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Conjectures about wheels in the join product with cycles

Staš M., Timková M.

By the crossing number $\mathrm{cr}(G)$ of a simple graph G we understand the minimum number of edge crossings over all possible drawings of G in the plane. The crossing numbers of the join product of two graphs have been studied for their wide application in practice. The main purpose of the paper is to establish $\mathrm{cr}\,(W_5 + C_n)$ for the wheel W_5 on six vertices, where C_n are the cycles on n vertices. W. Yue et. al. in [Comp. Eng. Appl. 2014, **50** (18), 79–84] conjectured that the crossing number of $W_m + C_n$ is equal to $Z(m+1)Z(n) + \big(Z(m)-1\big)\left\lfloor \frac{n}{2} \right\rfloor + n + \left\lceil \frac{m}{2} \right\rceil + 2$ for all integers $m, n \geq 3$. Here, the Zarankiewicz's number $Z(n) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ is defined for all $n \geq 1$. The mentioned conjecture was verified for $W_3 + C_n$ by M. Klešč, and for $W_4 + C_n$ by M. Staš and J. Valiska. We establish the validity of this conjecture for $W_5 + C_n$.

Key words and phrases: graph, join product, crossing number, wheel, cycle.

Faculty of Electrical Engineering and Informatics, Technical University of Košice, 042 00 Košice, Slovakia
E-mail: michal.stas@tuke.sk (Staš M.), maria.timkova@tuke.sk (Timková M.)

1 Introduction

Several experts have been dealing with the problem of reducing the number of crossings on the edges of simple graphs for a long time. Among the most popular areas in which the minimum number of crossings plays an important role is the implementation of VLSI layout. It caused significant progress in general, especially in the design of circuits and therefore has a relatively strong influence on multiple parallel calculations. Integer linear programming can also be used to formulate some exact algorithms to find provably optimal crossing numbers. Implementations of QuickCross heuristics also allow finding optimal or near-optimal embeddings of many graphs. In contrast, by M.R. Garey and D.S. Johnson [4] it is well-known that examining number of crossings of simple graphs is an NP-complete problem. Despite this knowledge, many researchers try to solve this problem at least on some class of graphs. Such a summarization of the known values of crossing numbers for some graph classes is published thanks to K. Clancy et. al. [3].

Let $G = \big(V(G), E(G)\big)$ be some simple graph. By a drawing of G we understand a representation of G in the plane in which its vertices $v_i \in V(G)$ are represented by different points and its edges $e_j \in E(G)$ by simple continuous arcs connecting the corresponding point pairs. The smallest number of edge crossings over all possible drawings of G in the plane is said to be the *crossing number* $\mathrm{cr}(G)$, see also M. Klešč [7]. A drawing is said to be *good* if no edge crosses itself, two edges can cross at most once, two edges incident with the same vertex do not cross each other, and no more than two edges cross at the same point. It can be very easily verified that an *optimal* drawing (a drawing with minimum number of crossings) is always the

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good drawing. That is why we will deal only with good drawings throughout the paper. Let D be a good drawing of the graph G. We denote the number of crossings in D by $\mathrm{cr}_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote the number of crossings between edges of G_i and edges of G_j by $\mathrm{cr}_D(G_i, G_j)$, and the number of crossings among edges of G_i in D by $\mathrm{cr}_D(G_i)$. Again, it can be very easily verified that both of the following equations are true

$$\mathrm{cr}_D(G_i \cup G_j) = \mathrm{cr}_D(G_i) + \mathrm{cr}_D(G_j) + \mathrm{cr}_D(G_i, G_j) \ , \tag{1}$$
$$\mathrm{cr}_D(G_i \cup G_j, G_k) = \mathrm{cr}_D(G_i, G_k) + \mathrm{cr}_D(G_j, G_k)$$

for any three mutually edge-disjoint subgraphs G_i, G_j, and G_k of G. Throughout the paper, we will also use the already known knowledge for the complete bipartite graphs $K_{m,n}$ thanks to D.J. Kleitman [6], i.e.

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \qquad \text{if} \quad \min\{m,n\} \le 6. \tag{2}$$

The join product of two graphs G_i and G_j consists from vertex-disjoint copies of G_i and G_j, and also by adding all edges between their vertex sets $V(G_i)$ and $V(G_j)$. The edge set of $G_i + G_j$ is formed by the union of the disjoint edge sets of both graphs G_i and G_j, and also the complete bipartite graph $K_{m,n}$ if the orders of graphs G_i and G_j are m and n, respectively. Let P_n and C_n be the *path* and the *cycle* on n vertices, respectively. The crossings numbers of $G + P_n$ and $G + C_n$ for all graphs G of order at most four are well-known by M. Klešč [8,9]. It is understandable that our immediate aim is to establish exact values for crossing numbers of $G + P_n$ and $G + C_n$ also for all graphs G of order five and six. Of course, some exact values are already determined [2,7,8,10–12,14,16,17,19,20]. In all these cases, the graph G is connected and usually contains at least one cycle. Note that $\mathrm{cr}(G + C_n)$ are known only for some disconnected graphs G on five or six vertices [13,15,18,21].

2 Preliminary results and cyclic permutations

The wheel W_5 consists of one dominating vertex of degree five and five vertices of degree three forming the subgraph isomorphic to the cycle C_5 (for brevity, we write $C_5^\star$). It is only necessary to suppose all possibilities of crossings between subdrawings of $C_5^\star$ and the five edges incident with the dominating vertex forming the subgraph isomorphic to the star S_5 on six vertices (also for brevity, we will write $S_5^\star$). In the rest of the paper, let $V(W_5) = \{v_1, \dots, v_6\}$, and let v_6 be the vertex notation of the dominating vertex in all considered good subdrawings of W_5.

We consider the join product of W_5 with the discrete graph D_n on n vertices. The graph $W_5 + D_n$ consists of one copy of the wheel W_5 and n vertices $t_1, t_2, \dots, t_n$. Any such vertex t_i is adjacent with any vertex of W_5. By the six edges incident with the vertex t_i is given a subgraph T^i. As $T^1 \cup \dots \cup T^n$ is isomorphic to $K_{6,n}$, we receive

$$W_5 + D_n = W_5 \cup \left(\bigcup_{i=1}^{n} T^i \right).$$

The above equality together with the crossing property (1) produce

$$\mathrm{cr}_D(W_5 + D_n) = \mathrm{cr}_D(W_5) + \mathrm{cr}_D\left(\bigcup_{i=1}^{n} T^i \right) + \mathrm{cr}_D\left(W_5, \bigcup_{i=1}^{n} T^i \right) \tag{3}$$

for all good drawings D of $W_5 + D_n$.

The graph $W_5 + P_n$ contains $W_5 + D_n$ as a subgraph, and therefore let P_n^* denote the path induced on n vertices of $W_5 + P_n$ not belonging to the subgraph W_5. The path P_n^* contains exactly n vertices $t_1, t_2, \ldots, t_n$ and $n-1$ edges t_it_{i+1} for $i = 1, 2, \ldots, n-1$, and thus

$$W_5 + P_n = W_5 \cup \left(\bigcup_{i=1}^{n} T^i \right) \cup P_n^*.$$

Similarly, the graph $W_5 + C_n$ contains both $W_5 + D_n$ and $W_5 + P_n$ as subgraphs. The subgraph of $W_5 + C_n$ induced on the vertices $t_1, t_2, \ldots, t_n$ is denoted by C_n^*. Therefore,

$$W_5 + C_n = W_5 \cup \left(\bigcup_{i=1}^{n} T^i \right) \cup C_n^*.$$

Let D be a good drawing of the join product $W_5 + D_n$. In the drawing D, a cyclic permutation recording the (cyclic) counterclockwise order in which the edges leave t_i is said to be the *rotation* $\mathrm{rot}_D(t_i)$ of a vertex t_i, see also C. Hernández-Vélez et. al. [5] or D.R. Woodall [25]. For example, if the counter-clockwise order the edges incident with the vertex t_i is t_iv_1, t_iv_2, t_iv_3, t_iv_4, t_iv_5 and t_iv_6, then we use the notation (123456). Besides, let $\overline{\mathrm{rot}}_D(t_i)$ denote the inverse permutation of $\mathrm{rot}_D(t_i)$. We separate all subgraphs T^i of $W_5 + D_n$ into five mutually-disjoint families of subgraphs according to the number of times the edges of W_5 are crossed by the subgraph T^i in D. Let $R_D = \{T^i : \mathrm{cr}_D(W_5, T^i) = 0\}$, $S_D = \{T^i : \mathrm{cr}_D(W_5, T^i) = 1\}$, $T_D = \{T^i : \mathrm{cr}_D(W_5, T^i) = 2\}$, and $U_D = \{T^i : \mathrm{cr}_D(W_5, T^i) = 3\}$. Every other subgraph T^i crosses edges of W_5 at least four times in D. Clearly, this idea of dividing all subgraphs T^i into five mentioned families of subgraphs will be also retained in all drawings of the join products $W_5 + P_n$ and $W_5 + C_n$.

The crossing numbers of $W_5 + D_n$ equal to $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor$ were established by Š. Berežný and M. Staš [1]. The result for $W_5 + D_n$ was independently proved by Y. Wang and Y.Q. Huang [22] around the same time.

Theorem 1 ([1, Theorem 3.1])**.** $\mathrm{cr}(W_5 + D_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor$ *for* $n \geq 1$.

Using the results of Š. Berežný and M. Staš [2], the crossing numbers of the graphs $W_5 + P_n$ have already been well-known for any $n \geq 2$.

Theorem 2 ([2, Theorem 3.3])**.** $\mathrm{cr}(W_5 + P_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor + 1$ *for* $n \geq 2$.

To date, the crossing number of $W_5 + C_n$ can only be given as a conjecture. This open problem will be solved in the next section.

3 The crossing number of $W_5 + C_n$

Lemma 1 ([11, Lemma 1])**.** *For $m \geq 1$, let G be a graph of order m. In an optimal drawing of the join product $G + C_n$, $n \geq 3$, the edges of C_n^* do not cross each other.*

As the graph $W_n + C_m$ is isomorphic to the graph $W_m + C_n$ for all integers $m, n \geq 3$, the proof of the much-needed Corollary 1 throughout this paper can thus be omitted.

Corollary 1. *In any optimal drawing of the join product $W_5 + C_n$, $n \geq 3$, the edges of $C_5^\star$ do not cross each other.*

We can always redraw a crossing of two edges of $C_5^\star$ in an effort to get a new drawing of $C_5^\star$ (with vertices in a different order) with less number of edge crossings. This idea has already been presented in the proof for $W_5 + D_n$ by Š. Berežný and M. Staš [1]. Based on the arguments above, we will assume that edges of $C_5^\star$ do not cross each other in all considered subdrawings $D(W_5)$ induced by a good drawing D of $W_5 + C_n$.

In Figure 1 below, the edges of $K_{6,n}$ cross each other $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times, each subgraph T^i, $i = 1, \dots, \lceil \frac{n}{2} \rceil$, on the right side crosses edges of the wheel W_5 once and each subgraph T^i, $i = \lceil \frac{n}{2} \rceil + 1, \dots, n$, on the left side crosses edges of W_5 exactly four times. The cycle C_n^* crosses W_5 five times, and so $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor + 5$ crossings appear among edges of the graph $W_5 + C_n$ in this drawing. Thus $\mathrm{cr}(W_5 + C_n) \leq 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor + 5$.

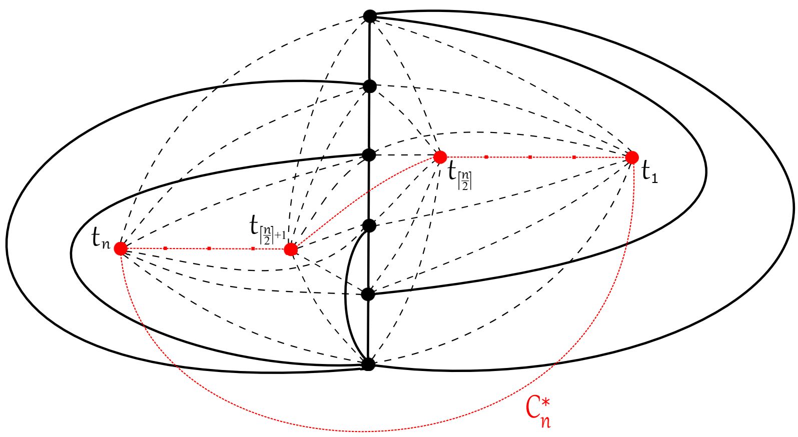

Figure 1. The good drawing of $W_5 + C_n$ with $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor + 5$ crossings

Our goal is to show that the crossing number of $W_5 + C_n$ is equal to this upper bound. Lemma 2 below confirms this result for $n = 3$ and $n = 4$, and its proof is strongly based on the isomorphisms mentioned above.

Lemma 2. $\mathrm{cr}(W_5 + C_3) = 17$ *and* $\mathrm{cr}(W_5 + C_4) = 27$.

Proof. M. Klešč in [9] showed that $\mathrm{cr}(W_3 + C_n) = 2\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 4$, and so we obtain the equalities $\mathrm{cr}(W_5 + C_3) = \mathrm{cr}(W_3 + C_5) = 17$. Similarly, by M. Staš and J. Valiska [20], the result $\mathrm{cr}(W_4 + C_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 4$ offers $\mathrm{cr}(W_5 + C_4) = \mathrm{cr}(W_4 + C_5) = 27$. □

Lemma 3. *For* $n \geq 3$*, let* D *be a good drawing of* $W_5 + C_n$*. If edges of* $C_5^\star$ *are crossed at least* $\lceil \frac{n}{2} \rceil + 2$ *times, then there are at least* $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor + 5$ *crossings in* D.

Proof. The wheel W_5 consists of two edge-disjoint subgraphs $C_5^\star$ and $S_5^\star$. By the assumption, let us consider that $\mathrm{cr}_D(C_5^\star) = 0$ and $\mathrm{cr}_D(C_5^\star, S_5^\star + C_n) \geq \lceil \frac{n}{2} \rceil + 2$ are fulfilling in the good drawing D of $W_5 + C_n$. The star $S_5^\star$ is isomorphic to the complete bipartite graph $K_{1,5}$ and the exact value for the crossing number of $K_{1,5} + C_n$ is given by M. Klešč et. al. [11], i.e. $\mathrm{cr}(K_{1,5} + C_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor + 3$. Since edges of $S_5^\star + C_n$ must be crossed at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor + 3$ times in D, we obtain

$$\mathrm{cr}_D(W_5 + C_n) = \mathrm{cr}_D(S_5^\star + C_n) + \mathrm{cr}_D(C_5^\star, S_5^\star + C_n)$$
$$\geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 4\left\lfloor \frac{n}{2} \right\rfloor + 3 + \left\lceil \frac{n}{2} \right\rceil + 2 = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + 3\left\lfloor \frac{n}{2} \right\rfloor + 5.$$

□

Let us first deal with the so far unknown subdrawing of the wheel W_5 in which edges of $C_5^\star$ do not cross each other and there is a possibility to obtain a subgraph T^i whose edges do not cross edges of W_5, that is, all six vertices of W_5 must be located on the boundary of one region $D(W_5)$. In the rest of the paper, let the dominating vertex v_6 of W_5 be placed in the inner region of $D(C_5^\star)$ and let also a subgraph $T^i \in R_D$ be represented by rotation $\mathrm{rot}_D(t_i) = (123456)$. This enforces that three edges v_2v_6, v_3v_6, and v_4v_6 have to cross the same edge v_1v_5 in $D(W_5)$. Taking into account the assumption that it does not matter which of the regions in $D(W_5)$ is unbounded in our considerations, we obtain one possible subdrawing of W_5 induced by D given in Figure 2.

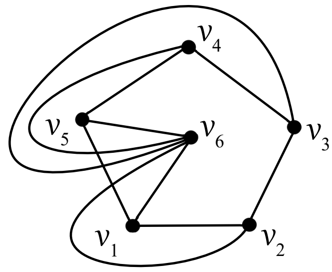

Figure 2. One possible drawing of the wheel W_5 in which edges of $C_5^\star$ do not cross each other and all six vertices of W_5 are located on boundary of one region

Lemma 4. *For $n \geq 3$, let D be a good drawing of $W_5 + C_n$ with the subdrawing of W_5 induced by D and one region with all six vertices of W_5 located on its boundary. If $|R_D| = 0$, then there are at least $6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + 3\left\lfloor\frac{n}{2}\right\rfloor + 5$ crossings in D.*

Proof. Let D be a good drawing of $W_5 + C_n$ with at most $6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + 3\left\lfloor\frac{n}{2}\right\rfloor + 4$ crossings in which the nonplanar subdrawing of W_5 can be obtained from the unique drawing (with respect to isomorphisms) in Figure 2. Each subgraph T^i crosses edges of the wheel W_5 at least twice, because we assume that the set R_D is empty and there is no possibility to add a new subgraph T^i into this drawing with just one additional crossing on edges of W_5. Note that each subgraph $T^i \in T_D$ crosses edges of the cycle $C_5^\star$ at least once. If $r = |R_D|$, $s = |S_D|$, and $t = |T_D|$, then $\mathrm{cr}_D(K_{6,n}) \geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$ thanks to (2) implies the following relation with respect to edge crossings of the wheel W_5 in D using (3):

$$\mathrm{cr}_D(W_5) + \mathrm{cr}_D\left(W_5, \bigcup_{i=1}^{n} T^i\right) \leq n + 3\left\lfloor\frac{n}{2}\right\rfloor + 4,$$

that is,

$$\mathrm{cr}_D(W_5) + 0r + 1s + 2t + 3(n - r - s - t) \leq n + 3\left\lfloor\frac{n}{2}\right\rfloor + 4.$$

The obtained above inequality enforces $3r + 2s + t \geq \left\lceil\frac{n}{2}\right\rceil + \mathrm{cr}_D(W_5) - 4$.

Further, if $\mathrm{cr}_D(W_5) = 3$ and $r = s = 0$, then $t \geq \left\lceil\frac{n}{2}\right\rceil - 1$. We remind that three other crossings on edges of $C_5^\star$ are included in $D(W_5)$. Since edges of $C_5^\star$ are crossed at least $\left\lceil\frac{n}{2}\right\rceil + 2$ times in D, Lemma 3 contradicts the assumption of D. □

Corollary 2. *For $n \geq 5$, let D be a good drawing of $W_5 + C_n$ with the subdrawing of W_5 induced by D and one region with all six vertices of W_5 located on its boundary. If $|R_D| \geq 1$ with at least one crossing on edges of the cycle C_n^*, then there are at least $6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + 3\left\lfloor\frac{n}{2}\right\rfloor + 5$ crossings in D.*

Proof. For a contradiction, suppose that n is a minimum integer at least five for which all assumptions of Corollary 2 are fulfilled, but the considered drawing D of $W_5 + C_n$ offers at most $6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + 3\left\lfloor\frac{n}{2}\right\rfloor + 4$ crossings. Let T^i be a subgraph from the nonempty set R_D and let also there be at least one crossing on edges of C_n^*.

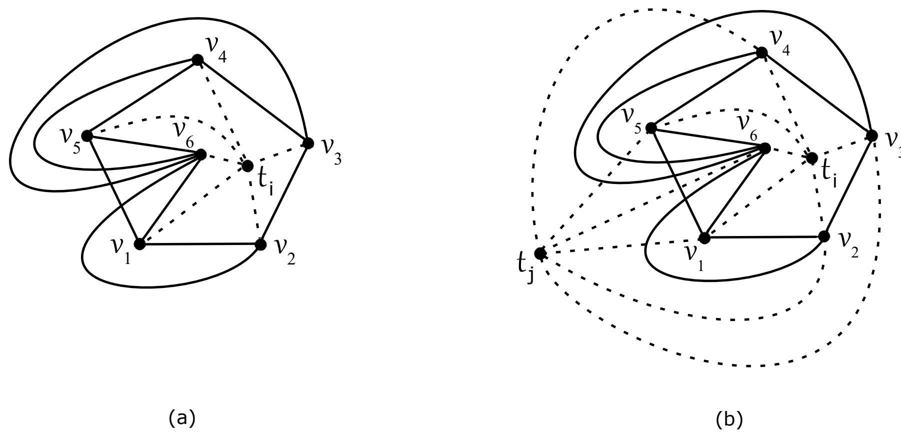

Figure 3. (a) the subdrawing of $W_5 \cup T^i$ for $T^i \in R_D$,
(b) the subdrawing of $W_5 \cup T^i \cup T^j$ for $T^i \in R_D$ and $\mathrm{cr}_D(W_5 \cup T^i, T^j) = 5$

The subgraph $W_5 \cup T^i$ given in Figure 3 (a) is represented by $\mathrm{rot}_D(t_i) = (123456)$ and it is easy to verify over all possible regions of $D(W_5 \cup T^i)$ that each of the $n-1$ remaining subgraphs T^j crosses its edges at least five times. If edges of $W_5 \cup T^i$ are crossed at least six times by each other T^j, by fixing the subgraph $W_5 \cup T^i$, we have

$$\begin{aligned}\mathrm{cr}_D(W_5 + C_n) &= \mathrm{cr}_D(K_{6,n-1}) + \mathrm{cr}_D(K_{6,n-1}, W_5 \cup T^i) \\ &\qquad + \mathrm{cr}_D(W_5 \cup T^i) + \mathrm{cr}_D(W_5 \cup K_{6,n}, C_n^*) + \mathrm{cr}_D(C_n^*) \\ &\geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 6(n-1) + 3 + 1 \geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + 3\left\lfloor\frac{n}{2}\right\rfloor + 5.\end{aligned}$$

Now, assume that there is some subgraph T^j such that $\mathrm{cr}_D(W_5 \cup T^i, T^j) < 6$ which causes that the vertex t_j must be placed in the outer region of the subdrawing $D(W_5)$ with two vertices v_2 and v_3 of W_5 on its boundary, and $\mathrm{cr}_D(W_5 \cup T^i, T^j) = 5$ enforces $\mathrm{cr}_D(T^i, T^j) = 0$, see Figure 3 (b). Thus, by fixing the subgraph $T^i \cup T^j$, we have

$$\begin{aligned}\mathrm{cr}_D(W_5 + C_n) &\geq \mathrm{cr}_D(W_5 + C_{n-2}) + \mathrm{cr}_D(T^i \cup T^j) + \mathrm{cr}_D(K_{6,n-2}, T^i \cup T^j) + \mathrm{cr}_D(W_5, T^i \cup T^j) \\ &\geq 6\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor + n - 2 + 3\left\lfloor\frac{n-2}{2}\right\rfloor + 5 + 0 + 6(n-2) + 5 \\ &= 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + 3\left\lfloor\frac{n}{2}\right\rfloor + 5,\end{aligned}$$

where edges of $T^i \cup T^j$ are crossed by each other subgraph T^k at least six times, using $\mathrm{cr}_D(K_{6,3}) \geq 6$ thanks to (2). In the last estimate, the lower bound of $\mathrm{cr}_D(W_5 + C_{n-2})$ is known by Lemma 4, if $|R_D| = 1$, and the result of Lemma 2 is strongly used in the case of $n = 5$ and $n = 6$.

Both subcases confirm a contradiction in D, and the proof of Corollary 2 is done. □

Corollary 2 enforces that the set R_D must be empty for any good drawing D of $W_5 + C_n$ with at most $6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + 3\left\lfloor\frac{n}{2}\right\rfloor + 4$ crossings and at least one crossing on edges of C_n^*. Thus, we obtain $\mathrm{cr}_D\left(W_5, \bigcup_{i=1}^n T^i\right) \geq n$ in D.

Corollary 3. *For $n \geq 5$, let D be a good drawing of $W_5 + C_n$ with the planar subdrawing of W_5 induced by D. If edges of C_n^* are crossed at least twice satisfying the restriction $\mathrm{cr}_D\left(W_5, C_n^*\right) = 0$, then there are at least $6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + 3\left\lfloor\frac{n}{2}\right\rfloor + 5$ crossings in D.*

Proof. Let D be a good drawing of $W_5 + C_n$ in which the planar subdrawing of W_5 is not crossed by any edge of the cycle C_n^*. Let us also suppose that edges of C_n^* do not cross each other using Lemma 1, but they are crossed at least twice by $\bigcup_{i=1}^n T^i$ in D. The edges of $C_5^\star$ also do not cross each other, and so the subdrawing of $C_5^\star$ induced by D divides the plane into two regions. The dominant vertex v_6 of the wheel W_5 can be placed in one of them. It is sufficient to discuss two possible positions of the vertex v_6 with five incident edges of W_5 with respect to the cycles $C_5^\star$ and C_n^*, see Figure 4. Clearly, it does not matter which of the regions in $D\left(W_5 \cup C_n^*\right)$ is unbounded in our considerations.

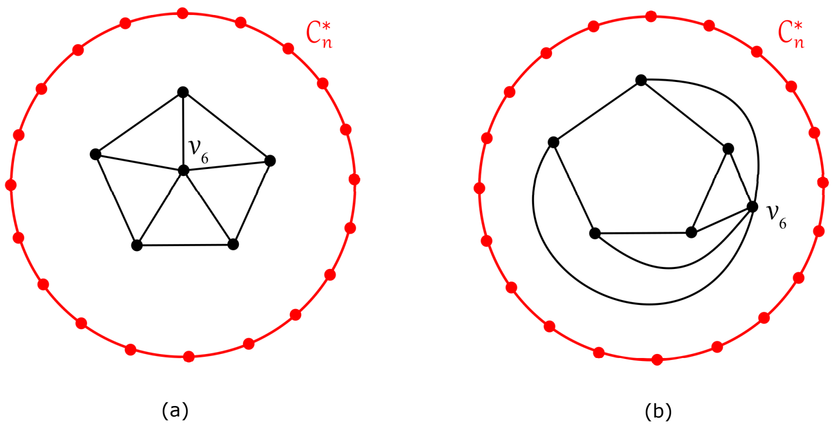

Figure 4. Two possible positions of W_5 and C_n^* in which $\mathrm{cr}_D(W_5) = 0$ and $\mathrm{cr}_D(W_5, C_n^*) = 0$

Assume the position of W_5 and C_n^* given in Figure 4 (a). For each $i = 1, \ldots, n$, the edge $t_i v_6$ crosses some edge of the cycle $C_5^\star$, and therefore we obtain at least n crossings on edges of $C_5^\star$. As $n \geq \left\lceil\frac{n}{2}\right\rceil + 2$, Lemma 3 enforces at least $6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + 3\left\lfloor\frac{n}{2}\right\rfloor + 5$ crossings in D.

Assume the position of W_5 and C_n^* given in Figure 4 (b). Any subgraph T^i crosses edges of W_5 at least three times because all vertices t_i of the cycle C_n^* are placed in the outer triangular region of $D(W_5)$. Taking into account the assumption that edges of C_n^* are crossed at least twice in D, we obtain

$$\begin{aligned}\mathrm{cr}_D\left(W_5 + C_n\right) &= \mathrm{cr}_D\left(K_{6,n}\right) + \mathrm{cr}_D\left(K_{6,n}, W_5\right) + \mathrm{cr}_D\left(K_{6,n}, C_n^*\right) \\ &\geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 3n + 2 \geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + 3\left\lfloor\frac{n}{2}\right\rfloor + 5.\end{aligned}$$

The obtained results in both subcases complete the proof of Corollary 3. □

Corollary 4. *For $n \geq 5$, let D be a good drawing of $W_5 + C_n$ with the planar subdrawing of W_5 induced by D. If $\mathrm{cr}_D\left(C_5^\star, C_n^*\right) \geq 2$, then there are at least $6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + 3\left\lfloor\frac{n}{2}\right\rfloor + 5$ crossings in D.*

Proof. For the planar subdrawing of W_5, it is obvious that the set R_D is empty because no region is incident with all vertices in $D(W_5)$. If some vertex t_i is placed in the pentagonal region of $D(W_5)$, then the edge t_iv_6 crosses one edge of $C_5^\star$ and the edge t_iv_j crosses either no or at least two edges of W_5 for $j = 1, \dots, 5$. Thus, the set T_D must also be empty. Since each subgraph $T^i \in S_D \cup U_D$ crosses some edge of $C_5^\star$ at least once, Lemma 3 using $\mathrm{cr}_D(C_5^\star, C_n^*) \geq 2$ confirms the desired result in the case of $|S_D \cup U_D| \geq \lceil \frac{n}{2} \rceil$.

In the following, let $|S_D \cup U_D| < \lceil \frac{n}{2} \rceil$. The edges of W_5 are crossed by $\bigcup_{i=1}^n T^i$ at least $n + 3\lfloor \frac{n}{2} \rfloor + 3$ times provided by

$$\mathrm{cr}_D\left(W_5, \bigcup_{i=1}^{n} T^i\right) \geq 1\,|S_D| + 3\,|U_D| + 4\big(n - |S_D| - |U_D|\,\big)$$
$$= 4n - 2\,|S_D| - |S_D| - |U_D| \geq 4n + 2\left(1 - \left\lceil \frac{n}{2} \right\rceil\right) + 1 - \left\lceil \frac{n}{2} \right\rceil = n + 3\left\lfloor \frac{n}{2} \right\rfloor + 3.$$

Finally, $\mathrm{cr}_D(C_5^\star, C_n^*) \geq 2$ together with $\mathrm{cr}_D(K_{6,n}) \geq 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ thanks to (2) imply at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor + 5$ crossings in D. □

Corollary 5. *For $n \geq 5$, let D be a good drawing of $W_5 + C_n$ with the nonplanar subdrawing of W_5 induced by D and one region with five vertices of W_5 located on its boundary. If edges of C_n^* are crossed at least twice satisfying the restriction $\mathrm{cr}_D(W_5, C_n^*) = 0$, then there are at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor + 5$ crossings in D.*

Proof. Let D be a good drawing of $W_5 + C_n$ with at most $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor + 4$ crossings in which the nonplanar subdrawing of W_5 is not crossed by any edge of C_n^*. Let us also suppose that edges of C_n^* do not cross each other again by Lemma 1, but they are crossed at least twice by $\bigcup_{i=1}^n T^i$ in D. Since edges of $S_5^\star$ do not cross each other, some edge of $C_5^\star$ must be crossed by v_iv_6 for $i \in \{1, \dots, 5\}$. Note that any region with five vertices of W_5 located on its boundary must be incident with the dominant vertex v_6 in $D(W_5)$. In the rest of the proof, let five vertices v_1, v_2, v_3, v_5, and v_6 of W_5 be located on the boundary of one region $D(W_5)$. Then the edge v_4v_6 do not cross any edge of $C_5^\star$, and the edges v_1v_6 and v_2v_6 cross either one or two different edges of $C_5^\star$ in D. Two obtained subdrawings of W_5 are given in Figure 5.

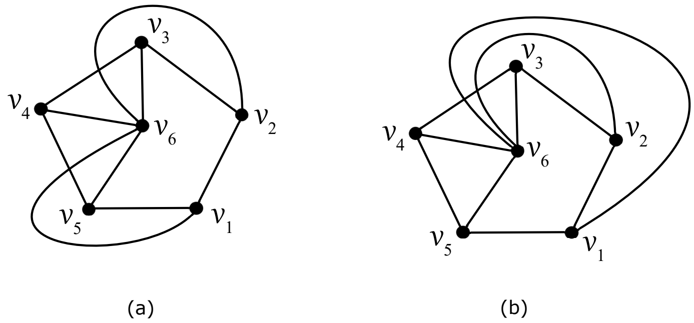

Figure 5. Two possible nonplanar drawings of W_5 with one region with five vertices of W_5 located on its boundary

At first, we consider the subdrawing of W_5 induced by D with the vertex notation in such a way as shown in Figure 5 (a). If all vertices t_i of C_n^* are not placed in the region of $D(W_5)$ with five vertices v_1, v_2, v_3, v_5, and v_6 of W_5 on its boundary, the same idea as in the second part of the proof of Corollary 3 can be used. In the following, we assume that they are located in the given pentagonal region of $D(W_5)$. For the considered subdrawing of W_5, the set R_D is empty but the set S_D can be nonempty. So, two possible cases may occur, as follows.

Case 1. Let S_D be the nonempty set, that is, only the edge v_1v_2 of W_5 is crossed by the edge t_iv_4 of any subgraph $T^i \in S_D$. The subgraph $W_5 \cup T^i$ given in Figure 6 is represented by $\mathrm{rot}_D(t_i) = (142365)$.

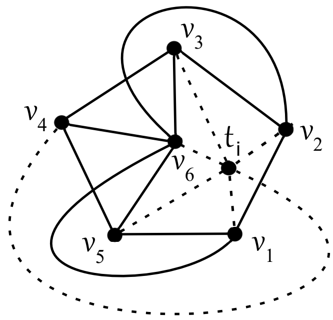

Figure 6. The subdrawing of $W_5 \cup T^i$ for $T^i \in S_D$

It is not difficult to verify over six possible regions of $D(W_5 \cup T^i)$ that each of the $n-1$ remaining subgraphs T^k crosses edges of $W_5 \cup T^i$ at least six times. By fixing the subgraph $W_5 \cup T^i$, we obtain

$$\begin{aligned}\mathrm{cr}_D(W_5 + C_n) &= \mathrm{cr}_D(K_{6,n-1}) + \mathrm{cr}_D(K_{6,n-1}, W_5 \cup T^i) + \mathrm{cr}_D(W_5 \cup T^i) + \mathrm{cr}_D(K_{6,n}, C_n^*) \\ &\geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 6(n-1) + 3 + 2 \geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + 3\left\lfloor\frac{n}{2}\right\rfloor + 5.\end{aligned}$$

Case 2. Let S_D be the empty set, that is, each subgraph T^i crosses edges of W_5 at least twice. We discuss two subcases. Let us first consider that there is a subgraph $T^i \in T_D$ whose edges do not cross edges of the cycle $C_5^\star$. The edges of $C_5^\star$ are not crossed by any $T^i \in T_D$ only if either $\mathrm{rot}_D(t_i) = (123465)$ or $\mathrm{rot}_D(t_i) = (123645)$, see Figure 7.

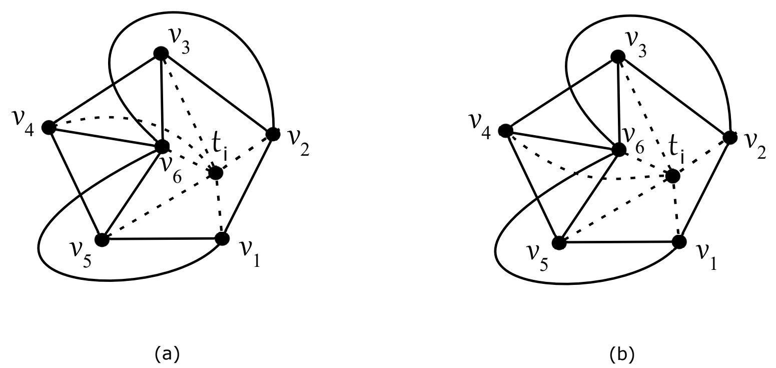

Figure 7. Two subdrawings of $W_5 \cup T^i$ for $T^i \in T_D$ with $\mathrm{cr}_D(C_5^\star, T^i) = 0$
(a) $\mathrm{rot}_D(t_i) = (123465)$ (b) $\mathrm{rot}_D(t_i) = (123645)$

In both cases, $\mathrm{cr}_D(W_5 \cup T^i, T^k) \geq 6$ is fulfilling for each other subgraph T^k, $k \neq i$, and therefore the same fixation as in the previous case can be applied.

In the following, suppose that there is no subgraph $T^i \in T_D$ satisfying $\mathrm{cr}_D\left(C_5^{\star}, T^i\right) = 0$. Now, let D' be the subdrawing of $W_5 + D_n$ induced by D without the edges of C_n^* with at most $6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + 3\left\lfloor\frac{n}{2}\right\rfloor + 2$ crossings. For easier reading, let $r = |R_D|$, $s = |S_D|$, and $t = |T_D|$. Of course, we also get the same cardinalities in the induced subdrawing D'. Then $\mathrm{cr}_{D'}(K_{6,n}) \geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$ thanks to (2) implies the following relation with respect to edge crossings of the wheel W_5 in D' using (3):

$$\mathrm{cr}_{D'}(W_5) + \mathrm{cr}_{D'}\left(W_5, \bigcup_{i=1}^{n} T^i\right) \leq n + 3\left\lfloor\frac{n}{2}\right\rfloor + 2,$$

that is,

$$\mathrm{cr}_{D'}(W_5) + 0r + 1s + 2t + 3(n - r - s - t) \leq n + 3\left\lfloor\frac{n}{2}\right\rfloor + 2.$$

The above inequality enforces $3r + 2s + t \geq \left\lceil\frac{n}{2}\right\rceil + \mathrm{cr}_{D'}(W_5) - 2$. Further, if $\mathrm{cr}_{D'}(W_5) = 2$ and $r = s = 0$, then $t \geq \left\lceil\frac{n}{2}\right\rceil$. We remind that two other crossings on edges of $C_5^{\star}$ are included in $D(W_5)$. Since edges of $C_5^{\star}$ are crossed at least $\left\lceil\frac{n}{2}\right\rceil + 2$ times in D, Lemma 3 contradicts the assumption of D.

Finally, we assume the subdrawing of the wheel W_5 with the vertex notation in such a way as shown in Figure 5 (b). The set R_D is empty, but the set S_D can be nonempty, and so the proof proceeds in a similar way as for the subdrawing of W_5 in Figure 5 (a). Only if $S_D \neq \varnothing$, then there are two possibilities of obtaining a subgraph $T^i \in S_D$ with either $\mathrm{rot}_D(t_i) = (123645)$ or $\mathrm{rot}_D(t_i) = (123654)$. □

By Corollaries 3, 4 and 5, the set S_D must also be empty for any good drawing D of $W_5 + C_n$ with at most $6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + 3\left\lfloor\frac{n}{2}\right\rfloor + 4$ crossings if $\mathrm{cr}_D(C_5^{\star}, C_n^*) \geq 2$ or edges C_n^* are crossed at least twice satisfying the restriction $\mathrm{cr}_D(W_5, C_n^*) = 0$. Thus, we obtain $\mathrm{cr}_D\left(W_5, \bigcup_{i=1}^{n} T^i\right) \geq 2n$ in D. In the following, we will also deal with another form of representation of subgraphs in drawings of $W_5 + C_n$.

Let $t_1, t_2, \ldots, t_n, t_1$ be the vertex notation of the n-cycle C_n for $n \geq 3$. The join product $W_5 + C_n$ consists of one copy of the graph W_5, one copy of the cycle C_n, and the edges joining each vertex of W_5 with each vertex of C_n. Let C_n^* be the cycle as a subgraph of $W_5 + C_n$ induced on the vertices of C_n not belonging to the subgraph W_5. The subdrawing $D(C_n^*)$ induced by any good drawing D of $W_5 + C_n$ represents some drawing of C_n. For the vertices $v_1, v_2, \ldots, v_6$ of the graph W_5, let T^{v_i} denote the subgraph induced by n edges joining the vertex v_i with n vertices of C_n^*. The edges joining the vertices of W_5 with the vertices of C_n^* form the complete bipartite graph $K_{6,n}$, and so

$$W_5 + C_n = W_5 \cup \left(\bigcup_{i=1}^{6} T^{v_i}\right) \cup C_n^*.$$

In the proof of the main theorem of this section, the following two statements related to some restricted subdrawings of $G + C_n$ will be also required.

Lemma 5 ([8, Lemma 2.2]). *For $m \geq 2$ and $n \geq 3$, let D be a good drawing of $D_m + C_n$ in which no edge of C_n^* is crossed, and C_n^* does not separate the other vertices of the graph. Then, for all $i, j = 1, 2, \ldots, m$, two different subgraphs T^{v_i} and T^{v_j} cross each other in D at least $\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$ times.*

Corollary 6 ([11, Corollary 4])**.** *For $m \geq 2$ and $n \geq 3$, let D be a good drawing of the join product $D_m + C_n$ in which the edges of C_n^* do not cross each other and C_n^* does not separate p vertices $v_1, v_2, \dots, v_p, 2 \leq p \leq m$. Let $T^{v_1}, T^{v_2}, \dots, T^{v_q}$, $q < p$, be the subgraphs induced on the edges incident with the vertices $v_1, v_2, \dots, v_q$ that do not cross C_n^*. If k edges of some subgraphs T^{v_j} induced on the edges incident with the vertex v_j, $j \in \{q+1, q+2, \dots, p\}$, cross the cycle C_n^*, then the subgraph T^{v_j} crosses each of the subgraphs $T^{v_1}, T^{v_2}, \dots, T^{v_q}$ at least $\lfloor \frac{n-k}{2} \rfloor \lfloor \frac{(n-k)-1}{2} \rfloor$ times in D.*

Now we are able to prove the main result of this paper.

Theorem 3. $\operatorname{cr}(W_5 + C_n) = 6 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3 \lfloor \frac{n}{2} \rfloor + 5$ *for* $n \geq 3$.

Proof. By Lemma 2, the result holds for $n = 3$ and $n = 4$. The drawing in Figure 1 shows that $\operatorname{cr}(W_5 + C_n) \leq 6 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3 \lfloor \frac{n}{2} \rfloor + 5$, and let suppose that there is an optimal drawing D of $W_5 + C_n$ such that

$$\operatorname{cr}_D(W_5 + C_n) \leq 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + 3 \left\lfloor \frac{n}{2} \right\rfloor + 4 \quad \text{for some } n \geq 5. \tag{4}$$

In the rest of the proof, let $I = \{1, 2, 3, 4, 5, 6\}$. By Theorem 1, at most four edges of the cycle C_n^* can be crossed in D, and we can also suppose that edges of C_n^* do not cross each other using Lemma 1. The subdrawing of C_n^* induced by D divides the plane into two regions with at least four vertices of W_5 in one of them, and so four possible cases may occur.

Case 1. There is at most one crossing on the edges of C_n^*. We have at least five distinct $i, j \in I$ such that any two different considered subgraphs T^{v_i} and T^{v_j} cross each other at least $\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times by Lemma 5.

Hence, there are at least $\binom{5}{2} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \geq 6 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3 \lfloor \frac{n}{2} \rfloor + 5$ crossings in D, which confirms a contradiction with the assumption (4).

In the next we assume, that each subgraph T^i crosses edges of the weel W_5, because $R_D = \varnothing$ thanks to Corollary 2.

Case 2. There are exactly two crossings on the edges of C_n^*. We discuss two subcases.

(a) If $\operatorname{cr}_D(W_5, C_n^*) = 2$ or $\operatorname{cr}_D(T^{v_i}, C_n^*) = 2$ for one vertex v_i, $i \in I$, then the same idea as in Case 1 contradicts the assumption (4).

(b) Let $\operatorname{cr}_D(T^{v_i}, C_n^*) = 1$ and $\operatorname{cr}_D(T^{v_j}, C_n^*) = 1$ for two different subgraphs T^{v_i}, T^{v_j}, where $i, j \in I$. By Lemma 5 and Corollary 6 for $p = 6$, $q = 4$, $k = 1$ we obtain at least $\binom{4}{2} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + (4+4) \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor + n \geq 6 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3 \lfloor \frac{n}{2} \rfloor + 5$ crossings in D, which confirms a contradiction with the assumption (4).

Case 3. There are exactly three crossings on the edges of C_n^*. We discuss three subcases.

(a) If $\operatorname{cr}_D(W_5, C_n^*) \neq 0$ or $\operatorname{cr}_D(C_n^*, T^{v_i}) = 3$ for one vertex v_i, $i \in I$, then the same idea as in Case 1 contradicts the assumption (4).

(b) Let $\operatorname{cr}_D(C_n^*, T^{v_i}) = 2$ and $\operatorname{cr}_D(C_n^*, T^{v_j}) = 1$ for $i, j \in I$, $i \neq j$. By Lemma 5, Corollary 6 for $p = 6$, $q = 4$, $k = 1$ and $p = 6$, $q = 4$, $k = 2$, we have at least $\binom{4}{2} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4 \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor + 4 \lfloor \frac{n-2}{2} \rfloor \lfloor \frac{n-3}{2} \rfloor + n \geq 6 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3 \lfloor \frac{n}{2} \rfloor + 5$ crossings in D, which confirms a contradiction with the assumption (4).

(c) Let $\mathrm{cr}_D\,(C_n^*, T^{v_i}) = 1$ for three distinct vertices v_i, $i \in I$. By Lemma 5, Corollary 6 for $p = 6$, $q = 3$, $k = 1$ and the fact that $S_D = \varnothing$ provided by Corollaries 3 and 5, we have at least $\binom{3}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + (3+3+3)\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 2n \geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + 3\left\lfloor\frac{n}{2}\right\rfloor + 5$ crossings in D, which confirms a contradiction with the assumption (4).

Case 4. There are exactly four crossings on the edges of C_n^*. We discuss eight subcases.

(a) Let $\mathrm{cr}_D\,(W_5, C_n^*) = 4$. If all six vertices of the wheel W_5 are placed in one region of $D\,(C_n^*)$, then the same idea as in Case 1 contradicts the assumption (4). If D is a good drawing of $W_5 + C_n$ with the planar subdrawing of W_5 induced by D, Corollary 4 contradicts the assumption (4). Now, we can assume that four vertices of the wheel W_5 are placed in one and two in the second region of $D\,(C_n^*)$ and D is a good drawing of $W_5 + C_n$ with the non-planar subdrawing of W_5 induced by D. We have at least $\left(\binom{4}{2} + \binom{2}{2}\right)\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + |S_D| + 2\big(n - |S_D|\big) + \mathrm{cr}_D\,(W_5) + 4 \geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + 3\left\lfloor\frac{n}{2}\right\rfloor + 5$ crossings in D using Lemma 5 and four crossings on edges of C_n^*. This confirms a contradiction with the assumption (4) for all $n \geq 7$ or $n \geq 5$ with $|S_D| \leq 4$. Note, that in the subcase, when $S_D \neq \varnothing$, we have only two possible drawings of W_5 given in Figure 5, where $\mathrm{cr}_D\,(W_5) = 2$. Finally, let $n \in \{5, 6\}$ and $|S_D| \geq 5$. If we consider the subdrawing of W_5 in Figure 5 (a), two different subgraphs $T^i, T^j \in S_D$ are crossed at least six times provided by $\mathrm{rot}_D\,(t_i) = \mathrm{rot}_D\,(t_j)$.[1] For the subdrawing of W_5 in Figure 5 (b), at least five interchanges of adjacent elements of (123645) are necessary to obtain cyclic permutation $\overline{(123654)} = (145632)$ because only one interchange of the adjacent elements of (123645) produces the cyclic permutation (123654). Since $\mathrm{cr}_D\,(T^i, T^j) \geq 5$ for every two distinct $T^i, T^j \in S_D$, we have at least $\binom{5}{2}5 + 4 + 2$ crossings in D which confirms a contradiction with the assumption (4).

(b) Let $\mathrm{cr}_D\,(W_5, C_n^*) = 3$. Five vertices of W_5, say v_2, v_3, v_4, v_5, v_6, are placed in one region of $D\,(C_n^*)$. If $\mathrm{cr}_D\,(C_n^*, T^{v_1}) = 1$, then the same idea as in Case 1 contradicts the assumption (4). If $\mathrm{cr}_D\,(C_n^*, T^{v_i}) = 1$ for $i \neq 1$, then by Lemma 5, Corollary 6 for $p = 5$, $q = 4$, $k = 1$ and four crossings on C_n^*, we have at least $\binom{4}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + n + 4 \geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + 3\left\lfloor\frac{n}{2}\right\rfloor + 5$ crossings in D, which also confirms a contradiction with the assumption (4).

(c) Let $\mathrm{cr}_D\,(W_5, C_n^*) = 2$. If $\mathrm{cr}_D\,(C_n^*, T^{v_i}) = 2$ for only one vertex v_i, $i \in I$, then the same idea as in Case 1 contradicts the assumption (4). If $\mathrm{cr}_D\,(C_n^*, T^{v_i}) = 1$ and $\mathrm{cr}_D\,(C_n^*, T^{v_j}) = 1$ for $i, j \in I$, $i \neq j$, then the same idea as in Case 2 (b) contradicts the assumption (4).

In the next we assume, that each subgraph T^i crosses edges of the wheel W_5 at least two times, because $R_D \cup S_D = \varnothing$ thanks to Corollaries 2, 3 and 5. Let us also suppose that all six vertices of W_5 are placed in the inner region of C_n^*.

(d) If $\mathrm{cr}_D\,(C_n^*, T^{v_i}) = 4$ for only one T^{v_i}, $i \in I$, then the same idea as in Case 1 contradicts the assumption (4).

---

[1]For $m \geq 3$, let t_x and t_y be two different vertices of degree m in any good drawing D of the graph $K_{m,2}$. Let T^x and T^y be two considered subgraphs represented by their $\mathrm{rot}_D\,(t_x)$ and $\mathrm{rot}_D\,(t_y)$ of the length m. If the minimum number of interchanges of adjacent elements of $\mathrm{rot}_D\,(t_x)$ required to produce $\mathrm{rot}_D\,(t_y)$ is at most z, then $\mathrm{cr}_D\,(T^x, T^y) \geq \left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor - z$ (see also [25]).

(e) Let $\mathrm{cr}_D(C_n^*, T^{v_i}) = 3$ and let $\mathrm{cr}_D(C_n^*, T^{v_j}) = 1$ for two distinct $i, j \in I$, then the same idea as in the second part of Case 4 (b) contradicts the assumption (4).

(f) Let $\mathrm{cr}_D(C_n^*, T^{v_i}) = 2$ and $\mathrm{cr}_D(C_n^*, T^{v_j}) = 2$ for two distinct $i, j \in I$. By Lemma 5, Corollary 6 for $p = 6$, $q = 4$, $k = 2$ and also due to four crossings on edges of C_n^*, we have at least $\binom{4}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + (4+4)\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor + 2n + 4 \geq 6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + n + 3\lfloor\frac{n}{2}\rfloor + 5$ crossings in D, which confirms a contradiction with the assumption (4).

(g) Let $\mathrm{cr}_D(C_n^*, T^{v_i}) = 1$, $\mathrm{cr}_D(C_n^*, T^{v_j}) = 1$, $\mathrm{cr}_D(C_n^*, T^{v_l}) = 2$ for three distinct $i, j, l \in I$. By Lemma 5, Corollary 6 for $p = 6$, $q = 3$, $k = 1$ and also $p = 6$, $q = 3$, $k = 2$ and four crossings on C_n^* we have at least $\binom{3}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + (3+3)\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor + 3\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor + 2n + 4 \geq 6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + n + 3\lfloor\frac{n}{2}\rfloor + 5$ crossings in D, which confirms a contradiction with the assumption (4).

(h) Let us suppose that $\mathrm{cr}_D(T^{v_i}, C_n^*) = 1$, $\mathrm{cr}_D(T^{v_j}, C_n^*) = 1$, $\mathrm{cr}_D(T^{v_l}, C_n^*) = 1$ and $\mathrm{cr}_D(T^{v_m}, C_n^*) = 1$ for four distinct $i, j, l, m \in I$. For such an index pair i, j, the subgraph $T^{v_i} \cup T^{v_j} \cup C_n^*$ is isomorphic to the graph $D_2 + C_n$. Consider $n - 2$ vertices of the cycle C_n^* incident with edges of T^{v_i} and T^{v_j} which do not cross C_n^*. Let us delete all edges of T^{v_i} and T^{v_j} which are not incident with these $n - 2$ vertices. The resulting subgraph is homeomorphic to the graph $D_2 + C_{n-2}$ and, in its subdrawing D' induced by D, we obtain $\mathrm{cr}_{D'}(T^{v_i}, T^{v_j}) \geq \lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$ thanks to Lemma 5. Clearly, the same holds for any of remaining index pairs from $\{i, j, l, m\}$. Thus, we have at least $\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + (2+2+2+2)\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor + \binom{4}{2}\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor + 2n + 5$ crossings in D using Lemma 5, Corollary 6 for $p = 6$, $q = 2$, $k = 1$, also due to four crossings on edges of C_n^* and one crossing in $D(W_5)$.

We have shown that there is no optimal drawing D of the graph $W_5 + C_n$ with fewer than $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + n + 3\lfloor\frac{n}{2}\rfloor + 5$ crossings, and this completes the proof of Theorem 3. □

4 Some consequence of the main result

W. Yue et. al. [24] conjectured that the crossing numbers of $W_m + C_n$ are given by $Z(m+1)Z(n) + \big(Z(m) - 1\big)\lfloor\frac{n}{2}\rfloor + n + \lceil\frac{m}{2}\rceil + 2$ for all integers $m, n \geq 3$, where W_m denotes the wheel of order $m + 1$. Here, $Z(n)$ known as the Zarankiewicz's number is defined by $\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$ for all $n \geq 1$. Recently, this conjecture was proved for $W_3 + C_n$ and $W_4 + C_n$ by M. Klešč [9], M. Staš and J. Valiska [20], respectively. As we mentioned earlier, the graph $W_n + C_m$ is isomorphic to the graph $W_m + C_n$ for all integers $m, n \geq 3$. Notice that $\mathrm{cr}(W_5 + C_n)$ thanks to the main Theorem 3 offers additional results for the join product of W_m with cycles on five vertices.

Theorem 4. $\mathrm{cr}(W_m + C_5) = 6\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + m + 3\lfloor\frac{m}{2}\rfloor + 5$ *for* $m \geq 3$.

Our next research will be devoted to showing the validity of Yue's conjecture for all integers $m, n \geq 3$ assuming the validity of Zarankiewicz's conjecture that $\mathrm{cr}(K_{m,n}) = Z(m)Z(n)$. Some useful remarks about Zarankiewicz's conjecture were also stated by X. Yang and Y. Wang [23].

5 Conclusions

We suppose that similar forms of discussions can be helpful to determine unknown values of the crossing numbers of other symmetric graphs $G = \big(V(G), E(G)\big)$ of order six with respect to the restriction $3|V(G)| \leq 2|E(G)|$ in the join products with cycles on n vertices. We expect the same for all graphs of order five, especially for the complete graph K_5 and the graph $K_5 \setminus e$ obtained by removing one edge from K_5.

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Під числом перетинів $\mathrm{cr}(G)$ простого графа G розуміють мінімальну кількість перетинів ребер серед усіх можливих зображень G на площині. Числа перетинів приєднаного добутку двох графів вивчають через їх широке практичне застосування. Основною метою статті є встановлення значення $\mathrm{cr}\,(W_5 + C_n)$ для колеса W_5 на шести вершинах, де C_n $-$ цикли на n вершинах. В. Юе та ін. у [Comp. Eng. Appl. 2014, **50** (18), 79–84] висунули гіпотезу, що число перетинів для $W_m + C_n$ дорівнює $Z(m+1)Z(n) + \big(Z(m)-1\big)\left\lfloor \frac{n}{2} \right\rfloor + n + \left\lceil \frac{m}{2} \right\rceil + 2$ для всіх цілих $m, n \geq 3$. Тут число Заранкевича $Z(n) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ визначене для всіх $n \geq 1$. Зазначену гіпотезу підтвердили для $W_3 + C_n$ $-$ М. Клешч, а для $W_4 + C_n$ $-$ М. Сташ та Ю. Валіска. У цій роботі ми доводимо справедливість цієї гіпотези для $W_5 + C_n$.

Ключові слова і фрази: граф, приєднаний добуток, число перетинів, колесо, цикл.