}u, T^{n+3}u)\\
&\qquad + \varsigma(T^{n+2}u, T^{n+2m}u)\varsigma(T^{n+3}u, T^{n+4}u)d_\varsigma(T^{n+3}u, T^{n+4}u)\\
&\qquad + \varsigma(T^{n+2}u, T^{n+2m}u)\varsigma(T^{n+4}u, T^{n+2m}u)d_\varsigma(T^{n+4}u, T^{n+2m}u)\\
&\qquad - \varsigma(T^{n+2}u, T^{n+2m}u)d_\varsigma(T^{n+3}u, T^{n+3}u)\\
&\qquad - \varsigma(T^{n+2}u, T^{n+2m}u)d_\varsigma(T^{n+4}u, T^{n+4}u)\\
&\quad\le \varsigma(T^n u, T^{n+1}u)\eta^n(\gamma(u)) + \varsigma(T^{n+1}u, T^{n+2}u)\eta^{n+1}(\gamma(u))\\
&\qquad + \varsigma(T^{n+2}u, T^{n+2m}u)\varsigma(T^{n+2}u, T^{n+3}u)\eta^{n+2}(\gamma(u))\\
&\qquad + \varsigma(T^{n+2}u, T^{n+2m}u)\varsigma(T^{n+3}u, T^{n+4}u)\eta^{n+3}(\gamma(u)) + \ldots\\
&\qquad + \varsigma(T^{n+2}u, T^{n+2m}u)\ldots\varsigma(T^{n+2m-2}u, T^{n+2m-1}u)\eta^{n+2m-2}(\gamma(u))\\
&\qquad + \varsigma(T^{n+2}u, T^{n+2m}u)\ldots\varsigma(T^{n+2m-1}u, T^{n+2m}u)\eta^{n+2m-1}(\gamma(u))\\
&\qquad + \varsigma(T^{n+2}u, T^{n+2m}u)\ldots\varsigma(T^{n+2m}u, T^{n+2m}u)\eta^{n+2m}(d_\varsigma(u,u))\\
&\qquad - d_\varsigma(T^{n+1}u, T^{n+1}u) - d_\varsigma(T^{n+2}u, T^{n+2}u)\\
&\qquad - \varsigma(T^{n+2}u, T^{n+2m}u)d_\varsigma(T^{n+3}u, T^{n+3}u)\\
&\qquad - \varsigma(T^{n+2}u, T^{n+2m}u)d_\varsigma(T^{n+4}u, T^{n+4}u) - \ldots\\
&\qquad - \varsigma(T^{n+2}u, T^{n+2m}u)\ldots\varsigma(T^{n+2m-2}u, T^{n+2m}u)d_\varsigma(T^{n+2m-1}u, T^{n+2m-1}u)\\
&\qquad - \varsigma(T^{n+2}u, T^{n+2m}u)\ldots\varsigma(T^{n+2m-2}u, T^{n+2m}u)d_\varsigma(T^{n+2m}u, T^{n+2m}u)\\
&\quad\le \varsigma(T^n u, T^{n+1}u)\eta^n(\gamma(u)) + \varsigma(T^{n+1}u, T^{n+2}u)\eta^{n+1}(\gamma(u))\\
&\qquad + \sum_{i=1}^{m-1}\prod_{j=1}^{i}\varsigma(T^{n+2j}u, T^{n+2m}u)\varsigma(T^{n+2i}u, T^{n+2i+1}u)\eta^{n+2i}(\gamma(u))\\
&\qquad + \sum_{i=1}^{m-1}\prod_{j=1}^{i}\varsigma(T^{n+2j}u, T^{n+2m}u)\varsigma(T^{n+2i+1}u, T^{n+2i+2}u)\eta^{n+2i+1}(\gamma(u))\\
&\qquad + \prod_{i=1}^{m}\varsigma(T^{n+2i}u, T^{n+2m}u)\eta^{n+2m}(\gamma(u)).
\end{aligned}
\tag{20}
$$

By using the property of function η and axiom *(ii)*, we conclude that the following two series

$$\sum_{i=1}^{\infty}\prod_{j=1}^{i}\varsigma(T^{n+2j}u, T^{n+2m}u)\varsigma(T^{n+2i}u, T^{n+2i+1}u)\eta^{n+2i}(\gamma(u)),$$

and

$$\sum_{i=1}^{\infty}\prod_{j=1}^{i}\varsigma(T^{n+2j}u,T^{n+2m}u)\varsigma(T^{n+2i+1}u,T^{n+2i+2}u)\eta^{n+2i+1}(\gamma(u)),$$

converge by the ratio test.

Let

$$\begin{aligned}
S&=\sum_{i=1}^{\infty}\prod_{j=1}^{i}\varsigma(T^{n+2j}u,T^{n+2m}u)\varsigma(T^{n+2i}u,T^{n+2i+1}u)\eta^{n+2i}(\gamma(u)),\\
S'&=\sum_{i=1}^{\infty}\prod_{j=1}^{i}\varsigma(T^{n+2j}u,T^{n+2m}u)\varsigma(T^{n+2i+1}u,T^{n+2i+2}u)\eta^{n+2i+1}(\gamma(u)),\\
S_n&=\sum_{i=1}^{n}\prod_{j=1}^{i}\varsigma(T^{n+2j}u,T^{n+2m}u)\varsigma(T^{n+2i}u,T^{n+2i+1}u)\eta^{n+2i}(\gamma(u)),\\
S'_n&=\sum_{i=1}^{n}\prod_{j=1}^{i}\varsigma(T^{n+2j}u,T^{n+2m}u)\varsigma(T^{n+2i+1}u,T^{n+2i+2}u)\eta^{n+2i+1}(\gamma(u)).
\end{aligned}$$

Then inequality (20) takes the following form

$$\begin{aligned}
d_\varsigma(T^nu,T^{n+2m}u)\le{}&\varsigma(T^nu,T^{n+1}u)\eta^n(\gamma(u))+\varsigma(T^{n+1}u,T^{n+2}u)\eta^{n+1}(\gamma(u))\\
&+[S_{m-1}-S_{n+1}]+[S'_{m-1}-S'_{n+1}]\\
&+\prod_{i=1}^{m-1}\varsigma(T^{n+2j}u,T^{n+2m}u))\eta^{n+2m}(\gamma(u)).
\end{aligned}$$

By taking limit in the above inequality as $n\to\infty$, we get $\lim_{n\to\infty} d_\varsigma(T^nu,T^{n+2m}u)=0$.

Hence, in both cases

$$\lim_{n\to\infty} d_\varsigma(T^nu,T^{n+\kappa}u)=0.$$

This shows that $\{T^nu\}$ is a Cauchy sequence. Since U is complete, there exists $\mu\in U$ such that

$$T^nu\to\mu \text{ as } n\to\infty. \tag{21}$$

We have $T^nu\in U_T$ for $u\in U_T$ and each $n\in\mathbb{N}$. Since $(u,Tu)\in\mathfrak{E}(\mathfrak{G})$, from axiom *(i)* there exists a subsequence $\{T^{k_n}u\}_{n\in\mathbb{N}}$ of $\{T^nu\}_{n\in\mathbb{N}}$ with $(T^{k_n}u,u)\in\mathfrak{E}(\mathfrak{G})$ for $n\in\mathbb{N}$. Hence, a path in $\mathfrak{G}$ can be formed by using the points $u,Tu,T^2u,\dots,T^{k_1}u,\mu$, and then $\mu\in[u]_{\tilde{\mathfrak{G}}}$. As T is orbitally $\mathfrak{G}$-continuous, so

$$T(T^{k_n}u)\to T\mu \text{ as } n\to\infty. \tag{22}$$

Now we show that μ is a fixed point of T. From axiom *(iv)* of Definition 2, we have

$$d_\varsigma(\mu,\mu)\le d_\varsigma(\mu,T\mu) \tag{23}$$

and

$$d_\varsigma(T\mu,T\mu)\le d_\varsigma(\mu,T\mu). \tag{24}$$

On the other way, from axiom *(v)* of Definition 2, we have

$$\begin{aligned} d_\varsigma(\mu, T\mu) &\leq \varsigma(\mu, T^{k_n}u)d_\varsigma(\mu, T^{k_n}u) + \varsigma(T^{k_n}u, T^{k_{n+1}}u)d_\varsigma(T^{k_n}u, T^{k_{n+1}}u) \\ &\quad + \varsigma(T^{k_{n+1}}u, T\mu)d_\varsigma(T^{k_{n+1}}u, T\mu) - d_\varsigma(T^{k_n}u, T^{k_n}u) - d_\varsigma(T^{k_{n+1}}u, T^{k_{n+1}}u) \\ &\leq \varsigma(\mu, T^{k_n}u)d_\varsigma(\mu, T^{k_n}u) + \varsigma(T^{k_n}u, T^{k_{n+1}}u)d_\varsigma(T^{k_n}u, T^{k_{n+1}}u) \\ &\quad + \varsigma(T^{k_{n+1}}u, T\mu)d_\varsigma(T^{k_{n+1}}u, T\mu). \end{aligned}$$

By using equations (18), (21) and (22), (24) in the above inequality, we get

$$d_\varsigma(\mu, T\mu) \leq \lim_{n\to\infty} \varsigma(\mu, T^{k_n}u)d_\varsigma(\mu, \mu) + \lim_{n\to\infty} \varsigma(T^{k_{n+1}}u, T\mu)d_\varsigma(\mu, T\mu)$$

and

$$[1 - \lim_{n\to\infty} \varsigma(T^{k_{n+1}}u, T\mu)]d_\varsigma(\mu, T\mu) \leq \lim_{n\to\infty} \varsigma(\mu, T^{k_n}u)d_\varsigma(\mu, \mu)d_\varsigma(\mu, \mu). \tag{25}$$

Since for each $u \in U$ and $i \geq 1$ we have $\lim_{n\to\infty} \varsigma(T^i u, T^n u) < 1$. Hence, (25) implies that

$$d_\varsigma(\mu, T\mu) \leq \frac{\lim_{n\to\infty} \varsigma(\mu, T^{k_n}u)}{1 - \lim_{n\to\infty} \varsigma(T^{k_{n+1}}u, T\mu)} d_\varsigma(\mu, \mu) = \lim_{n\to\infty} \frac{\varsigma(\mu, T^{k_n}u)}{1 - \varsigma(T^{k_{n+1}}u, T\mu)} d_\varsigma(\mu, \mu).$$

Then by using inequality (16), we obtain $d_\varsigma(\mu, T\mu) = 0$. Using this equality in (23) and (24), we get

$$d_\varsigma(\mu, \mu) = d_\varsigma(\mu, T\mu) = d_\varsigma(T\mu, T\mu).$$

Hence, $T\mu = \mu$ and μ is a fixed point of $T\big|_{[u]_{\tilde{\mathfrak{G}}}}$. □

Example 5. *Let $U = [0, \infty)$. Define a mapping $d_\varsigma : U \times U \to [0, \infty)$ by*

$$d_\varsigma(u, \mathfrak{V}) = \max\{u, \mathfrak{V}\}$$

for all $u, \mathfrak{V} \in U$. Also define the function $\varsigma : U \times U \to [1, \infty)$ by

$$\varsigma(u, \mathfrak{V}) = u + \mathfrak{V} + 1$$

for all $u, \mathfrak{V} \in U$. Then (U, d_ς) is a complete controlled rectangular partial metric space.

Let us define a mapping $T : U \to U$ and $\eta : [0, \infty) \to [0, \infty)$ by

$$T(u) = \frac{u}{2}$$

for all $u \in U$, and

$$\eta(t) = \frac{t}{t+1}$$

for all $t \in [0, \infty)$. Let

$$\mathfrak{E}(\mathfrak{G}) = \{(u, \mathfrak{V}) \in [0, \infty) \times [0, \infty) \ : \ u \geq \mathfrak{V}\}.$$

Then it is easy to prove that T is a graphic $\mathfrak{G}_\eta$-contraction and satisfies all the axioms of Theorem 4. Hence, T has a fixed point $\mu = 0$.

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Received 15.06.2023

Аламґір Н., Кіран К., Алі Й., Айді Г. *Деякі результати про нерухому точку в контрольованих прямокутних часткових метричних просторах з графовою структурою* // Карпатські матем. публ. — 2025. — Т.17, №1. — С. 343–363.

У цій роботі ми вводимо поняття контрольованого прямокутного часткового метричного простору, яке є узагальненням контрольованого прямокутного метричного простору та контрольованого метричного простору. Крім того, ми отримуємо деякі результати про нерухому точку за відповідних умов у цих просторах. Як застосування, ми формулюємо теорему про нерухому точку для графічних стискань у контрольованому прямокутному частковому метричному просторі через графову структуру.

Ключові слова і фрази: контрольований прямокутний частковий метричний простір, нерухома точка, граф.