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Some bounds for distance signless Laplacian energy-like invariant of networks

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For a graph or network G, denote by $D(G)$ the distance matrix and $Tr(G)$ the diagonal matrix of vertex transmissions. The distance signless Laplacian matrix of G is $D^Q(G) = Tr(G) + D(G)$. We introduce the distance signless Laplacian energy-like invariant as $DEL(G) = \sum_{i=1}^{n} \sqrt{\rho_i}$, where $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ are the eigenvalues of distance signless Laplacian matrix. In this paper, we obtain new upper and lower bounds for $DEL(G)$. These bounds involve some important invariants including diameter, minimum and maximum transmission degree, distance signless Laplacian spectral radius and the Wiener index. Additionally, we characterize the extremal graphs attaining these bounds. Finally, we establish some relations between different versions of distance signless Laplacian energy of graphs.

Key words and phrases: distance signless Laplacian matrix, distance signless Laplacian energy-like invariant, spectral radius, distance signless Laplacian energy, Wiener index.

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1 Introduction

Let $G(V, E)$ be a network (or simple graph) having the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E(G)$. The *distance matrix* of G is $D(G) = \left[d_G(v_i, v_j)\right]_{v_i, v_j \in V(G)}$ (see [9]), where $d_G(v_i, v_j)$, or d_{ij} for short, represents the distance between two vertices v_i and v_j. The *transmission degree* $Tr_G(v_i)$ of a vertex v_i is defined as

$$Tr_G(v_i) = \sum_{j=1}^{n} d_G(v_i, v_j), \quad j \neq i.$$

For simplicity, we write $Tr_i = Tr_G(v_i)$.

The *transmission degree sequence* of vertices $\{v_1, v_2, \ldots, v_n\}$ is $\{Tr_1, Tr_2, \ldots, Tr_n\}$ and the weighted sum $T_i = \sum_{j=1}^{n} d_{ij} Tr_j$ is called the *second transmission degree*. It is well known that $W(G) = \frac{1}{2}\sum_{i=1}^{n} Tr_i$ is the *Wiener index*, also known as the *transmission* of G. A graph G is called *transmission regular* when every transmission is equal. The transmission regular graphs are essentially distance balanced graphs [24].

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Let $Tr(G) = diag\,(Tr_1, Tr_2, \ldots, Tr_n)$ represent the diagonal matrix of vertex transmissions. The *distance Laplacian matrix* of G is denoted by $D^L(G) = Tr(G) - D(G)$. Analogously, the *distance signless Laplacian matrix* is denoted by $D^Q(G) = Tr(G) + D(G)$. Since $D^Q(G)$ is symmetric, non-negative and irreducible for connected graph G, the eigenvalues of $D^Q(G)$ can be ordered as $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. The eigenvalue ρ_1 is termed as the *distance signless Laplacian spectral radius*. In the sequel, $\rho(G)$ is used to replace ρ_1. Note that $D^Q(G)$ is an irreducible matrix. It follows from the Perron-Frobenius theorem, $\rho(G) > 0$ is a simple eigenvalue (its algebraic multiplicity is equal to 1). Moreover, there exists only one positive unit eigenvector X associated with $\rho(G)$. It is known as the *distance signless Laplacian Perron vector*. For standard graph theoretical notations, such as the degree of vertex $\deg_G(v)$ and the neighborhood of vertex $N(v)$, we refer to [17,29,37]. In particular, we will use $\overline{G}$ to represent the *complement* of G. Both distance Laplacian spectra [10–12,22,23,34,35,41,42] and distance signless Laplacian spectra [1–8,10–14,18,19,36,43] have been widely studied in spectral graph theory.

Suppose that the adjacency matrix $A(G)$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. We define *energy* of G [21] as $E(G) = \sum_{i=1}^{n} |\lambda_i|$. If $\delta_1, \delta_2, \ldots, \delta_n$ are the eigenvalues of $D(G)$, distance energy $E_D(G)$ [26] is defined as $E_D(G) = \sum_{i=1}^{n} |\delta_i|$. Note that δ_i's satisfy

$$\sum_{i=1}^{n} \delta_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{n} {\delta_i}^2 = 2s, \tag{1}$$

where

$$s = \sum_{1 \leq i < j \leq n} (d_{ij})^2. \tag{2}$$

In [45], the energy metric is defined using distance Laplacian eigenvalues, which was shown to maintain the many properties of distance energy. Bearing in mind the identities in (1), some auxiliary eigenvalues $\zeta_i = \delta_i^L - \frac{2W(G)}{n}$ were introduced, where δ_i^L is the ith eigenvalue of $D^L(G)$, $i = 1, 2, \ldots, n$. It has also been revealed that these auxiliary eigenvalues follow similar properties regarding (1).

In [45], it is states that these auxiliary eigenvalues exhibit properties similar to (1), as follows.

Suppose that G is connected and has n vertices. We have $\sum_{i=1}^{n} \zeta_i = 0$ and $\sum_{i=1}^{n} {\zeta_i}^2 = 2S$, where $S = s + \frac{1}{2}\left(\sum_{i=1}^{n} Tr_i - \frac{2W(G)}{n}\right)^2$ and s is given by (2).

Let ξ_i be the auxiliary eigenvalues associated to the distance signless Laplacian eigenvalues with $\xi_i = \rho_i - \frac{2W(G)}{n}$. The following quantity is introduced as the *distance signless Laplacian energy* [6,19,30]:

$$E_{D^Q}(G) = \sum_{i=1}^{n} \left| \rho_i - \frac{2W(G)}{n} \right|.$$

Define another type of distance signless Laplacian based graph metric as $DEL(G) = \sum\limits_{i=1}^{n} \sqrt{\rho_i}$. This is termed as *distance signless Laplacian energy-like invariant*.

It is well-known that the $DEL(G)$ is related to molecular descriptors of the molecules that modeled by the chemical graphs. Since the distance Laplacian (or its signless version) are based on metric properties of the graph there may be some (geo)metrical motivation for this number, which can be investigated. In Section 4, we have obtained relationships between $E_{D^Q}(G)$ and $DEL(G)$ which can help in the derivation of one from another.

The paper is organized as follows. Some preliminary results, which will be used later, are provided in Section 2. In Section 3, we obtain upper and lower bounds for $DEL(G)$ in connected graphs and identify extremal graphs that attain these bounds. Section 4 is devoted to new bounds for the distance signless Laplacian energy-like invariant, and we further establish some relations for different versions of the distance (signless) Laplacian energy.

2 Preliminaries

The following lemmas will be used for developing the main results in subsequent sections.

Lemma 1 ([43]). *Suppose G is connected and has the vertex set $\{v_1, v_2, \ldots, v_n\}$. We have $\rho(G) \geq \frac{4\sigma(G)}{n}$, with the equality holding if and only if G becomes transmission regular.*

Lemma 2 ([6]). *Suppose that the transmission degree sequence of G is $\{Tr_1, Tr_2, \ldots, Tr_n\}$. We have*

$$\sum_{i=1}^{n} \rho_i = 2W(G) \tag{3}$$

and

$$\sum_{i=1}^{n} \rho_i^2 = 2 \sum_{1 \leq i < j \leq n} \left(d_{ij}\right)^2 + \sum_{i=1}^{n} Tr_i^2. \tag{4}$$

From (3), (4) and the definition of ξ_i's, we see that $\sum_{i=1}^{n} \xi_i = 0$ and $\sum_{i=1}^{n} \xi_i^2 = 2\mathcal{S}$, where

$$\mathcal{S} = s + \frac{1}{2} \sum_{i=1}^{n} {Tr_i}^2 - \frac{2W(G)^2}{n},$$

and s is given by (2). Similar properties as (1) can be derived for the auxiliary eigenvalues.

Lemma 3 ([32]). *Suppose that the non-negative matrix $A \in \mathbb{R}^{n \times n}$ has spectral radius $\lambda(A)$ and $r_1, r_2, \ldots, r_n$ are the row sums. We have*

$$\min_{1 \leq i \leq n} r_i \leq \lambda(A) \leq \max_{1 \leq i \leq n} r_i.$$

Further, given A to be irreducible, the above two equalities can be attained if and only if all row sums are identical.

It is clear that $r_i = Tr_i + \sum_{j=1}^{n} d_{ij} = 2Tr_i$ is the ith row sum of $D^Q(G)$. By Lemma 3, the observation below is immediate.

Corollary 1. *Suppose that G is connected and has the vertex set $\{v_1, v_2, \ldots, v_n\}$. Let $Tr_{\max}$ and $Tr_{\min}$ represent its largest and least transmissions. We have $2Tr_{\min} \leq \rho(G) \leq 2Tr_{\max}$. The two equalities are attained if and only if G becomes transmission regular.*

In [40], J. Radon showed the following inequality.

Lemma 4 ([40]). *If a_k, x_k, $p > 0$, $k \in \{1, 2, \ldots, r\}$, we have*

$$\sum_{k=1}^{r} \frac{x_k^{p+1}}{a_k^p} \geq \frac{\left(\sum_{k=1}^{r} x_k\right)^{p+1}}{\left(\sum_{k=1}^{r} a_k\right)^p}.$$

The two equalities are attained if and only if

$$\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_r}{a_r}.$$

Lemma 5 ([39]). *Let $b_1, b_2, \ldots, b_n$ be positive numbers. Assume $a_1, a_2, \ldots, a_n$ are real numbers. We have*

$$\frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \le \max_{1\le i\le n} \frac{a_i}{b_i},$$

where the equality is attained if and only if the ratios $\frac{a_i}{b_i}$ are identical.

Lemma 6 ([27]). *Suppose that $x_1, x_2, \ldots, x_n$ form a list of on-negative numbers. Denote by $\alpha = \frac{1}{N}\sum_{i=1}^{N} x_i$ and $\beta = \left(\prod_{i=1}^{N} x_i\right)^{\frac{1}{N}}$ the arithmetic and geometric averages, respectively. We have*

$$\frac{1}{N(N-1)}\sum_{i<j}\left(\sqrt{x_i} - \sqrt{x_j}\right)^2 \le \alpha - \beta \le \frac{1}{N}\sum_{i<j}\left(\sqrt{x_i} - \sqrt{x_j}\right)^2,$$

where the equality is attained if and only if x_i is identical for any i.

Lemma 7 ([20]). *Assume that a_i and b_i, $1 \le i \le n$, are two lists of non-negative numbers. Suppose there are $r, R \in \mathbb{R}$ satisfying $r \le \frac{b_i}{a_i} \le R$, $a_i \ne 0$, for every i. We have*

$$\sum_{i=1}^{n} b_i^2 + rR\sum_{i=1}^{n} a_i^2 \le (r+R)\sum_{i=1}^{n} a_i b_i,$$

where the equality is attained if and only if $b_i = a_i r$ or $b_i = a_i R$ for some $i \in \{1, 2, \cdots, n\}$.

Lemma 8. *Suppose G is connected and has exactly two distinct distance signless Laplacian eigenvalues. Then G is complete.*

Proof. First, let the graph G admit exactly two distinct D^Q-eigenvalues. In a similar way as in [25, Lemma 2], we can prove that the diameter of G is one. Conversely, if G is a complete graph with vertex set $\{v_1, v_2, \ldots, v_n\}$, then $D^Q(G) = D(G) + (n-1)I$. It then can be seen that G admits exactly two distinct D^Q-eigenvalues, $2n-2$ and $n-2$. □

Lemma 9 ([38]). *Suppose we have two lists of positive numbers a_i and b_i, $1 \le i \le n$. Then we obtain*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \le \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2 \left(\sum_{i=1}^{n} a_i b_i\right)^2,$$

in which $m_1 = \min_{1\le i\le n} a_i$, $m_2 = \min_{1\le i\le n} b_i$, $M_1 = \max_{1\le i\le n} a_i$, and $M_2 = \max_{1\le i\le n} b_i$.

Lemma 10 (Power mean inequality). *Suppose we have a list of positive numbers $a_1, a_2, \ldots, a_p$ and $s, t \in \mathbb{R}$ with $s > t$. Then $M_s \ge M_t$, where $M_r = \left(\frac{a_1^r + \cdots + a_p^r}{p}\right)^{\frac{1}{r}}$. The equality is attained if and only if $a_1 = a_2 = \cdots = a_p$.*

Lemma 11 ([33]). *Suppose we have two lists of nonnegative numbers a_i and b_i, $1 \le i \le n$. Then we have*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \frac{n^2}{4}\left(M_1M_2 - m_1m_2\right)^2,$$

where $m_1 = \min_{1\le i\le n} a_i$, $m_2 = \min_{1\le i\le n} b_i$, $M_1 = \max_{1\le i\le n} a_i$, and $M_2 = \max_{1\le i\le n} b_i$.

Lemma 12 ([15]). *Suppose that we have two lists of nonnegative numbers a_i and b_i, $1 \le i \le n$. There are $a, b, A, B \in \mathbb{R}$ satisfying $a \le a_i \le A$ and $b \le b_i \le B$ for any i and*

$$\left| n \sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \le \alpha(n)(A-a)(B-b),$$

where $\alpha(n) = n\left[\frac{n}{2}\right]\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right)$ and $[x]$ represents the floor rounding of x. The equality is attained if and only if $a_1 = a_2 = \cdots = a_n$ and $b_1 = b_2 = \cdots = b_n$.

3 Bounds on distance signless Laplacian energy-like invariant

In this section, we obtain several inequalities involving the distance signless Laplacian energy-like invariant $DEL(G)$. The following first result gives an inequality using maximum transmission degree $Tr_{\max}(G)$, minimum transmission degree $Tr_{\min}(G)$ and distance signless Laplacian energy-like invariant $DEL(G)$.

Theorem 1. *If G is a connected graph having maximum transmission degree $Tr_{\max}(G)$ and minimum transmission degree $Tr_{\min}(G)$, then*

$$\left(DEL(G) - \sqrt{2Tr_{\min}}\right)^2 \left(S - \frac{1}{2Tr_{\max}}\right) \ge (n-1)^3, \tag{5}$$

where $S = \sum_{i=1}^{n} \frac{1}{\rho_i}$. The equality is attained if and only if $G \cong K_n$.

Proof. We have

$$\begin{aligned} \sum_{i=2}^{n} \sqrt{\rho_i} \sum_{i=2}^{n} \frac{1}{\sqrt{\rho_i}} &= n - 1 + \sum_{2 \le i < j \le n} \left(\sqrt{\frac{\rho_i}{\rho_j}} + \sqrt{\frac{\rho_j}{\rho_i}}\right) \\ &= n - 1 + \sum_{2 \le i < j \le n} \sqrt{\left(\sqrt{\frac{\rho_i}{\rho_j}} + \sqrt{\frac{\rho_j}{\rho_i}}\right)^2 + 4} \\ &\ge n - 1 + (n-1)(n-2) \qquad \text{as} \quad \left(\sqrt{\frac{\rho_i}{\rho_j}} - \sqrt{\frac{\rho_j}{\rho_i}}\right)^2 \ge 0 = (n-1)^2. \end{aligned} \tag{6}$$

Recalling Cauchy-Schwarz inequality, it can be seen that

$$\left(\sum_{i=2}^{n} \frac{1}{\sqrt{\rho_i}}\right)^2 \le (n-1) \sum_{i=2}^{n} \frac{1}{\rho_i} = (n-1)\left(S - \frac{1}{\rho_1}\right).$$

Therefore, it follows from the definition of distance signless Laplacian energy-like invariant that

$$DEL(G) = \sum_{i=1}^{n} \sqrt{\rho_i} \ge \sqrt{\rho_1} + \frac{(n-1)^2}{\sqrt{(n-1)\left(S - \frac{1}{\rho_1}\right)}} \ge \sqrt{2Tr_{\min}} + \sqrt{\frac{(n-1)^3}{\left(S - \frac{1}{2Tr_{\max}}\right)}}, \tag{7}$$

which leads to (5).

Next, suppose equality holds in (5). The inequalities in the above argument will become equalities. Particularly, using (6), we obtain $\left(\sqrt{\frac{\rho_i}{\rho_j}} - \sqrt{\frac{\rho_j}{\rho_i}}\right)^2 = 0$ for $2 \le i < j \le n$, or $\rho_2 = \cdots = \rho_n$. Also, from equality in (7), we get $\rho_1 = 2Tr_{\min}$. Hence, applying Corollary 1, we observe that G is a transmission regular graph. Therefore, equality holds in (5) if and only if G is a transmission regular graph having no more than two distinct distance signless Laplacian eigenvalues. In other words, by Lemma 8, the equality is attained if and only if $G \cong K_n$. On the other hand, if $G \cong K_n$, the equality holds in (5) immediately. □

Now, we obtain an inequality using minimum transmission degree $Tr_{\min}(G)$ and distance signless Laplacian energy-like invariant $DEL(G)$.

Theorem 2. *Suppose that G is a connected graph and it has the vertex set $\{v_1, v_2, \ldots, v_n\}$ and minimum transmission degree $Tr_{\min}(G)$. Then*

$$\left(S - \frac{1}{2Tr_{\min}}\right)\left(2\sum_{1\le i<j\le n}(d_{ij})^2 + \sum_{i=1}^{n} Tr_i^2 - 4Tr_{\min}^2\right) \ge \left(DEL(G) - \sqrt{2Tr_{\min}}\right)^2, \tag{8}$$

where $S = \sum_{i=1}^{n} \frac{1}{\rho_i}$. The equality is attained if and only if $G \cong K_n$.

Proof. Setting $p = 1$, $r = n-1$, $x_k = \sqrt{\rho_k}$ and $a_k = \frac{n}{\rho_k}$, $k = 2, 3, \ldots, n$, in Lemma 4, we get

$$\sum_{k=2}^{n} \frac{\rho_k^2}{n} \ge \frac{\left(\sum_{k=2}^{n}\sqrt{\rho_k}\right)^2}{\sum_{k=2}^{n}\frac{n}{\rho_k}},$$

where equality holds if and only if $\rho_2 = \cdots = \rho_n$. From the above results, we arrive at

$$n\left(DEL(G) - \sqrt{\rho_1}\right)^2 \le n\left(S - \frac{1}{\rho_1}\right)\left(2\sum_{1\le i<j\le n}(d_{ij})^2 + \sum_{i=1}^{n} Tr_i^2 - \rho_1^2\right),$$

where the equality is attained if and only if $\rho_2 = \cdots = \rho_n$. We introduce the function

$$f(x) = n\left(S - \frac{1}{x}\right)\left(2\sum_{1\le i<j\le n}(d_{ij})^2 + \sum_{i=1}^{n} Tr_i^2 - x^2\right) - n\left(DEL(G) - \sqrt{x}\right)^2,$$

for $2Tr_{\min} \le x \le 2Tr_{\max}$. Then

$$f'(x) = -2nx\left(S - \frac{1}{x}\right) + \frac{n}{x^2}\left(2\sum_{1\le i<j\le n}(d_{ij})^2 + \sum_{i=1}^{n} Tr_i^2 - x^2\right) + \frac{n}{\sqrt{x}}\left(DEL(G) - \sqrt{x}\right).$$

Now,

$$\begin{aligned}
-2n\rho_1\left(S - \frac{1}{\rho_1}\right) &+ \frac{n}{\rho_1^2}\left(2\sum_{1\le i<j\le n}(d_{ij})^2 + \sum_{i=1}^{n} Tr_i^2 - \rho_1^2\right) + \frac{n}{\sqrt{\rho_1}}\left(DEL(G) - \sqrt{\rho_1}\right)\\
&= -2n\rho_1 S + \frac{n}{\rho_1^2}\left(2\sum_{1\le i<j\le n}(d_{ij})^2 + \sum_{i=1}^{n} Tr_i^2\right) + \frac{n}{\sqrt{\rho_1}}DEL(G)\\
&= n\sum_{i=1}^{n}\left(\frac{\rho_i^2}{\rho_1^2} - \frac{2\rho_1}{\rho_i} + \sqrt{\frac{\rho_i}{\rho_1}}\right) = n\sum_{i=1}^{n}\left(\left(\sqrt{\frac{\rho_i}{\rho_1}} - \frac{\rho_1}{\rho_i}\right) + \left(\frac{\rho_i^2}{\rho_1^2} - \frac{\rho_1}{\rho_i}\right)\right) \le 0.
\end{aligned}$$

Clearly, $f(x)$ is a decreasing function over $2Tr_{\min} \le x \le 2Tr_{\max}$. Therefore

$$\begin{aligned}
0 &\le f(\rho_1) \le f(2Tr_{\min})\\
&= n\left(S - \frac{1}{2Tr_{\min}}\right)\left(2\sum_{1\le i<j\le n}(d_{ij})^2 + \sum_{i=1}^{n} Tr_i^2 - 4Tr_{\min}^2\right) - n\left(DEL(G) - \sqrt{2Tr_{\min}}\right)^2,
\end{aligned}$$

that is,

$$n\left(S - \frac{1}{2Tr_{\min}}\right)\left(2\sum_{1\le i<j\le n}(d_{ij})^2 + \sum_{i=1}^{n} Tr_i^2 - 4Tr_{\min}^2\right) \ge n\left(DEL(G) - \sqrt{2Tr_{\min}}\right)^2.$$

The first half of the proof is complete.

Assume that the equality in (8) is attained. The equalities must be attained. Thus, we get $\rho_2 = \rho_3 = \cdots = \rho_n$. By Lemma 8, we derive that $G \cong K_n$. □

The next proposition gives a lower bound for distance signless Laplacian energy-like invariant $DEL(G)$ involving transmission degrees as well as $W(G)$.

Proposition 1. *Suppose that G has the transmission degree sequence* $\{Tr_1, Tr_2, \dots, Tr_n\}$. *Then*

$$DEL(G) \geq \sqrt{\frac{8W^3(G)}{2\sum_{1\leq i<j\leq n}\left(d_{ij}\right)^2 + \sum_{i=1}^n Tr_i^2}}. \tag{9}$$

The equality is attained if and only if $G \cong K_1$.

Proof. From Hölder's inequality, it is clear that

$$\sum_{i=1}^n (\sqrt{\rho_i})^2 = \sum_{i=1}^n (\sqrt{\rho_i})^{\frac{4}{3}} (\sqrt{\rho_i})^{\frac{2}{3}} \leq \left[\sum_{i=1}^n \left((\sqrt{\rho_i})^{\frac{4}{3}}\right)^3\right]^{\frac{1}{3}} \left[\sum_{i=1}^n \left((\sqrt{\rho_i})^{\frac{2}{3}}\right)^{\frac{3}{2}}\right]^{\frac{2}{3}}$$
$$= \left(\sum_{i=1}^n (\sqrt{\rho_i})^4\right)^{\frac{1}{3}} \left(\sum_{i=1}^n \sqrt{\rho_i}\right)^{\frac{2}{3}}.$$

Hence, $\sum_{i=1}^n \sqrt{\rho_i} \geq \sqrt{\frac{(\sum_{i=1}^n \rho_i)^3}{\sum_{i=1}^n \rho_i^2}}$, and the equality is attained if and only if $\sqrt{\rho_1} = \cdots = \sqrt{\rho_n}$. Using Lemma 2, we get

$$DEL(G) = \sum_{i=1}^n \sqrt{\rho_i} \geq \sqrt{\frac{8W^3(G)}{2\sum_{1\leq i<j\leq n}\left(d_{ij}\right)^2 + \sum_{i=1}^n Tr_i^2}}.$$

On the basis of the above discussion, the equality in (9) us attained if and only if $\sqrt{\rho_1} = \cdots = \sqrt{\rho_n}$, that is, if and only if $G \cong K_1$. □

From Proposition 1 we have the following observation.

Corollary 2. *Suppose that G is connected and has the vertex set* $\{v_1, v_2, \dots, v_n\}$ *and diameter* d, *then*

$$DEL(G) \geq 2n(n-1)\sqrt{\frac{1}{4d^2 + n^2(n-1)}}.$$

Proof. First, recall that $d_{ij} \leq d$ for $i \neq j$. There are $\frac{n(n-1)}{2}$ pairs of vertices in G. In view of the lower bound proved in Proposition 1, we obtain

$$DEL(G) \geq \sqrt{\frac{8W^3(G)}{2\sum_{1\leq i<j\leq n}\left(d_{ij}\right)^2 + \sum_{i=1}^n Tr_i^2}} \geq \sqrt{\frac{8\frac{n^3(n-1)^3}{8}}{2\frac{n(n-1)}{2}d^2 + \frac{n^3(n-1)^2}{4}}}$$
$$= 2n(n-1)\sqrt{\frac{1}{4d^2 + n^2(n-1)}}.$$

□

Let $a = \{a_1, a_2, \dots, a_n\}$ be a list of positive numbers. The mean of products of k tuples of this family, W_k, can be defined as follows:

$$\begin{aligned} W_1 &= \frac{1}{n}\left(a_1 + a_2 + \cdots + a_n\right), \\ W_2 &= \frac{1}{\frac{1}{2}n(n-1)}\left(a_1a_2 + a_1a_3 + \cdots + a_1a_n + a_2a_3 + \cdots + a_{n-1}a_n\right), \\ &\vdots \\ W_n &= a_1a_2\dots a_n. \end{aligned}$$

It can be seen that the arithmetic average value is W_1 and the geometric average value becomes $W_n^{\frac{1}{n}}$.

The MacLaurin symmetric average value inequality is the following.

Lemma 13 ([16]). *If $a_1, a_2, \dots, a_n$ are given as above, then*

$$W_1 \geq W_2^{\frac{1}{2}} \geq W_3^{\frac{1}{3}} \geq \cdots \geq W_n^{\frac{1}{n}},$$

where the equality is attained if and only if $a_1 = a_2 = \cdots = a_n$.

Lemma 14 ([28]). *Given $a, b \geq 0$ and $z \in [0,1]$, we have*

$$(1-z)a + zb \geq a^{1-z}b^z + r\left(\sqrt{a} - \sqrt{b}\right)^2. \tag{10}$$

Here, $r := \min\{z, 1-z\}$.

Proposition 2. *Suppose that G is connected and has the vertex set $\{v_1, v_2, \dots, v_n\}$. We have*

$$DEL(G) \geq \sqrt{(n-1)\left(nP^{\frac{1}{n}} - \tfrac{2}{n}W(G)\right)}, \tag{11}$$

where $P = \prod_{i=1}^{n} \rho_i$ and the equality is attained if and only if $G \cong K_1$.

Proof. By invoking Lemma 13, we obtain

$$\sum_{i=1}^{n} \rho_i^{\frac{n-1}{2n}} \geq n\left(\prod_{i=1}^{n} \rho_i\right)^{\frac{n-1}{2n^2}} \tag{12}$$

and

$$\sum_{j=1}^{n} \rho_j^{\frac{n+1}{2n}} \geq n\left(\prod_{j=1}^{n} \rho_j\right)^{\frac{n+1}{2n^2}}. \tag{13}$$

Setting $z = \frac{n+1}{2n}$, $a = \rho_i$, $b = \rho_j$ and $r = \frac{n-1}{2n}$ in (10), we get

$$\frac{n-1}{2n}\rho_i + \frac{n+1}{2n}\rho_j \geq \rho_i^{\frac{n-1}{2n}}\rho_j^{\frac{n+1}{2n}} + \frac{n-1}{2n}\left(\rho_i + \rho_j - 2\sqrt{\rho_i\rho_j}\right).$$

Moreover,

$$\frac{n-1}{2n}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_i + \frac{n+1}{2n}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_j \geq \sum_{i=1}^{n}\rho_i^{\frac{n-1}{2n}}\sum_{j=1}^{n}\rho_j^{\frac{n+1}{2n}} + \frac{n-1}{2n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\rho_i + \rho_j - 2\sqrt{\rho_i\rho_j}\right),$$

hence from (12) and (13), we deduce that

$$2nW(G) \geq n^2 \left(\prod_{i=1}^{n} \rho_i \right)^{\frac{n-1}{2n^2}} \left(\prod_{j=1}^{n} \rho_j \right)^{\frac{n+1}{2n^2}} + 2W(G)(n-1) - \left(\frac{n-1}{n} \right) DEL^2(G).$$

This implies

$$DEL^2(G) \geq \left(\frac{n-1}{n} \right) \left(n^2 P^{\frac{1}{n}} - 2W(G) \right),$$

which leads to the lower bound (11).

If equality holds in (11), all equalities above will be attained. From Lemma 13, we get $\rho_1 = \rho_2 = \cdots = \rho_n$, and therefore $G \cong K_1$.

Furthermore, one can easily examine the fact that when $G \cong K_1$, the equality holds in (11). □

Now, we apply Lemma 11 to derive a simple lower bound for $DEL(G)$.

Proposition 3. *Suppose that G is connected and has the vertex set $\{v_1, v_2, \ldots, v_n\}$. We have*

$$DEL(G) \geq \sqrt{2nW(G) - \frac{n^2}{4}\left(\sqrt{\rho_1} - \sqrt{\rho_n}\right)^2}.$$

Proposition 4. *Suppose that G is connected and has the vertex set $\{v_1, v_2, \ldots, v_n\}$. We have*

$$DEL(G) \geq \sqrt{2nW(G) - \alpha(n)\left(\sqrt{\rho_1} - \sqrt{\rho_n}\right)^2}, \tag{14}$$

where the equality is attained if and only if $G \cong K_1$.

Proof. For $a_i = \sqrt{\rho_i}$, $b_i = \sqrt{\rho_i}$, $a = b = \sqrt{\rho_n}$, $A = B = \sqrt{\rho_1}$, $i = 1, 2, \ldots, n$, apply Lemma 12, it follows that

$$\left| n \sum_{i=1}^{n} (\sqrt{\rho_i})^2 - \left(\sum_{i=1}^{n} \sqrt{\rho_i} \right)^2 \right| \leq \alpha(n) \left(\sqrt{\rho_1} - \sqrt{\rho_n}\right)^2.$$

By Lemma 2, the above inequality becomes

$$2nW(G) - DEL^2(G) \leq \alpha(n) \left(\sqrt{\rho_1} - \sqrt{\rho_n}\right)^2,$$

and so the inequality (14) follows.

Equality of (14) holds if and only if $\rho_1 = \rho_2 = \cdots = \rho_n$. This is equivalent to $G \cong K_1$. □

Since $\alpha(n) = n \left[\frac{n}{2}\right] \left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right) \leq \frac{n^2}{4}$, we obtain

$$\sqrt{2n\sigma(G) - \alpha(n)\left(\sqrt{\rho_1} - \sqrt{\rho_n}\right)^2} \geq \sqrt{2n\sigma(G) - \frac{n^2}{4}\left(\sqrt{\rho_1} - \sqrt{\rho_n}\right)^2}.$$

Thus, we have the following observation.

Remark 1. *The lower bound shown in Proposition 4 is sharper than the lower bound given in Proposition 3.*

Now, we consider to investigate some upper bounds for $DEL(G)$. The next theorem presents an upper bound involving the transmission degrees of vertices.

Proposition 5. *Suppose that G has the transmission degree sequence $\{Tr_1, Tr_2, \ldots, Tr_n\}$. Then*

$$DEL(G) \leq \frac{1}{2}\Big(\sqrt{\frac{n}{2}} + \sqrt{\frac{2}{n}}\Big)\Big(\sum_{i=1}^{n} \sqrt{Tr_i}\Big).$$

Proof. In the light of the Cauchy-Schwarz inequality, we derive $\left(\sum_{i=1}^{n} \sqrt{\rho_i}\right)^2 \leq n\left(\sum_{i=1}^{n} \rho_i\right)$. For every $1 \leq i \leq n$, we know that $n-1 \leq Tr_i \leq \frac{n(n-1)}{2}$. So, by Lemma 9, we arrive at

$$n\Big(\sum_{i=1}^{n} Tr_i\Big) \leq \frac{1}{4}\Bigg(\sqrt{\frac{\frac{n(n-1)}{2}}{n-1}} + \sqrt{\frac{n-1}{\frac{n(n-1)}{2}}}\Bigg)^2 \Big(\sum_{i=1}^{n} \sqrt{Tr_i}\Big)^2.$$

Since $n\left(\sum_{i=1}^{n} \rho_i\right) = n\left(\sum_{i=1}^{n} Tr_i\right)$, we get $DEL(G) \leq \frac{1}{2}\Big(\sqrt{\frac{n}{2}} + \sqrt{\frac{2}{n}}\Big)\Big(\sum\limits_{i=1}^{n} \sqrt{Tr_i}\Big)$. This finishes the proof of the theorem. □

Next, we derive another upper bound for $DEL(G)$ involving order n, maximum transmission degree $Tr_{\max}(G)$ and minimum transmission degree $Tr_{\min}(G)$ of a graph G.

Theorem 3. *Suppose that G is a connected graph and has the maximum transmission degree $Tr_{\max}(G)$ and minimum transmission degree $Tr_{\min}(G)$. We have*

$$DEL(G) \leq \sqrt{2Tr_{\max}} + (n-1)\left(\frac{2\sum_{1\leq i<j\leq n}\left(d_{ij}\right)^2 + \sum_{i=1}^{n} Tr_i^2 - 4Tr_{\min}^2}{n-1}\right)^{\frac{1}{4}}, \tag{15}$$

where the equality is attained if and only if $G \cong K_n$.

Proof. Setting $p = n-1$, $a_i = \rho_i$, $i = 2, 3, \ldots, n$ and $s = 2$, $t = \frac{1}{2}$ in Lemma 10, we get

$$\left(\frac{\sum_{i=2}^{n} \rho_i^2}{n-1}\right)^{\frac{1}{2}} \geq \left(\frac{\sum_{i=2}^{n} \sqrt{\rho_i}}{n-1}\right)^2. \tag{16}$$

Therefore,

$$\left(\frac{2\sum_{1\leq i<j\leq n}\left(d_{ij}\right)^2 + \sum_{i=1}^{n} Tr_i^2 - \rho_1^2}{n-1}\right)^{\frac{1}{2}} \geq \left(\frac{DEL(G) - \sqrt{\rho_1}}{n-1}\right)^2.$$

So,

$$\begin{aligned} DEL(G) &\leq \sqrt{\rho_1} + (n-1)\left(\frac{2\sum_{1\leq i<j\leq n}\left(d_{ij}\right)^2 + \sum_{i=1}^{n} Tr_i^2 - \rho_1^2}{n-1}\right)^{\frac{1}{4}} \\ &\leq \sqrt{2Tr_{\max}} + (n-1)\left(\frac{2\sum_{1\leq i<j\leq n}\left(d_{ij}\right)^2 + \sum_{i=1}^{n} Tr_i^2 - 4Tr_{\min}^2}{n-1}\right)^{\frac{1}{4}}. \end{aligned} \tag{17}$$

Now, if the equality in (15) is attained, then inequalities above turn out to be equalities. Particularly, from equality in (16), by Lemma 10, we get $\rho_2 = \cdots = \rho_n$. Also, from equality in (17), we have $\rho_1 = 2Tr_{\min}$, so by Corollary 1, G is a transmission regular graph. Therefore, equality holds in (15) if and only if G is a transmission regular graph having at most two distinct distance signless Laplacian eigenvalues. That is, by Lemma 8, if and only if $G \cong K_n$. On the other hand, one can readily show that the equality is attained in (15) for the complete graph K_n. □

If minimum degree is δ, maximum degree is Δ and diameter is d, then

$$Tr_p = \sum_{j=1}^{n} d_{jp} \leq d_p + 2 + 3 + \cdots + (d-1) + d\big(n - 1 - d_p - (d-2)\big)$$
$$= dn - \frac{d(d-1)}{2} - 1 - d_p(d-1)$$

for all $p = 1, 2, \ldots, n$. Also, for any $i = 1, \ldots, n$, we have

$$Tr_i \geq d_i + 2\,(n - d_i - 1) = 2n - d_i - 2 \geq 2(2n - \Delta - 2). \tag{18}$$

Using Theorem 3, we derive the the following result.

Corollary 3. *Suppose that G has the minimum degree δ and the maximum degree Δ. If the diameter of G is d, then*

$$DEL(G) \leq \sqrt{2\Big(dn - \frac{d(d-1)}{2} - 1 - \delta(d-1)\Big)} + (n-1)\left(\frac{2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \sum_{i=1}^{n} Tr_i^2 - 4(2n-\Delta-2)^2}{n-1}\right)^{\frac{1}{4}}.$$

The next result presents an upper bound for $DEL(G)$ involving maximum transmission $Tr_{\max}(G)$, minimum transmission $Tr_{\min}(G)$ and Wiener index $W(G)$.

Theorem 4. *Suppose that G is connected and has the vertex set $\{v_1, v_2, \ldots, v_n\}$ and $\rho_n \geq k > 0$. We have*

$$DEL(G) \leq \frac{2\sqrt{2Tr_{\max}k} + (n-1)k + 2W(G) - 2Tr_{\min}}{2\sqrt{k}}, \tag{19}$$

where the equality is attained if and only if $G \cong K_n$.

Proof. Denote by $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n \geq k > 0$ the distance signless Laplacian spectrum. Since $2Tr_{\min} \leq \rho_1 \leq 2Tr_{\max}$, so we see that

$$DEL(G) = \sum_{i=1}^{n} \sqrt{\rho_i} = \sqrt{\rho_1} + (n-1)\sqrt{\rho_n} + \sum_{i=2}^{n-1}\left(\frac{\rho_i - \rho_n}{\sqrt{\rho_i} + \sqrt{\rho_n}}\right)$$
$$\leq \sqrt{\rho_1} + (n-1)\sqrt{\rho_n} + \sum_{i=2}^{n-1}\left(\frac{\rho_i - \rho_n}{2\sqrt{\rho_n}}\right)$$
$$= \frac{2\sqrt{\rho_1\rho_n} + (n-1)\rho_n + 2W(G) - \rho_1}{2\sqrt{\rho_n}}$$
$$\leq \frac{2\sqrt{2Tr_{\max}\rho_n} + (n-1)\rho_n + 2W(G) - 2Tr_{\min}}{2\sqrt{\rho_n}}.$$

For $x \geq k$, we examine the following function

$$f(x) = \frac{2\sqrt{2Tr_{\max}x} + (n-1)x + 2W(G) - 2Tr_{\min}}{2\sqrt{x}}.$$

Accordingly, we have

$$f'(x) = \frac{x(n-1) + 2Tr_{\min} - 2W(G)}{4x\sqrt{x}}$$

for all $x \geq k$. Since $2Tr_{\min} + (n-1)\rho_n \leq \rho_1 + \rho_2 + \cdots + \rho_n = 2W(G)$, it follows that $f(x)$ is a decreasing function for $x \geq k$. Therefore,

$$f(x) \leq f(k) = \frac{2\sqrt{2Tr_{\max}k} + (n-1)k + 2W(G) - 2Tr_{\min}}{2\sqrt{k}},$$

which gives

$$DEL(G) \leq \frac{2\sqrt{2Tr_{\max}k} + (n-1)k + 2W(G) - 2Tr_{\min}}{2\sqrt{k}}.$$

Equality occurs in (19) if and only if $2Tr_{\min} = \rho_1 = 2Tr_{\max}$, $\rho_2 = \cdots = \rho_n = k$. In other words, it happens if and only if G is transmission regular having at most two distinct distance signless Laplacian eigenvalues. In view of Lemma 8, the equality is attained if and only if $G \cong K_n$. Moreover, we can easily derive the equality in (19) for complete graph K_n in a direct manner. □

For each $1 \leq i \leq n$, we obtain $n-1 \leq Tr_i \leq \frac{n(n-1)}{2}$. The following result is immediate.

Corollary 4. *Suppose that G is connected and has n vertices with $\rho_n \geq k > 0$. We have*

$$DEL(G) \leq \frac{2\sqrt{kn(n-1)} + (n-1)(k-2) + 2W(G)}{2\sqrt{k}}.$$

Next, we establish some bounds for $DEL(G)$ in terms of the Wiener index $W(G)$.

Proposition 6. *Suppose G is connected and has the vertex set $\{v_1, v_2, \ldots, v_n\}$. Then*

$$\sqrt{2W(G)} \leq DEL(G) \leq \sqrt{2nW(G)}. \tag{20}$$

Proof. Let $S = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sqrt{\rho_i} - \sqrt{\rho_j}\right)^2$. Then simple computations show that

$$S = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sqrt{\rho_i} - \sqrt{\rho_j}\right)^2 = 2\sum_{i=1}^{n} 2W(G) - 2\left(\sum_{i=1}^{n}\sqrt{\rho_i}\right)\left(\sum_{j=1}^{n}\sqrt{\rho_j}\right) = 4nW(G) - 2DEL^2(G).$$

Since $S \geq 0$, we get $DEL(G) \leq \sqrt{2nW(G)}$. On the other hand, we have

$$DEL^2(G) = \sum_{i=1}^{n}(\sqrt{\rho_i})^2 + 2\sum_{i\neq j}\sqrt{\rho_i}\sqrt{\rho_j} \geq \sum_{i=1}^{n}\rho_i = 2W(G).$$

□

The following results provides bounds for $DEL(G)$ in terms of the two extreme distance signless Laplacian eigenvalues ρ_n and ρ_1 as well as $W(G)$.

Theorem 5. *Assume that G is connected and has vertex set $\{v_1, \ldots, v_n\}$. Then*

$$\frac{n\sqrt{\rho_1\rho_n} + 2nW(G)}{\sqrt{\rho_1} + \sqrt{\rho_n}} \leq DEL(G) \leq \sqrt{2nW(G) - \frac{n}{2}\left(\sqrt{\rho_1} - \sqrt{\rho_n}\right)^2}, \tag{21}$$

where the equalities are attained if and only if $G \cong K_1$.

Proof. We consider the lower bound first. For $b_i = \sqrt{\rho_i}$, $a_i = 1$, $r = \sqrt{\rho_n}$, $R = \sqrt{\rho_1}$, $i = 1, \ldots, n$, using Lemma 7, we get

$$\sum_{i=1}^{n} (\sqrt{\rho_i})^2 + \sqrt{\rho_1}\sqrt{\rho_n}\sum_{i=1}^{n} 1 \leq (\sqrt{\rho_1} + \sqrt{\rho_n}) \sum_{i=1}^{n} \sqrt{\rho_i}.$$

Therefore, by Lemma 2 and simple calculations, we get the desired lower bound in (21).

If for some i, $ra_i = b_i = Ra_i$, then the equality $b_i = r = R$ holds. In other words, for any j, $j \neq i$, we have $\sqrt{\rho_i} \leq \sqrt{\rho_j} \leq \sqrt{\rho_i}$. Therefore, equality on left-hand side of (21) is true if and only if $\rho_1 = \cdots = \rho_n$, that is, if and only if $G \cong K_1$.

Now, we obtain the given upper bound. As

$$n\sum_{i=1}^{n} \rho_i - \left(\sum_{i=1}^{n} \sqrt{\rho_i}\right)^2 = \sum_{1\leq i<j\leq n} (\sqrt{\rho_i} - \sqrt{\rho_j})^2,$$

so we get

$$n\sum_{i=1}^{n} \rho_i - \left(\sum_{i=1}^{n} \sqrt{\rho_i}\right)^2 \geq \sum_{i=2}^{n-1} \left((\sqrt{\rho_1} - \sqrt{\rho_i})^2 + (\sqrt{\rho_i} - \sqrt{\rho_n})^2\right) + (\sqrt{\rho_1} - \sqrt{\rho_n})^2. \tag{22}$$

Also, in view of Jensen's inequality [31], it can be seen that

$$\sum_{i=2}^{n-1} \left((\sqrt{\rho_1} - \sqrt{\rho_i})^2 + (\sqrt{\rho_i} - \sqrt{\rho_n})^2\right) \geq \frac{n-2}{2}(\sqrt{\rho_1} - \sqrt{\rho_n})^2. \tag{23}$$

By Lemma 2 and inequalities (22) and (23), we obtain the desired upper bound in (21). Equality in (22) holds if and only if $\sqrt{\rho_2} = \sqrt{\rho_3} = \cdots = \sqrt{\rho_n}$, whereas in (23) if and only if $\sqrt{\rho_1} - \sqrt{\rho_i} = \sqrt{\rho_i} - \sqrt{\rho_n}$ for each $i = 2, \ldots, n-1$. This means that equality in right-hand side of (21) holds if and only if $\sqrt{\rho_1} = \sqrt{\rho_2} = \cdots = \sqrt{\rho_n}$, that is, if and only if $G \cong K_1$. □

If G has order n and is connected with the property that $\sqrt{\rho_1} \neq \sqrt{\rho_n}$, then

$$\sqrt{2nW(G) - \frac{n}{2}(\sqrt{\rho_1} - \sqrt{\rho_n})^2} \leq \sqrt{2nW(G)}.$$

Therefore, we conclude the following remark.

Remark 2. *The upper bound given in (21) improves the given upper bound in (20).*

Another simple upper bound for $DEL(G)$ is as follows.

Corollary 5. *Suppose that G has order n and is connected with the property $\sqrt{\rho_1} \neq \sqrt{\rho_n}$. Then*

$$DEL(G) \leq \frac{\sqrt{2n}W(G)}{\sqrt{\rho_1} - \sqrt{\rho_n}}. \tag{24}$$

Proof. On the basis of the upper bound of Theorem 5, we obtain

$$DEL^2(G) + \frac{n}{2}(\sqrt{\rho_1} - \sqrt{\rho_n})^2 \leq 2nW(G).$$

From the well-known A-G mean inequality [31], it follows that

$$2\sqrt{\frac{n}{2}DEL^2(G)(\sqrt{\rho_1} - \sqrt{\rho_n})^2} \leq 2nW(G),$$

from where the inequality (24) follows. □

Let $n \geq 3$, $P = \prod_{i=1}^{n} \rho_i$, and $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n > 0$ the distance signless Laplacian eigenvalues of G. The following result gives a lower and an upper bound for $DEL(G)$ using P, two extreme transmission degree $Tr_{\max}$ and $Tr_{\min}$, as well as $W(G)$.

Theorem 6. *If G is connected, then*

$$\sqrt{2Tr_{\min}} + \sqrt{2W(G) - 2Tr_{\max} + (n-1)(n-2)\left(\frac{P}{2Tr_{\max}}\right)^{\frac{1}{n-1}}} \leq DEL(G) \leq \sqrt{2Tr_{\max}} + \sqrt{(n-2)\big(2W(G) - 2Tr_{\min}\big) + (n-1)\left(\frac{P}{2Tr_{\min}}\right)^{\frac{1}{n-1}}}, \tag{25}$$

where the equalities are attained if and only if $G \cong K_n$.

Proof. In view of Lemma 6, if we set $N = n-1$ and $x_i = \rho_i$, $i = 2, 3, \ldots, n$, we arrive at

$$\frac{\sum_{2\leq i<j\leq n}\left(\sqrt{\rho_i} - \sqrt{\rho_j}\right)^2}{(n-1)(n-2)} \leq \frac{2W(G) - \rho_1}{n-1} - \left(\frac{P}{\rho_1}\right)^{\frac{1}{n-1}} \leq \frac{\sum_{2\leq i<j\leq n}\left(\sqrt{\rho_i} - \sqrt{\rho_j}\right)^2}{n-1}.$$

We have

$$\begin{aligned}\sum_{2\leq i<j\leq n}\left(\sqrt{\rho_i} - \sqrt{\rho_j}\right)^2 &= (n-2)\sum_{i=2}^{n}\rho_i - 2\sum_{2\leq i<j\leq n}\sqrt{\rho_i\rho_j} \\ &= (n-2)\big(2W(G) - \rho_1\big) - \left(\sum_{i=2}^{n}\sqrt{\rho_i}\right)^2 + \sum_{i=2}^{n}\rho_i \\ &= (n-1)\big(2W(G) - \rho_1\big) - \big(DEL(G) - \sqrt{\rho_1}\big)^2,\end{aligned}$$

from which

$$\begin{aligned}\frac{(n-1)(2W(G) - \rho_1) - \big(DEL(G) - \sqrt{\rho_1}\big)^2}{(n-1)(n-2)} &\leq \frac{2W(G) - \rho_1}{n-1} - \left(\frac{P}{\rho_1}\right)^{\frac{1}{n-1}} \\ &\leq \frac{(n-1)(2W(G) - \rho_1) - \big(DEL(G) - \sqrt{\rho_1}\big)^2}{n-1}.\end{aligned}$$

This implies that

$$DEL(G) \leq \sqrt{\rho_1} + \sqrt{(n-2)\big(2W(G) - \rho_1\big) + (n-1)\left(\frac{P}{\rho_1}\right)^{\frac{1}{n-1}}} \tag{26}$$

and

$$DEL(G) \geq \sqrt{\rho_1} + \sqrt{2W(G) - \rho_1 + (n-1)(n-2)\left(\frac{P}{\rho_1}\right)^{\frac{1}{n-1}}}. \tag{27}$$

Let

$$f(x) = \sqrt{(n-2)\big(2W(G) - x\big) + (n-1)\left(\frac{P}{x}\right)^{\frac{1}{n-1}}}.$$

Notice that $f(x)$ is decreasing with respect to $x \geq 2Tr_{\min}$. Consequently,

$$f(x) \leq f(2Tr_{\min}) = \sqrt{(n-2)\big(2W(G) - 2Tr_{\min}\big) + (n-1)\left(\frac{P}{2Tr_{\min}}\right)^{\frac{1}{n-1}}}.$$

Since $\rho_1 \leq 2Tr_{\max}$, from (26), it follows that

$$\begin{aligned} DEL(G) &\leq \sqrt{\rho_1} + \sqrt{(n-2)\big(2W(G) - 2Tr_{\min}\big) + (n-1)\left(\frac{P}{2Tr_{\min}}\right)^{\frac{1}{n-1}}} \\ &\leq \sqrt{2Tr_{\max}} + \sqrt{(n-2)\big(2W(G) - 2Tr_{\min}\big) + (n-1)\left(\frac{P}{2Tr_{\min}}\right)^{\frac{1}{n-1}}}. \end{aligned}$$

Similarly,

$$g(x) = \sqrt{2W(G) - x + (n-1)(n-2)\left(\frac{P}{x}\right)^{\frac{1}{n-1}}}$$

is also decreasing for $\rho_1 \leq 2Tr_{\max}$ and consequently

$$g(x) \geq g(2Tr_{\max}) = \sqrt{2W(G) - 2Tr_{\max} + (n-1)(n-2)\left(\frac{P}{2Tr_{\max}}\right)^{\frac{1}{n-1}}}.$$

Then, from (27), we obtain

$$DEL(G) \geq \sqrt{2Tr_{\min}} + \sqrt{2W(G) - 2Tr_{\max} + (n-1)(n-2)\left(\frac{P}{2Tr_{\max}}\right)^{\frac{1}{n-1}}}.$$

The first part is finished here.

Next, if the equality holds in (25), equalities will be attained in the above arguments. By Lemma 6 and if equalities holds in (25), we have $\rho_2 = \rho_3 = \cdots = \rho_n$, and hence applying Lemma 8, we see that $G \cong K_n$.

On the other hand, we can directly verify that the equalities hold in (25) for the complete graph K_n. □

4 Relations between versions of distance signless Laplacian energy

In this section, we establish relations between distance signless Laplacian energy $E_{DQ}(G)$ and distance signless Laplacian energy-like invariant $DEL(G)$ of a graph G.

Theorem 7. *Assume that G is connected and has the vertex set $\{v_1, \ldots, v_n\}$. We have*

$$\frac{E_{DQ}(G)}{DEL(G)} \leq \max\left\{\sqrt{\rho_1} - \frac{2W(G)}{n\sqrt{\rho_1}}, \frac{2W(G)}{n\sqrt{\rho_n}} - \sqrt{\rho_n}\right\}, \tag{28}$$

where equality holds if and only if $G = K_1$ or $\rho_1 = \rho_2 = \cdots = \rho_\beta$ and $\rho_{\beta+1} = \rho_{\beta+2} = \cdots = \rho_n$ with $\left(\sqrt{\rho_1} + \sqrt{\rho_n}\right)\left(\sqrt{\rho_1\rho_n} - \frac{2W(G)}{n}\right) = 0$, where β, $1 \leq \beta \leq n$, is the number of distance signless Laplacian eigenvalues of G, which are no less than $\frac{2W(G)}{n}$.

Proof. Assume that β is the integer satisfying $\rho_\beta \geq \frac{2W(G)}{n}$ and $\rho_{\beta+1} < \frac{2W(G)}{n}$. In the light of Lemma 5, we have

$$\begin{aligned}\frac{E_{DQ}(G)}{DEL(G)} &= \frac{\sum_{i=1}^{n}\left|\rho_i - \frac{2W(G)}{n}\right|}{\sum_{i=1}^{n}\sqrt{\rho_i}} \leq \max_{1\leq i\leq n}\left\{\frac{\left|\rho_i - \frac{2W(G)}{n}\right|}{\sqrt{\rho_i}}\right\}\\ &= \left\{\max_{1\leq i\leq \beta}\left\{\sqrt{\rho_i} - \frac{2W(G)}{n\sqrt{\rho_i}}\right\}, \max_{\beta+1\leq i\leq n}\left\{\frac{2W(G)}{n\sqrt{\rho_i}} - \sqrt{\rho_i}\right\}\right\}.\end{aligned} \tag{29}$$

A line of calculation shows that $f(x) = \sqrt{x} - \frac{2W(G)}{n\sqrt{x}}$ is an increasing function for $0 < \rho_\beta \leq x \leq \rho_1$ and $g(x) = \frac{2W(G)}{n\sqrt{x}} - \sqrt{x}$ is a decreasing function for $0 < \rho_n \leq x \leq \rho_{\beta+1}$. Consequently,

$$\sqrt{\rho_i} - \frac{2W(G)}{n\sqrt{\rho_i}} \leq \sqrt{\rho_1} - \frac{2W(G)}{n\sqrt{\rho_1}}, \qquad \frac{2W(G)}{n\sqrt{\rho_i}} - \sqrt{\rho_i} \leq \frac{2W(G)}{n\sqrt{\rho_n}} - \sqrt{\rho_n}.$$

Hence

$$\frac{E_{DQ}(G)}{DEL(G)} \leq \max\left\{\sqrt{\rho_1} - \frac{2\sigma(G)}{n\sqrt{\rho_1}}, \frac{2\sigma(G)}{n\sqrt{\rho_n}} - \sqrt{\rho_n}\right\}.$$

This concludes the first half.

Assume that the equality holds in (28). In this case all inequalities above turn out to be equalities. Using Lemma 5, it is immediate that the equality holds in (29) if and only if

$$\frac{\left|\rho_1 - \frac{2\sigma(G)}{n}\right|}{\sqrt{\rho_1}} = \frac{\left|\rho_2 - \frac{2\sigma(G)}{n}\right|}{\sqrt{\rho_2}} = \cdots = \frac{\left|\rho_n - \frac{2\sigma(G)}{n}\right|}{\sqrt{\rho_n}}.$$

In the sequel, we split the proof into the following two scenarios.

Case (i): $1 \leq \beta \leq n-1$. Then we have

$$\frac{\rho_1 - \frac{2\sigma(G)}{n}}{\sqrt{\rho_1}} = \cdots = \frac{\rho_\beta - \frac{2\sigma(G)}{n}}{\sqrt{\rho_\beta}} = \frac{\frac{2\sigma(G)}{n} - \rho_{\beta+1}}{\sqrt{\rho_{\beta+1}}} = \cdots = \frac{\frac{2\sigma(G)}{n} - \rho_n}{\sqrt{\rho_n}},$$

that is,

$$\sqrt{\rho_1} - \frac{2\sigma(G)}{n\sqrt{\rho_1}} = \cdots = \sqrt{\rho_\beta} - \frac{2\sigma(G)}{n\sqrt{\rho_\beta}} = \frac{2\sigma(G)}{n\sqrt{\rho_{\beta+1}}} - \sqrt{\rho_{\beta+1}} = \cdots = \frac{2\sigma(G)}{n\sqrt{\rho_n}} - \sqrt{\rho_n}.$$

For $1 \leq i \leq j \leq \beta \leq n-1$, we know

$$\sqrt{\rho_i} - \frac{2\sigma(G)}{n\sqrt{\rho_i}} = \sqrt{\rho_j} - \frac{2\sigma(G)}{n\sqrt{\rho_j}},$$

and thus

$$\left(\sqrt{\rho_i} - \sqrt{\rho_i}\right)\left(\sqrt{\rho_i\rho_j} + \frac{2\sigma(G)}{n}\right) = 0 \quad \Longrightarrow \quad \rho_i = \rho_j.$$

For $\beta + 1 \leq i \leq j \leq n-1$, we have

$$\frac{2\sigma(G)}{n\sqrt{\rho_i}} - \sqrt{\rho_i} = \frac{2\sigma(G)}{n\sqrt{\rho_j}} - \sqrt{\rho_j},$$

therefore, similarly

$$\left(\sqrt{\rho_i}-\sqrt{\rho_i}\right)\left(\sqrt{\rho_i\rho_j}+\frac{2\sigma(G)}{n}\right)=0 \quad\Longrightarrow\quad \rho_i=\rho_j.$$

From the above observations, we conclude that $\rho_1=\rho_2=\cdots=\rho_\beta$, $\rho_{\beta+1}=\rho_{\beta+2}=\cdots=\rho_n$ with

$$\sqrt{\rho_1}-\frac{2\sigma(G)}{n\sqrt{\rho_1}}=\frac{2\sigma(G)}{n\sqrt{\rho_n}}-\sqrt{\rho_n}, \qquad \text{that is,} \qquad \left(\sqrt{\rho_i}+\sqrt{\rho_i}\right)\left(\sqrt{\rho_i\rho_j}-\frac{2\sigma(G)}{n}\right)=0.$$

Case (ii): $\beta=n$. In this case, we have

$$\sqrt{\rho_1}-\frac{2\sigma(G)}{n\sqrt{\rho_1}}=\sqrt{\rho_2}-\frac{2\sigma(G)}{n\sqrt{\rho_2}}=\cdots=\sqrt{\rho_n}-\frac{2\sigma(G)}{n\sqrt{\rho_n}},$$

that is, $\rho_1=\rho_2=\cdots=\rho_n$, hence $G\cong K_1$.

On the other hand, suppose that G satisfies $\rho_1=\rho_2=\cdots=\rho_\beta$, $\rho_{\beta+1}=\rho_{\beta+2}=\cdots=\rho_n$ and $\left(\sqrt{\rho_i}+\sqrt{\rho_i}\right)\left(\sqrt{\rho_i\rho_j}-\frac{2\sigma(G)}{n}\right)=0$. Then

$$E_{DQ}(G)=\sum_{i=1}^{n}\left|\rho_i-\frac{2\sigma(G)}{n}\right|=\beta\sqrt{\rho_1}\left(\sqrt{\rho_1}-\frac{2\sigma(G)}{n\sqrt{\rho_1}}\right)+(n-\beta)\sqrt{\rho_n}\left(\frac{2\sigma(G)}{n\sqrt{\rho_n}}-\sqrt{\rho_n}\right)$$

and

$$DEL(G)=\beta\sqrt{\rho_1}+(n-\beta)\sqrt{\rho_n}.$$

Hence

$$\frac{E_{DQ}(G)}{DEL(G)}=\sqrt{\rho_1}-\frac{2\sigma(G)}{n\sqrt{\rho_1}}=\frac{2\sigma(G)}{n\sqrt{\rho_n}}-\sqrt{\rho_n},$$

and the proof is complete. □

The next result is an upper bound for $DEL(G)-E_{DQ}(G)$ involving minimum transmission degree $Tr_{\min}(G)$ and Wiener index $W(G)$.

Theorem 8. *For a connected graph G with the vertex set $\{v_1,v_2,\ldots,v_n\}$, we have*

$$DEL(G)-E_{DQ}(G)<\sqrt{\frac{2W(G)}{n}-1}+\sqrt{(n-1)\left(\frac{2(n-1)W(G)}{n}+1\right)}-\frac{4\left(nTr_{\min}-W(G)\right)}{n(n-1)}.$$

Proof. Consider a function

$$g(x)=\sqrt{x}+\sqrt{(n-1)\left(2W(G)-x\right)}, \quad x\le\frac{2W(G)}{n}-1.$$

The first derivative of g is

$$g'(x)=\frac{1}{2\sqrt{x}}-\frac{\sqrt{n-1}}{2\sqrt{2W(G)-x}}, \quad x\le\frac{2W(G)}{n}-1.$$

Thus $g(x)$ is increasing with respect to $x\le\frac{2W(G)}{n}-1$, and then

$$\sqrt{x}+\sqrt{(n-1)\left(2W(G)-x\right)}\le\sqrt{\frac{2W(G)}{n}-1}+\sqrt{(n-1)\left(\frac{2(n-1)W(G)}{n}+1\right)}.$$

Using Cauchy-Schwarz inequality and the fact $\rho_n \leq \frac{2W(G)}{n} - 1$ (see [44]), we have

$$\begin{aligned} DEL(G) = \sum_{i=1}^{n} \sqrt{\rho_i} \leq \sqrt{\rho_n} + \sqrt{(n-1)\sum_{i=1}^{n-1}\rho_i} \leq \sqrt{\rho_n} + \sqrt{(n-1)\big(2W(G)-\rho_n\big)} \\ \leq \sqrt{\frac{2W(G)}{n} - 1} + \sqrt{(n-1)\left(\frac{2(n-1)W(G)}{n} + 1\right)}. \end{aligned} \tag{30}$$

Suppose β is the integer such that $\rho_\beta \geq \frac{2W(G)}{n}$ and $\rho_{\beta+1} < \frac{2W(G)}{n}$. Note that $\sum_{i=1}^{n} \rho_i = 2W(G)$ and $2W(G) \geq \rho_1 + (n-1)\rho_n$. Also, by Corollary 1, we have $2Tr_{\min} \leq \rho_1 \leq 2Tr_{\max}$. Then we get $\rho_n \leq \frac{2W(G)-\rho_1}{n-1} \leq \frac{2W(G)-2Tr_{\min}}{n-1}$. Moreover,

$$\begin{aligned} E_{DQ}(G) &= 2\max_{1\leq j\leq n}\left(\sum_{i=1}^{j}\rho_i - \frac{2jW(G)}{n}\right) \geq 2\left(\sum_{i=1}^{n-1}\rho_i - \frac{2(n-1)W(G)}{n}\right) \\ &= 2\left(2W(G) - \rho_n - \frac{2(n-1)W(G)}{n}\right) = \frac{4W(G)}{n} - 2\rho_n \\ &\geq \frac{4W(G)}{n} - \frac{4W(G) - 4Tr_{\min}}{n-1} = \frac{4\big(nTr_{\min} - W(G)\big)}{n(n-1)}. \end{aligned}$$

Now, we have the following claim.

Claim. *The following inequality*

$$E_{DQ}(G) > \frac{4\big(nTr_{\min} - W(G)\big)}{n(n-1)}. \tag{31}$$

holds.

Proof of Claim. By using the method of contradiction, we can prove the strict inequality in (31). Assume that the equality holds in (31). As a result, the above inequalities become equalities. Therefore,

$$\max_{1\leq j\leq n}\left(\sum_{i=1}^{j}\rho_i - \frac{2jW(G)}{n}\right) = \sum_{i=1}^{n-1}\rho_i - \frac{2(n-1)W(G)}{n} \tag{32}$$

and

$$\rho_n = \frac{2W(G)-\rho_1}{n-1} = \frac{2W(G)-2Tr_{\min}}{n-1}. \tag{33}$$

From (32), we conclude $\beta = n-1$. Also, from (33), we obtain $\rho_2 = \cdots = \rho_n$ and from Corollary 1, G is a transmission regular graph. Recall that $\beta = n-1$. We derive that $\rho_n < \frac{2W(G)}{n} \leq \rho_{n-1} = \rho_n$, which is a contradiction. □

Now, by inequality (30) and using the Claim, we get

$$DEL(G) - E_{DQ}(G) < \sqrt{\frac{2W(G)}{n} - 1} + \sqrt{(n-1)\left(\frac{2(n-1)W(G)}{n} + 1\right)} - \frac{4\big(nTr_{\min} - W(G)\big)}{n(n-1)},$$

which leads to the desired conclusion. □

The following result gives another relationship between the two versions of distance signless Laplacian energy.

Theorem 9. *Suppose that G is connected. Then*

$$\left(E_{DQ}(G)+\frac{2W(G)}{n}-2Tr_{\max}\right)^2 \le \left(4W(G)-\frac{2W(G)}{n}-2Tr_{\min}\right)^2 - \frac{8W(G)}{n}\left(DEL(G)-\sqrt{2Tr_{\max}}\right)^2, \tag{34}$$

where the equality is attained if and only if $G \cong K_n$.

Proof. By employing the Cauchy-Schwarz inequality, we know that

$$\begin{aligned} E_{DQ}(G) &= \sum_{i=1}^{n}\left|\rho_i-\frac{2W(G)}{n}\right| = \rho_1-\frac{2W(G)}{n}+\sum_{i=2}^{n}\left|\rho_i-\frac{2W(G)}{n}\right|\rho_1-\frac{2W(G)}{n} \\ &\quad+\sum_{i=2}^{n}\left(\sqrt{\rho_i}+\sqrt{\frac{2W(G)}{n}}\right)\left|\sqrt{\rho_i}-\sqrt{\frac{2W(G)}{n}}\right| \\ &\le 2Tr_{\max}-\frac{2W(G)}{n} \\ &\quad+\sqrt{\left(4W(G)-\frac{2W(G)}{n}-2Tr_{\min}\right)^2-\frac{8W(G)}{n}\left(DEL(G)-\sqrt{2Tr_{\max}}\right)^2}, \end{aligned}$$

which gives the required inequality in (34). The equality occurs if and only if the Cauchy-Schwarz inequality attains the equality and $2Tr_{\min} = \rho_1 = 2Tr_{\max}$. That is, if and only if $2Tr_{\min} = \rho_1 = 2Tr_{\max}$ and $\rho_2 = \cdots = \rho_n$. That is, by Corollary 1, if and only if G is a transmission regular graph with $\rho_2 = \cdots = \rho_n$. Namely, if and only if G is a transmission regular graph having at most two distinct distance signless Laplacian eigenvalues. It then follows from Lemma 8 that equality is again equivalent to $G \cong K_n$.

Moreover, if $G \cong K_n$ then the equality occurs in (34). □

We conclude with the following observation.

Corollary 6. *Suppose that G is connected. Then*

$$\left(E_{DQ}(G)+\frac{2W(G)}{n}-2N\right)^2 \le \left(4W(G)-\frac{2W(G)}{n}-2M\right)^2-\frac{8W(G)}{n}\left(DEL(G)-\sqrt{2N}\right)^2,$$

where $M = 2n-\Delta-2$ and $N = dn-\frac{d(d-1)}{2}-1-\delta(d-1)$.

Proof. Follows from Theorem 9 and inequality (18). □

5 Conclusions

We have defined distance signless Laplaian energy-like invariant $DEL(G)$ and we have obtained several bounds for $DEL(G)$ in terms of some established graph invariants. We have derived relationships between distance signless Laplacian energy $E_{DQ}(G)$ and distance signless Laplacian energy-like invariant $DEL(G)$. These relations can be helpful in the further investigation of $E_{DQ}(G)$. Further, this type of graph energy can find importance in the study of molecular structure, which can be an interesting case of future investigation.

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Альхеваз А., Багіпур М., Пірзада С., Шанг Ю. *Деякі оцінки для лапласіанового енергоподібного інваріанта із беззнаковою відстанню у мережах* // Карпатські матем. публ. — 2025. — Т.17, №1. — С. 255–276.

Нехай G — граф або мережа, $D(G)$ — матриця відстаней, а $Tr(G)$ — діагональна матриця трансмісій вершин. Лапласіанова матриця із беззнаковою відстанню мережі G має вигляд $D^Q(G) = Tr(G) + D(G)$. Ми вводимо лапласіановий енергоподібний інваріант із беззнаковою відстанню як $DEL(G) = \sum_{i=1}^{n} \sqrt{\rho_i}$, де $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ — власні значення лапласіанової матриці із беззнаковою відстанню. У цій статті ми отримуємо нові верхню та нижню межі для $DEL(G)$. Ці межі включають деякі важливі інваріанти, такі як діаметр, мінімальний та максимальний степінь трансмісії, лапласіановий спектральний радіус із беззнаковою відстанню та індекс Вінера. Крім того, ми характеризуємо екстремальні графи, що досягають цих меж. Нарешті, ми встановлюємо деякі співвідношення між різними версіями лапласіанової енергії із беззнаковою відстанню графів.

Ключові слова і фрази: лапласіанова матриця із беззнаковою відстанню, лапласіановий енергоподібний інваріант із беззнаковою відстанню, спектральний радіус, лапласіанова енергія із беззнаковою відстанню, індекс Вінера.