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Nagy type inequalities in metric measure spaces and some applications

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We obtain a sharp Nagy type inequality in a metric space (X, ρ) with measure μ that estimates the uniform norm of a function using its $\|\cdot\|_{H^\omega}$-norm determined by a modulus of continuity ω, and a seminorm that is defined on a space of locally integrable functions. We consider charges ν that are defined on the set of μ-measurable subsets of X and are absolutely continuous with respect to μ. Using the obtained Nagy type inequality, we prove a sharp Landau-Kolmogorov type inequality that estimates the uniform norm of a Radon-Nikodym derivative of a charge via a $\|\cdot\|_{H^\omega}$-norm of this derivative, and a seminorm defined on the space of such charges. We also prove a sharp inequality for a hypersingular integral operator. In the case $X = \mathbb{R}^m_+ \times \mathbb{R}^{d-m}$, $0 \le m \le d$, we obtain inequalities that estimate the uniform norm of a mixed derivative of a function using the uniform norm of the function and the $\|\cdot\|_{H^\omega}$-norm of its mixed derivative.

Key words and phrases: Nagy type inequality, Landau-Kolmogorov type inequality, Stechkin's problem, charge, modulus of continuity, mixed derivative.

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1 Introduction

Inequalities that estimate norms of the intermediate derivatives of univariate or multivariate functions using the norms of the functions and their derivatives of higher order play an important role in many branches of mathematics, including analysis, approximation theory, differential equations, theory of ill-posed problems, numeric methods and many others. It appears that the richest applications are obtained from sharp inequalities of this kind, which attracts much interest to the inequalities with the smallest possible constants. For univariate functions, the results by E. Landau [23], A.N. Kolmogorov [19] and B.Sz. Nagy [29] are among the brightest ones in this topic. A survey on the results for univariate and multivariate functions for the case of derivatives of integer and fractional order, discussions about applications and relations to other extremal problems, and further references can be found in [2,7,8,13]. The article [12] (see also [13, Chapter 2]) contains a periodic analogue of Nagy's inequality, some recent results on the Nagy type inequalities are contained in [18]. Some inequalities of Landau-Kolmogorov type for Radon-Nikodym derivative of charges defined on Lebesgue measurable subsets of an open cone $C \subset \mathbb{R}^d$ that are absolutely continuous with respect to the Lebesgue measure were obtained in [10].

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In this article, we study functional classes that are defined in terms of a majorant for modulus of continuity of the functions. The moduli of continuity as independent functions and classes of functions with given majorants of moduli of continuity of functions or of their derivatives were introduced by S.M. Nikol'skii in [26]. Such classes were studied by many authors (see, e.g., [20, Chapter 7]). Extremal problems for such classes of non-real valued functions were considered in [3–6, 9, 21, 22]. Extremal problems for various hyper-singular integral operators on classes of univariate and multivariate functions defined by a majorant on their modulus of continuity were considered in [11, 14, 15, 27].

Let X, Y, and Z be linear spaces equipped with seminorm $\|\cdot\|_X$, norm $\|\cdot\|_Y$, and seminorm $\|\cdot\|_Z$, respectively. A linear operator $S\colon X\to Y$ is called bounded, if

$$\|S\| = \|S\|_{X\to Y} = \sup_{\|x\|_X\le 1} \|Sx\|_Y < \infty.$$

Otherwise the operator S is called unbounded. By $\mathcal{L}(X,Y)$ we denote the space of all linear bounded operators $S\colon X\to Y$.

Let $A\colon X\to Y, B\colon X\to Z$ be homogeneous operators (not necessarily linear) with the domains $D_A, D_B\subset X$, $D_B\subset D_A$. Let also $\mathfrak{M} = \left\{x\in D_B : \|Bx\|_Z\le 1\right\}$. For the operator A and an operator $S\in\mathcal{L}(X,Y)$ we set

$$U(A,S;\mathfrak{M}) = \sup\left\{\, \|Ax-Sx\|_Y : x\in\mathfrak{M}\right\}.$$

Note that for each $x\in D_B$ one has

$$\|Ax-Sx\|_Y \le U\left(A,S;\mathfrak{M}\right)\|Bx\|_Z\,.$$

The Stechkin problem of approximation of a generally speaking unbounded operator by linear bounded operators on the class $\mathfrak{M}$ is formulated as follows. For a given number N find the quantity

$$E_N\left(A,\mathfrak{M}\right) = \inf\left\{U\left(A,S;\mathfrak{M}\right) : S\in\mathcal{L}(X,Y), \|S\|\le N\right\}. \tag{1}$$

The statement of a somewhat more general problem, first important results, and solutions to this problem for differential operators of small orders were presented in [28]. For a survey of further results on this problem see [2]. The Stechkin problem, in turn, is intimately connected to Landau-Kolmogorov type inequalities. The following well-known theorem (which we formulate in a convenient for us form) describes this connection.

Theorem 1. *For any $x\in D_B$ and arbitrary $S\in\mathcal{L}(X,Y)$ the following Landau-Kolmogorov-Nagy type inequality holds*

$$\|Ax\|_Y \le \|Ax-Sx\|_Y + \|S\|\|x\|_X \le U(A,S;\mathfrak{M})\|Bx\|_Z + \|S\|\cdot\|x\|_X, \tag{2}$$

and, therefore,

$$\forall\, x\in D_A,\ \forall\, N>0,\quad \|Ax\|_Y \le E_N(A,\mathfrak{M})\|Bx\|_Z + N\|x\|_X.$$

If in addition there exist $\overline{S}\in\mathcal{L}(X,Y)$ and $\overline{x}\in\mathfrak{M}$ such that

$$\|A\overline{x}\|_Y = \left\|A\overline{x}-\overline{S}\overline{x}\right\|_Y + \left\|\overline{S}\right\|\|\overline{x}\|_X = U\left(A,\overline{S};\mathfrak{M}\right) + \left\|\overline{S}\right\|\cdot\|\overline{x}\|_X\,,$$

then

$$E_{\left\|\overline{S}\right\|}(A,\mathfrak{M}) = U(A,\overline{S};\mathfrak{M}) = \|A\overline{x}\| - \left\|\overline{S}\right\|\|\overline{x}\|_X,$$

and the operator $\overline{S}$ is optimal for problem (1).

Remark 1. *In S.B. Stechkin's article [28] it is assumed that X and Y are Banach spaces. However, as it is easy to see, completeness and even presence of a norm in X is not necessary. It is sufficient to have a seminorm in X. Completeness of Y is also not necessary.*

In Section 2, we give necessary notations and definitions. In Section 3, we obtain a sharp Nagy type inequality in a metric space (X, ρ) with measure μ that estimates the uniform norm of a function using its $\|\cdot\|_{H^\omega}$-norm determined by a modulus of continuity ω, and a seminorm that is defined on a space of locally integrable functions. In Section 4, we prove a sharp inequality of Nagy type in the context of metric Sobolev spaces. Using the inequality from Section 3, in Section 5 we prove a sharp Landau-Kolmogorov type inequality that estimates the uniform norm of a Radon-Nikodym derivative of a charge from a particular class of charges via a $\|\cdot\|_{H^\omega}$-norm of this derivative, and a seminorm defined on the space of charges. In Section 6, we obtain a sharp Landau-Kolmogorov type inequality for generalized hypersingular operators. Finally, in Section 7, we suppose that $X = \mathbb{R}^m_+ \times \mathbb{R}^{d-m}$, $0 \leq m \leq d$, and obtain inequalities that estimate the uniform norm of a mixed derivative of a function $f\colon X \to \mathbb{R}$ using the uniform norm of the function and the norm of its mixed derivative which is defined with the help of some modulus of continuity.

We use the following scheme to obtain the main results of the article. We define an appropriate bounded operator S, give an estimate for the quantity $U(A, S, \mathfrak{M})$, and plugging it into (2), we obtain a Nagy or Landau-Kolmogorov type inequality. Then we prove its sharpness. Theorem 1 shows that we simultaneously obtain a solution to the corresponding Stechkin problem.

2 Notations and definitions

Let (X, ρ) be a metric space with a Borel measure μ. Assume that X is a commutative monoid (i.e. an associative and commutative binary operation $+$ is defined on X, and there exists an element $\theta \in X$ such that $x + \theta = \theta + x = x$ for all $x \in X$) such that for each measurable set $Q \subset X$ and each $x \in X$ one has

$$\mu(x + Q) = \mu(Q).$$

Suppose that for all $x, y \in X$,

$$\rho(x + y, x) \leq \rho(y, \theta).$$

Everywhere below $B_h = B_h(\theta)$ is an open ball of radius $h > 0$ with center θ. We suppose that $0 < \mu(B_h) < \infty$ and $B_h \neq \{\theta\}$ for all $h > 0$.

An invariant Haar measure on a locally compact group with metrical topology (see, e.g., [25]) is an important example.

For a measurable set $Q \subset X$ by $L_1(Q)$ $\big($respectively $L_\infty(Q)\big)$ we denote the space of functions $f\colon Q \to \mathbb{R}$ integrable (respectively essentially bounded) on Q with the corresponding norm. By $L_{\rm loc}(X)$ we denote the space of all functions $f\colon X \to \mathbb{R}$ that are integrable on each open ball of X. In the space $L_{\rm loc}(X)$ we introduce a family of seminorms

$$\rfloor f \lceil_h = \sup_{x \in X} \left| \int_{x + B_h} f(u) d\mu(u) \right|, \quad h > 0,$$

and a seminorm

$$\rfloor f \lceil = \sup_{h > 0} \rfloor f \lceil_h.$$

By $L_{\rfloor\cdot\lceil_h}(X)$ $\big($respectively $L_{\rfloor\cdot\lceil}(X)\big)$ we denote the family of functions $f \in L_{\rm loc}(X)$ with a finite seminorm $\rfloor \cdot \lceil_h$ $\big($respectively $\rfloor \cdot \lceil\big)$. It is clear that the space $L_1(X)$ is contained in each of these sets.

By $C(X)$ we denote the space of all continuous functions $f\colon X \to \mathbb{R}$, by $C_b(X)$ the space of functions $f \in C(X)$ with a finite norm

$$\|f\|_{C(X)} = \sup_{x\in X} |f(x)|,$$

by $\mathcal{B}(X)$ we denote the space of bounded functions $f\colon X \to \mathbb{R}$ with a norm

$$\|f\|_{\mathcal{B}(X)} = \sup_{x\in X} |f(x)|.$$

Everywhere below we assume that the measure μ is such that $C(X) \subset L_{\rm loc}(X)$.

Let ω be a modulus of continuity, i.e. a non-negative, non-decreasing, semi-additive function $\omega\colon [0,\infty) \to [0,\infty)$ such that $\omega(0) = 0$. By $H^\omega(X)$ we denote the space of functions $f\colon X \to \mathbb{R}$ such that

$$\|f\|_{H^\omega(X)} := \sup_{x,y\in X, x\neq y} \frac{|f(x) - f(y)|}{\omega\big(\rho(x,y)\big)} < \infty.$$

3 A Nagy type inequality

For each $h > 0$ we define an operator $S_h\colon L_{\rfloor\cdot\lceil_h}(X) \to \mathcal{B}(X)$ by the following rule

$$S_h f(x) = \frac{1}{\mu(B_h)} \int_{B_h} f(x+u)\, d\mu(u).$$

It is clear that this operator is bounded and

$$\|S_h\|_{L_{\rfloor\cdot\lceil_h}(X)\to\mathcal{B}(X)} = \frac{1}{\mu(B_h)}. \tag{3}$$

We need the following result, which is sometimes called an Ostrowski type inequality. A rather general version of such kind of results is contained in [6, Theorem 2].

Lemma 1. *If $f \in H^\omega(X)$, then for each $h > 0$ one has*

$$\big\|f(\cdot) - S_h f(\cdot)\big\|_{\mathcal{B}(X)} \le \frac{\|f\|_{H^\omega(X)}}{\mu(B_h)} \int_{B_h} \omega\big(\rho(u,\theta)\big)\, d\mu(u). \tag{4}$$

Inequality (4) *is sharp and becomes equality for all functions*

$$f_\omega(x) = c \pm \omega\big(\rho(u,\theta)\big), \quad c \in \mathbb{R}.$$

Proof. For each $x \in X$ we have

$$\begin{aligned}\big|f(x) - S_h f(x)\big| &= \left| f(x) - \frac{1}{\mu(B_h)} \int_{B_h} f(x+u)\, d\mu(u)\right| \le \frac{1}{\mu(B_h)} \int_{B_h} \big|f(x) - f(x+u)\big|\, d\mu(u)\\ &\le \frac{\|f\|_{H^\omega(X)}}{\mu(B_h)} \int_{B_h} \omega\big(\rho(x+u,x)\big)\, d\mu(u) \le \frac{\|f\|_{H^\omega(X)}}{\mu(B_h)} \int_{B_h} \omega\big(\rho(u,\theta)\big)\, d\mu(u)\end{aligned}$$

and inequality (4) is proved. For the function f_ω one has

$$\|f_\omega\|_{H^\omega(X)} = 1. \tag{5}$$

Indeed, for all $x, y \in X$, we obtain

$$|f_\omega(x) - f_\omega(y)| = |\omega(\rho(x,\theta)) - \omega(\rho(y,\theta))| \leq \omega(|\rho(x,\theta) - \rho(y,\theta)|) \leq \omega(\rho(x,y)),$$

hence $\|f_\omega\|_{H^\omega(X)} \leq 1$. Since there exists $y \in B_h \setminus \{\theta\}$, we obtain

$$\sup_{x\in X,\, x\neq\theta} |f_\omega(x) - f_\omega(\theta)| \geq |f_\omega(y) - f_\omega(\theta)| = \omega(\rho(y,\theta)),$$

which implies $\|f_\omega\|_{H^\omega(X)} \geq 1$, and hence (5) holds. Moreover,

$$\|f(\cdot) - S_h f(\cdot)\|_{\mathcal{B}(X)} \geq |f(\theta) - S_h f(\theta)| = \frac{1}{\mu(B_h)} \int_{B_h} \omega(\rho(u,\theta))\, d\mu(u),$$

which together with (5) implies that inequality (4) is sharp. □

The following theorem contains a variant of the Nagy type inequality. For $\alpha \in \mathbb{R}$ we set $\alpha_+ := \max\{\alpha, 0\}$.

Theorem 2. *If $h > 0$ and $f \in H^\omega(X) \cap L_{\rfloor\cdot\lceil_h}(X)$, then*

$$\begin{aligned}\|f\|_{\mathcal{B}(X)} &\leq \|f - S_h f\|_{\mathcal{B}(X)} + \|S_h\|_{L_{\rfloor\cdot\lceil_h}(X)\to\mathcal{B}(X)} \|f\|_{L_{\rfloor\cdot\lceil_h}(X)} \\ &\leq \frac{\|f\|_{H^\omega(X)}}{\mu(B_h)} \int_{B_h} \omega(\rho(u,\theta))\, d\mu(u) + \frac{\rfloor f\lceil_h}{\mu(B_h)}.\end{aligned} \tag{6}$$

The inequality is sharp and turns into equality for the function

$$f_{e,h}(x) = \big(\omega(h) - \omega(\rho(x,\theta))\big)_+. \tag{7}$$

Moreover, $f_{e,h} \in H^\omega(X) \cap L_{\rfloor\cdot\lceil}(X)$, $\rfloor f_{e,h}\lceil = \rfloor f_{e,h}\lceil_h$, and hence for each $h > 0$ the inequality

$$\|f\|_{\mathcal{B}(X)} \leq \frac{\|f\|_{H^\omega(X)}}{\mu(B_h)} \int_{B_h} \omega(\rho(u,\theta))\, d\mu(u) + \frac{\rfloor f\lceil}{\mu(B_h)} \tag{8}$$

holds and it is sharp on the class $H^\omega(X) \cap L_{\rfloor\cdot\lceil}(X)$.

Remark 2. *In the case, when $\omega(t) = t^\alpha$, $0 < \alpha \leq 1$, and the space (X, ρ, μ) satisfies the following s-regularity type condition*

$$\exists\, b > 0,\ \exists\, s > 0,\ \forall\, h > 0,\ \mu(B_h) \geq bh^s,$$

inequality (8) can be written in a multiplicative form. We do not adduce the details.

Proof. For each $x \in X$, due to Lemma 1, the definition of the operator S_h, and the equality (3), we have

$$\begin{aligned}|f(x)| \leq |f(x) - S_h f(x)| + |S_h f(x)| &\leq \|f(x) - S_h f(x)\|_{B(X)} + \|S_h\|_{L_{\rfloor\cdot\lceil_h}(X)\to B(X)} \rfloor f\lceil_h \\ &\leq \frac{\|f\|_{H^\omega(X)}}{\mu(B_h)} \int_{B_h} \omega(\rho(u,\theta))\, d\mu(u) + \frac{\rfloor f\lceil_h}{\mu(B_h)},\end{aligned}$$

which implies inequality (6). Inequality (8) is a consequence of inequality (6).

For the function $f_{e,h}$ we have $\|f_{e,h}\|_{\mathcal{B}(X)} = f_{e,h}(\theta) = \omega(h)$, $\|f_{e,h}\|_{H^\omega(X)} = 1$, and

$$\rfloor f_{e,h} \lceil_h = \omega(h)\mu(B_h) - \int_{B_h} \omega\big(\rho(u,\theta)\big)\, d\mu(u).$$

Indeed, on the one hand,

$$\rfloor f_{e,h} \lceil_h \geq \int_{\theta+B_h} f_{e,h}(u)\, d\mu(u) = \omega(h)\mu(B_h) - \int_{B_h} \omega\big(\rho(u,\theta)\big)\, d\mu(u),$$

and on the other hand, for each $x \in X$, due to monotonicity of ω,

$$\begin{aligned}\int_{x+B_h} f_{e,h}(u)\, d\mu(u) &= \int_{(x+B_h)\cap B_h} f_{e,h}(u)\, d\mu(u)\\ &\leq \int_{B_h} f_{e,h}(u)\, d\mu(u) = \omega(h)\mu(B_h) - \int_{B_h} \omega\big(\rho(u,\theta)\big)\, d\mu(u).\end{aligned}$$

Direct computations now show that inequality (6) becomes equality on the function $f_{e,h}$.

Finally, the same arguments as during computation of the quantity $\rfloor f_{e,h} \lceil_h$ show that $\rfloor f_{e,h} \lceil = \rfloor f_{e,h} \lceil_h$, and hence inequality (8) is also sharp. □

Corollary 1. *If $h > 0$ and $f \in H^\omega(X) \cap L_1(X)$, then*

$$\|f(x)\|_{\mathcal{B}(X)} \leq \frac{\|f\|_{H^\omega(X)}}{\mu(B_h)} \int_{B_h} \omega\big(\rho(u,\theta)\big)\, d\mu(u) + \frac{1}{\mu(B_h)} \|f\|_{L_1(X)}.$$

The inequality is sharp. It becomes equality on the function $f_{e,h}$ defined by (7).

Proof. Since $L_1(X) \subset L_{\rfloor\cdot\lceil_h}(X)$ and $\|f_{e,h}\|_{L_1(X)} = \rfloor f_{e,h} \lceil_h$, the statement of the corollary follows from Theorem 2. □

4 Nagy type inequalities in metric Sobolev spaces

In this section, we consider metric Sobolev spaces with essentially bounded upper gradients, which are defined as follows. Let (X,ρ_X) and (Y,ρ_Y) be metric spaces such as in the previous sections. For a modulus of continuity ω we define the space $W^{1,\omega}(X,Y)$ as the space of all functions $f\colon X \to Y$ with the following property (cf. [1, Chapter 5] and [17, Chapter 10.2]). There exists a non-negative function $G = G_f \in L_\infty(X)$ and a set $N = N_f \subset X$ such that $\mu(N) = 0$ and

$$\rho_Y\big(f(x), f(y)\big) \leq \big(G(x) + G(y)\big) \cdot \omega\big(\rho_X(x,y)\big) \quad \text{for all} \quad x,y \in X \setminus N. \tag{9}$$

We call G an upper gradient of f.

Let (Y,ρ_Y) be the space of reals with the usual metric. The technique developed in the proof of the previous theorem allows to prove the following result, which thus can be in some sense considered as a corollary of Theorem 2.

Theorem 3. *Let $f \in W^{1,\omega}(X,\mathbb{R}) \cap L_{\rfloor\cdot\lceil_h}(X)$ and G_f be an upper gradient of f. Then for any $h > 0$ the following inequality holds*

$$\|f\|_{L_\infty(X)} \leq \frac{2\|G_f\|_{L_\infty(X)}}{\mu(B_h)} \int_{B_h} \omega\big(\rho_X(\theta,u)\big)\, d\mu(u) + \frac{\rfloor f \lceil_h}{\mu(B_h)}. \tag{10}$$

Inequality (10) *is sharp in the sense that there exists a function f and its upper gradient G_f for which the inequality becomes equality.*

Proof. For almost all $x \in X$, we have

$$\begin{aligned}
|f(x)| &\le \left| f(x) - \frac{1}{\mu(B_h)} \int_{x+B_h} f(y)\, d\mu(y) \right| + \frac{1}{\mu(B_h)} \left| \int_{x+B_h} f(y)\, d\mu(y) \right| \\
&\le \frac{1}{\mu(B_h)} \int_{x+B_h} |f(x) - f(y)|\, d\mu(y) + \frac{\rfloor f \lceil_h}{\mu(B_h)} \\
&\le \frac{1}{\mu(B_h)} \int_{B_h} |G_f(\theta) - G_f(u)| \omega\big(\rho_X(\theta, u)\big)\, d\mu(u) + \frac{\rfloor f \lceil_h}{\mu(B_h)} \\
&\le \frac{2\|G_f\|_{L_\infty(X)}}{\mu(B_h)} \int_{B_h} \omega\big(\rho_X(\theta, u)\big)\, d\mu(u) + \frac{\rfloor f \lceil_h}{\mu(B_h)},
\end{aligned}$$

and inequality (10) is proved. The inequality becomes equality for the function $f = f_{e,h}$ defined in (7) and its upper gradient $G \equiv \frac{1}{2}$. Indeed, the fact that inequality (10) becomes equality for such functions f and G can be verified directly. The fact that G is indeed an upper gradient for f (with $N = \varnothing$) can be proved as follows. Let $x, y \in X$, $u = \rho_X(x, \theta)$ and $v = \rho_X(y, \theta)$. We can assume that $u \le v$. If $h \le u \le v$, then $f(x) = f(y) = 0$ and inequality (9) holds. If $0 \le u < h \le v$, then

$$|f(x) - f(y)| = \omega(h) - \omega(u) \le \omega(h-u) \le \omega(v-u) = \omega\big(\rho_X(y,\theta) - \rho_X(x,\theta)\big) \le \omega\big(\rho_X(x,y)\big).$$

The case $0 \le u \le v \le h$ can be considered similarly. $\square$

Similarly to Corollary 1 one can prove the following result.

Corollary 2. *Let $f \in W^{1,\omega}(X, \mathbb{R}) \cap L_1(X)$ and G_f be an upper gradient of f. Then for any $h > 0$ the following inequality holds*

$$\|f\|_{L_\infty(X)} \le \frac{2\|G_f\|_{L_\infty(X)}}{\mu(B_h)} \int_{B_h} \omega\big(\rho_X(\theta, u)\big)\, d\mu(u) + \frac{\|f\|_{L_1(X)}}{\mu(B_h)}.$$

The inequality is sharp in the sense that there exists a function f and its upper gradient G_f for which the inequality becomes equality.

5 Landau-Kolmogorov type inequalities for charges

By $\mathfrak{N}(X)$ we denote the family of charges ν defined on the family of all μ-measurable subsets of X and that are absolutely continuous with respect to the measure μ (see, e.g., [16, Chapter 5]). By the Radon-Nikodym theorem, for a charge $\nu \in \mathfrak{N}(X)$ there exists an integrable function $f\colon X \to \mathbb{R}$ such that for an arbitrary measurable set $Q \subset X$ we have

$$\nu(Q) = \int_Q f(x)\, d\mu(x). \tag{11}$$

This function f is called the Radon-Nikodym derivative of the charge ν with respect to the measure μ and will be denoted by $D_\mu \nu$. The family $\mathfrak{N}(X)$ is a linear space with respect to the standard addition and multiplication by a real number. Define a family of seminorms $\{\rceil \cdot \lfloor_h : h > 0\}$ by

$$\rceil \nu \lfloor_h = \big\| \nu(\cdot + B_h) \big\|_{B(X)}.$$

It is clear that if a charge ν and a function f are related via (11), then

$$\rceil \nu \lfloor_h = \rfloor f \lceil_h.$$

For $h > 0$ by $\mathfrak{N}_{\rceil \cdot \lfloor_h}(X)$ we denote the set of charges $\nu \in \mathfrak{N}(X)$ with a finite seminorm $\rceil \cdot \lfloor_h$.

Theorem 4. *If $h > 0$ and $\nu \in \mathfrak{N}_{\rceil\cdot\lfloor_h}(X)$ is such that $D_\mu \nu \in H^\omega(X)$, then*

$$\begin{aligned}\|D_\mu \nu\|_{\mathcal{B}(X)} &\le \|D_\mu \nu - \overline{S}_h \nu\|_{\mathcal{B}(X)} + \|\overline{S}_h\|_{\mathfrak{N}_{\rceil\cdot\lfloor_h}(X)\to\mathcal{B}(X)} \rceil \nu \lfloor_h \\ &\le \frac{\|D_\mu \nu\|_{H^\omega(X)}}{\mu(B_h)} \int_{B_h} \omega\big(\rho(u,\theta)\big)\, d\mu(u) + \frac{\rceil \nu \lfloor_h}{\mu(B_h)},\end{aligned}$$

where the operator $\overline{S}_h \colon \mathfrak{N}_{\rceil\cdot\lfloor_h}(X) \to \mathcal{B}(X)$ is defined by

$$\overline{S}_h \nu(x) = \frac{\nu(x + B_h)}{\mu(B_h)}.$$

The inequality is sharp, and becomes an equality for the charge $\nu_{e,h}$ such that $D_\mu \nu_{e,h} = f_{e,h}$, where the function $f_{e,h}$ is defined by (7).

Remark 3. *This theorem generalizes the result of [10, Theorem 3].*

Proof. It is enough to apply Theorem 2 to the function $f = D_\mu \nu$ and notice that after a change of variables in the integral, we obtain $\overline{S}_h \nu = S_h f$. ☐

Corollary 3. *If $h > 0$, $\nu \in \mathfrak{N}_{\rceil\cdot\lfloor_h}(X)$ is such that $D_\mu \nu \in W^{1,\omega}(X, \mathbb{R})$, and G_ν is an arbitrary upper gradient of $D_\mu \nu$, then the following inequality holds*

$$\|D_\mu \nu\|_{L_\infty(X)} \le \frac{2\|G_\nu\|_{L_\infty(X)}}{\mu(B_h)} \int_{B_h} \omega\big(\rho_X(\theta, u)\big)\, d\mu(u) + \frac{\rfloor \nu \lceil_h}{\mu(B_h)}.$$

The inequality is sharp in the sense that there exists a charge ν and an upper gradient G_ν for which the inequality becomes equality.

Proof. The theorem follows from Theorem 3. ☐

6 Inequalities for generalized hypersingular integrals

Let $P \colon \mathbb{R}_+ \to \mathbb{R}_+$ be a locally integrable function such that

$$\int_{X \setminus B_h} P\big(\rho(u,\theta)\big)\, d\mu(u) < \infty \quad \text{for some } \ h > 0.$$

Define the following operator

$$\mathfrak{D}_P f(x) = \int_X \big(f(x) - f(x+u)\big) P\big(\rho(u,\theta)\big)\, d\mu(u),$$

which can be considered as a hypersingular integral. We also consider the following truncated hypersingular integral

$$\overline{\mathfrak{D}}_{P,h} f(x) = \int_{X \setminus B_h} \big(f(x) - f(x+u)\big) P\big(\rho(u,\theta)\big)\, d\mu(u).$$

It is easy to see that

$$\|\overline{\mathfrak{D}}_{P,h}\|_{C_b(X)\to C_b(X)} = 2 \int_{X \setminus B_h} P\big(\rho(u,\theta)\big)\, d\mu(u).$$

Theorem 5. *Let $f \in H^\omega(X) \cap C_b(X)$. If for some $h > 0$*

$$\int_{B_h} \omega\big(\rho(u,\theta)\big) P\big(\rho(u,\theta)\big)\, d\mu(u) < \infty \quad \text{and} \quad \int_{X\setminus B_h} P\big(\rho(u,\theta)\big)\, d\mu(u) < \infty,$$

then

$$\begin{aligned} \|\mathfrak{D}_P f\|_{\mathcal{B}(X)} &\le \|\mathfrak{D}_P f - \overline{\mathfrak{D}}_{P,h} f\|_{\mathcal{B}(X)} + \|\overline{\mathfrak{D}}_{P,h}\|_{C_b(X)\to C_b(X)} \|f\|_{C_b(X)} \\ &\le \|f\|_{H^\omega(X)} \int_{B_h} \omega\big(\rho(u,\theta)\big) P\big(\rho(u,\theta)\big) d\mu(u) + 2\|f\|_{C_b(X)} \int_{X\setminus B_h} P\big(\rho(u,\theta)\big)\, d\mu(u). \end{aligned} \tag{12}$$

The inequality is sharp and turns into equality for the function

$$f_{e,\omega}(u) = \begin{cases} \omega\big(\rho(u,\theta)\big) - \frac{1}{2}\omega(h), & \rho(u,\theta) \le h, \\ \frac{1}{2}\omega(h), & \rho(u,\theta) \ge h. \end{cases}$$

Proof. We have

$$\begin{aligned} \|\mathfrak{D}_P f - \overline{\mathfrak{D}}_{P,h} f\|_{\mathcal{B}(X)} &= \sup_{x\in X} \left| \int_{B_h} \big(f(x) - f(x+u)\big) P\big(\rho(u,\theta)\big)\, d\mu(u) \right| \\ &\le \sup_{x\in X} \|f\|_{H^\omega(X)} \int_{B_h} \omega\big(\rho(x, x+u)\big) P\big(\rho(u,\theta)\big)\, d\mu(u) \\ &\le \|f\|_{H^\omega(X)} \int_{B_h} \omega\big(\rho(u,\theta)\big) P\big(\rho(u,\theta)\big)\, d\mu(u). \end{aligned}$$

Thus

$$\begin{aligned} \big\|\mathfrak{D}_P f(x)\big\|_{\mathcal{B}(X)} &\le \|\mathfrak{D}_P f - \overline{\mathfrak{D}}_{P,h} f\|_{\mathcal{B}(X)} + \|\overline{\mathfrak{D}}_{P,h}\|_{C_b(X)\to C_b(X)} \|f\|_{C_b(X)} \\ &\le \|f\|_{H^\omega(X)} \int_{B_h} \omega\big(\rho(u,\theta)\big) P\big(\rho(u,\theta)\big)\, d\mu(u) + 2\|f\|_{C_b(X)} \int_{X\setminus B_h} P\big(\rho(u,\theta)\big)\, d\mu(u), \end{aligned}$$

and the inequality (12) is proved. For the function $f_{e,\omega}$, it is clear that $\|f_{e,\omega}\|_{C_b(X)} = \frac{1}{2}\omega(h)$, $\|f_{e,\omega}\|_{H^\omega(X)} = 1$, and

$$\|\mathfrak{D}_P f_{e,\omega}\|_{\mathfrak{B}(X)} = -\mathfrak{D}_P f_{e,\omega}(\theta) = \int_{B_h} \omega\big(\rho(u,\theta)\big) P\big(\rho(u,\theta)\big)\, d\mu(u) + \omega(h) \int_{X\setminus B_h} P\big(\rho(u,\theta)\big)\, d\mu(u),$$

thus the inequality becomes equality for the function $f_{e,\omega}$. □

7 Inequalities for mixed derivatives

Assume that $X = \mathbb{R}^d_{m,+} := \mathbb{R}^m_+ \times \mathbb{R}^{d-m}, 0 \le m \le d$, μ is the Lebesgue measure in $\mathbb{R}^d_{m,+}$, $\rho(x,y) = \max\limits_{i=1,\dots,d} |x_i - y_i|$, so that $B_h = (0,h)^m \times (-h,h)^{d-m}$. In this section, for brevity we write dx instead of $d\mu(x)$. For a locally integrable function $f\colon X \to \mathbb{R}$ set $\mathbf{I} = (1,\dots,1) \in \mathbb{R}^d$ and

$$\partial_{\mathbf{I}} f = \frac{\partial^d f}{\partial x_1 \dots \partial x_d},$$

where the derivatives are understood in the distributional sense.

Let $\{e_i\}$ be the standard basis in $\mathbb{R}^d$. For $i = 1,\dots,d$ and $h > 0$ we set

$$\Delta^+_{i,h} f(x) := f(x + he_i) - f(x) \quad \text{and} \quad \Delta_{i,h} f(x) := f(x + he_i) - f(x - he_i).$$

In virtue of the Fubini theorem, for almost all $x \in \mathbb{R}^d_{m,+}$ one has

$$\int_{x+B_h} \partial_{\mathbf{I}} f(u)\, du = (\Delta^+_{1,h} \circ \ldots \circ \Delta^+_{m,h} \circ \Delta_{m+1,h} \circ \ldots \circ \Delta_{d,h}) f(x). \tag{13}$$

Define an operator $\mathfrak{S}_h \colon L_\infty(C) \to L_\infty(C)$, setting

$$\mathfrak{S}_h f(x) = \frac{1}{2^{d-m}h^d} \left(\Delta^+_{1,h} \circ \ldots \circ \Delta^+_{m,h} \circ \Delta_{m+1,h} \circ \ldots \circ \Delta_{d,h}\right) f(x), \quad h > 0.$$

Theorem 6. *If $h > 0$ and $f \in \mathcal{B}(\mathbb{R}^d_{m,+})$ is such that $\partial_{\mathbf{I}} f \in H^\omega(\mathbb{R}^d_{m,+})$, then*

$$\begin{aligned} \|\partial_{\mathbf{I}} f\|_{\mathcal{B}(\mathbb{R}^d_{m,+})} &\le \|\partial_{\mathbf{I}} f - \mathfrak{S}_h f\|_{\mathcal{B}(\mathbb{R}^d_{m,+})} + \|\mathfrak{S}_h\| \|f\|_{\mathcal{B}(\mathbb{R}^d_{m,+})} \\ &\le \frac{\|\partial_{\mathbf{I}} f\|_{H^\omega(\mathbb{R}^d_{m,+})}}{2^{d-m}h^d} \int_{B_h} \omega\big(\rho(u,\theta)\big)\, du + \frac{2^m}{h^d} \|f\|_{\mathcal{B}(\mathbb{R}^d_{m,+})}. \end{aligned} \tag{14}$$

In the case, when $\omega(t) = t^\alpha$, $\alpha \in (0,1]$, the following multiplicative inequality

$$\|\partial_{\mathbf{I}} f\|_{\mathcal{B}(\mathbb{R}^d_{m,+})} \le 2^{\frac{m\alpha}{d+\alpha}} \left(\frac{d+\alpha}{\alpha}\right)^{\frac{\alpha}{d+\alpha}} \cdot \|f\|^{\frac{\alpha}{d+\alpha}}_{\mathcal{B}(\mathbb{R}^d_{m,+})} \cdot \|\partial_{\mathbf{I}} f\|^{\frac{d}{d+\alpha}}_{H^\omega(\mathbb{R}^d_{m,+})} \tag{15}$$

holds. For $m = 0$ and $m = 1$ these inequalities are sharp.

Remark 4. *This theorem generalizes the result [10, Theorem 5].*

Proof. Applying inequality (6) to $\partial_{\mathbf{I}} f$ and taking into account that $\mu(B_h) = 2^{d-m}h^d$, we obtain

$$\begin{aligned} \|\partial_{\mathbf{I}} f\|_{\mathcal{B}(\mathbb{R}^d_{m,+})} &\le \|\partial_{\mathbf{I}} f - \mathfrak{S}_h f\|_{\mathcal{B}(\mathbb{R}^d_{m,+})} + \|\mathfrak{S}_h\| \|f\|_{\mathcal{B}(\mathbb{R}^d_{m,+})} \\ &\le \frac{\|\partial_{\mathbf{I}} f\|_{H^\omega(\mathbb{R}^d_{m,+})}}{2^{d-m}h^d} \int_{B_h} \omega\big(\rho(u,\theta)\big)\, du + \frac{1}{2^{d-m}h^d} \sup_{x \in \mathbb{R}^d_{m,+}} \left| \int_{x+B_h} \partial_{\mathbf{I}} f(u) du \right|. \end{aligned}$$

Representation (13) implies

$$\frac{1}{2^{d-m}h^d} \sup_{x \in \mathbb{R}^d_{m,+}} \left| \int_{x+B_h} \partial_{\mathbf{I}} f(u)\, du \right| \le \frac{2^d}{2^{d-m}h^d} \|f\|_{\mathcal{B}(\mathbb{R}^d_{m,+})} = \frac{2^m}{h^d} \|f\|_{\mathcal{B}(\mathbb{R}^d_{m,+})},$$

which implies inequality (14).

In the case $\omega(t) = t^\alpha$, using the layer cake representation (see, e.g., [24, Theorem 1.13]), symmetry considerations, and writing $|x|_\infty$ instead of $\rho(x,\theta)$, we obtain

$$\begin{aligned} \int_{B_h} \omega\big(\rho(u,\theta)\big)\, du &= 2^{d-m} \int_{(0,h)^d} |u|^\alpha_\infty\, du = 2^{d-m} \int_0^\infty \mu\left\{v \in (0,h)^d \colon |u|^\alpha_\infty > t\right\} dt \\ &= 2^{d-m} \int_0^{h^\alpha} \left(h^d - t^{\frac{d}{\alpha}}\right) dt = 2^{d-m} h^{d+\alpha} \left(1 - \frac{\alpha}{d+\alpha}\right) = \frac{d \cdot 2^{d-m}}{d+\alpha} h^{d+\alpha}. \end{aligned}$$

So that the right-hand side of (14) becomes

$$\frac{d}{d+\alpha} \|\partial_{\mathbf{I}} f\|_{H^\omega(\mathbb{R}^d_{m,+})} h^\alpha + \frac{2^m}{h^d} \|f\|_{\mathcal{B}(\mathbb{R}^d_{m,+})}.$$

Minimizing this expression with respect to $h > 0$, i.e. choosing

$$h = 2^{\frac{m}{d+\alpha}} \left(\frac{d+\alpha}{\alpha} \cdot \frac{\|f\|_{\mathcal{B}(\mathbb{R}^d_{m,+})}}{\|\partial_{\mathbf{I}} f\|_{H^\omega(\mathbb{R}^d_{m,+})}} \right)^{\frac{1}{d+\alpha}},$$

we obtain the right-hand side of (15).

Next we prove sharpness of inequality (14) for the case $m = 0$. Consider the function

$$g_{e,h}(x) = \int_0^{x_1} \dots \int_0^{x_d} f_{e,h}(u)\, du,$$

where $f_{e,h}$ is defined in (7). Then $\partial_{\mathbf{I}} g_{e,h} = f_{e,h}$, and hence

$$\|\partial_{\mathbf{I}} g_{e,h}\|_{\mathcal{B}(\mathbb{R}^d)} = \omega(h), \quad \|\partial_{\mathbf{I}} g_{e,h}\|_{H^\omega(\mathbb{R}^d)} = 1.$$

Moreover,

$$\|g_{e,h}\|_{\mathcal{B}(\mathbb{R}^d)} = \int_0^h \dots \int_0^h \Big(\omega(h) - \omega\big(|u|_\infty\big)\Big)\, du = h^d \omega(h) - \int_0^h \dots \int_0^h \omega\big(|u|_\infty\big)\, du$$

and due to symmetricity of $[-h, h]^d$, we get

$$\int_{B_h} \omega\left(|u|_\infty\right)\, du = 2^d \int_0^h \dots \int_0^h \omega\left(|u|_\infty\right)\, du.$$

Direct computations now show that inequalities (14) and (15) become equality for the function $g_{e,h}$.

Finally, we prove sharpness of inequality (14) in the case $m = 1$. In this case, we have $B_h = (0, h) \times (-h, h)^{d-1}$. There exists $0 < a < h$ such that

$$\begin{aligned}\int_{\{x \in B_h\colon\, x_1 < a\}} \Big(\omega(h) - \omega\big(|x|_\infty\big)\Big)\, dx &= \int_{\{x \in B_h\colon\, x_1 > a\}} \Big(\omega(h) - \omega\big(|x|_\infty\big)\Big)\, dx \\ &= \frac{1}{2} \int_{B_h} \Big(\omega(h) - \omega\big(|x|_\infty\big)\Big)\, dx.\end{aligned}$$

The set B_h consists of 2^{d-1} equal cubes with edge lengths equal to h; $\theta = \theta_d$ is one of the vertices for each of these cubes. The hyperplane $x_1 = a$ divides these cubes into pieces $c_1^-, \dots, c_{2^{d-1}}^-$ that have θ among vertices, and pieces $c_1^+, \dots, c_{2^{d-1}}^+$ that have (a, θ_{d-1}) among their vertices.

It is clear that

$$\int_{\{x \in B_h\colon\, x_1 < a\}} \Big(\omega(h) - \omega\big(|x|_\infty\big)\Big)\, dx = \sum_{i=1}^{2^{d-1}} \int_{c_i^-} \Big(\omega(h) - \omega\big(|x|_\infty\big)\Big)\, dx$$

and

$$\int_{\{x \in B_h\colon\, x_1 > a\}} \Big(\omega(h) - \omega\big(|x|_\infty\big)\Big)\, dx = \sum_{i=1}^{2^{d-1}} \int_{c_i^+} \Big(\omega(h) - \omega\big(|x|_\infty\big)\Big)\, dx.$$

From the symmetry considerations it follows that each of 2^{d-1} summands in each of the right-hand sides of these equalities are equal. Thus for each $i, j = 1, \dots, 2^{d-1}$

$$\int_{c_i^-} \Big(\omega(h) - \omega\big(|x|_\infty\big)\Big)\, dx = \int_{c_j^+} \Big(\omega(h) - \omega\big(|x|_\infty\big)\Big)\, dx = \frac{1}{2^d} \int_{B_h} \Big(\omega(h) - \omega\big(|x|_\infty\big)\Big)\, dx.$$

An extremal function in this case we define as follows

$$G_{e,h}(x) = \int_a^{x_1} \int_0^{x_2} \dots \int_0^{x_d} \partial_{\mathbf{I}} f_{e,h}(u)\, du_d \dots du_1,$$

where $f_{e,h}$ is defined in (7). For this function we have

$$\|G_{e,h}\|_{\mathcal{B}(\mathbb{R}^d_{1,+})} = \frac{1}{2^d} \int_{B_h} \Big(\omega(h) - \omega\big(|x|_\infty\big) \Big)\, dx = \frac{h^d}{2} \omega(h) - \frac{1}{2^d} \int_{B_h} \omega\big(|x|_\infty\big)\, dx,$$

$$\|\partial_{\mathbf{I}} G_{e,h}\|_{\mathcal{B}(\mathbb{R}^d_{1,+})} = \omega(h), \quad \|\partial_{\mathbf{I}} G_{e,h}\|_{H^\omega(\mathbb{R}^d_{1,+})} = 1.$$

Direct computations now show that inequalities (14) and (15) with $m = 1$ become equalities on the function $G_{e,h}$. □

Remark 5. *It is not clear, whether inequality* (14) *is sharp for* $m = 2, \dots, d$*, even in the case* $d = 2$ *and* $\omega(t) = t$.

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Бабенко В.Ф., Бабенко В.В., Коваленко О.В., Парфінович Н.В. *Нерівності типу Надя у метричних просторах з мірою і деякі застосування* // Карпатські матем. публ. — 2023. — Т.15, №2. — C. 563–575.

Ми доводимо точну нерівність типу Надя у метричному просторі (X,ρ) з мірою μ, яка оцінює рівномірну норму функції за допомогою її $\|\cdot\|_{H^\omega}$-норми, що визначена модулем неперервності ω, і напівнормою, яка визначена у просторі локально інтегровних функцій. Для зарядів ν, визначених на множині μ-вимірних підмножин простору X, і які є абсолютно неперервними по відношенню до міри μ, використовуючи отриману нерівність типу Надя, ми доводимо точну нерівність типу Ландау-Колмогорова, яка оцінює рівномірну норму похідної Радона-Нікодима заряду за допомогою $\|\cdot\|_{H^\omega}$-норми цієї похідної і напівнорми, що визначені на множині таких зарядів. Ми також доводимо точну нерівність для гіперсингулярних інтегральних операторів. У випадку $X=\mathbb{R}^m_+\times\mathbb{R}^{d-m}$, $0\le m\le d$, ми отримали нерівність, що оцінює рівномірну норму мішаної похідної функції за допомогою рівномірної норми функції і $\|\cdot\|_{H^\omega}$-норми її мішаної похідної.

Ключові слова і фрази: нерівність типу Надя, нерівність типу Ландау-Колмогорова, задача Стєчкіна, заряд, модуль неперервності, мішана похідна.