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Comparative growth of an entire function and the integrated counting function of its zeros

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Let (ζ_n) be a sequence of complex numbers such that $\zeta_n \to \infty$ as $n \to \infty$, $N(r)$ be the integrated counting function of this sequence, and let α be a positive continuous and increasing to $+\infty$ function on $\mathbb{R}$ for which $\alpha(r) = o(\log(N(r)/\log r))$ as $r \to +\infty$. It is proved that for any set $E \subset (1, +\infty)$ satisfying $\int_E r^{\alpha(r)}dr = +\infty$, there exists an entire function f whose zeros are precisely the ζ_n, with multiplicities taken into account, such that the relation

$$\liminf_{r\in E,\, r\to+\infty} \frac{\log\log M(r)}{\log r \log(N(r)/\log r)} = 0$$

holds, where $M(r)$ is the maximum modulus of the function f. It is also shown that this relation is best possible in a certain sense.

Key words and phrases: entire function, maximum modulus, Nevanlinna characteristic, zero, counting function, integrated counting function.

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1 Introduction and results

Let $\mathcal{Z}$ be the class of all complex sequences $\zeta = (\zeta_n)$ such that $0 < |\zeta_1| \le |\zeta_2| \le \dots$ and $\zeta_n \to \infty$ as $n \to \infty$. For any sequence ζ belonging to the class $\mathcal{Z}$, by $\mathcal{E}(\zeta)$ we denote the class of all entire functions whose zeros are precisely the ζ_n, where a complex number that occurs m times in the sequence ζ corresponds to a zero of multiplicity m. For every $r \ge 0$, let $n_\zeta(r)$ and $N_\zeta(r)$ be the counting function and the integrated counting function of this sequence, respectively, that is

$$n_\zeta(r) = \sum_{|\zeta_n|\le r} 1, \qquad N_\zeta(r) = \int_0^r \frac{n_\zeta(t)}{t}dt.$$

For an entire function f and every $r \ge 0$, we denote by $M_f(r)$ and $T_f(r)$ the maximum modulus and the Nevanlinna characteristic of the function f, respectively, i.e.

$$M_f(r) = \max\Big\{\big|f(z)\big| : |z| = r\Big\}, \qquad T_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \big|f\big(re^{i\theta}\big)\big| d\theta.$$

By L we denote the class of all positive continuous and increasing to $+\infty$ functions on $\mathbb{R}$.

Let $\zeta \in \mathcal{Z}$. It is well known that the growth of functions $f \in \mathcal{E}(\zeta)$, which is usually identified with the growth of $\log M_f(r)$ or $T_f(r)$, can be arbitrarily rapid compared to the

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growth of the functions $n_\zeta(r)$ and $N_\zeta(r)$. Many authors (see, for example, [1–5, 7, 8, 13, 15, 16]) investigated the opposite, in a certain sense, and more interesting question: how slow can the growth of a function $f \in \mathcal{E}(\zeta)$ be with respect to $n_\zeta(r)$ or $N_\zeta(r)$? In particular, the following two theorems that generalize two results of W. Bergweiler [8] were proved in [4].

Theorem A ([4]). *Let $l \in L$. Then for any sequence $\zeta \in \mathcal{Z}$ such that $n_\zeta(r) \geq l(r)$ for all large r and for every unbounded set $E \subset (1, +\infty)$ there exists a function $f \in \mathcal{E}_\zeta$ such that*

$$\liminf_{r\in E,\, r\to+\infty} \frac{\log\log M_f(r)}{\log n_\zeta(r) \log l^{-1}\big(n_\zeta(r)\big)} = 0. \tag{1}$$

Theorem B ([4]). *Let $l \in L$, and let ψ be a positive function on $\mathbb{R}$ satisfying*

$$\liminf_{x\to+\infty} \frac{\psi\big(\log[x]\big)}{\log x} = 0. \tag{2}$$

Then there exist a sequence $\zeta \in \mathcal{Z}$ such that $n_\zeta(r) \geq l(r)$ for all large r and $n_\zeta(r-0) = l(r)$ on an unbounded from above set of values of r and a set $F \subset (1, +\infty)$ of upper logarithmic density 1 such that for an arbitrary function $f \in \mathcal{E}_\zeta$ we have

$$\psi\big(\log n_\zeta(r)\big) \log l^{-1}\big(n_\zeta(r)\big) = o\big(\log\log M_f(r)\big)$$

as $r \to +\infty$ through the set F.

It is clear that if (2) is not satisfied for some positive function ψ on $\mathbb{R}$, then $\log n_\zeta(r)$ in (1) can be replaced by $\psi\big(\log n_\zeta(r)\big)$. According to Theorem B, the same replacement is impossible when (2) is satisfied. Therefore, Theorem B shows that relation (1) is best possible.

Note also that the proof of Theorem B given in [4] shows that the set F in this theorem can be expressed in the form $F = \bigcup_{k=1}^{\infty}(s_k, r_k)$, where

$$1 < s_1 < r_1 < s_2 < r_2 < \ldots, \qquad \lim_{n\to\infty} \frac{\log r_k}{\log s_k} = +\infty. \tag{3}$$

In this article, we establish results of the type of Theorems A and B for the function $N_\zeta(r)$ instead of $n_\zeta(r)$.

First of all, we note that, by Theorem B, the growth of each function f from the class $\mathcal{E}_\zeta$ can be arbitrarily rapid compared to the growth of the counting function $n_\zeta(r)$ along a large set of values r. In other words, for an arbitrary function $h \in L$ there exist a sequence $\zeta \in \mathcal{Z}$ and a set $F \subset (1, +\infty)$ of upper logarithmic density 1 such that for every function $f \in \mathcal{E}_\zeta$ we have

$$h\big(n_\zeta(r)\big) = o\big(\log\log M_f(r)\big)$$

as $r \to +\infty$ through the set F.

A similar situation is impossible for the integrated counting function $N_\zeta(r)$. This follows from the following theorem of A.A. Gol'dberg.

Theorem C ([13]). *Let $\zeta \in \mathcal{Z}$ be an arbitrary sequence. Then there exists a function $f \in \mathcal{E}_\zeta$ such that*

$$\log\log M_f(r) = o\big(N_\zeta(r)\big) \tag{4}$$

as $r \to +\infty$ outside an exceptional set $E \subset (1, +\infty)$ of finite logarithmic measure.

This result is best possible in the following sense.

Theorem D ([13]). *Let $\psi \in L$ be an arbitrary function satisfying $\psi(x) = o(x)$ as $x \to +\infty$. Then there exist a sequence $\zeta \in \mathcal{Z}$ and a set $F \subset (0, r)$ of upper linear density 1 such that for every function $f \in \mathcal{E}_\zeta$ we have*

$$\psi\big(N_\zeta(r)\big) = o\big(\log\log M_f(r)\big) \tag{5}$$

as $r \to +\infty$ through the set F.

The estimates for the sizes of the sets E and F in Theorems C and D can be sharpened.

Theorem E ([1]). *Let $\zeta \in \mathcal{Z}$ be an arbitrary sequence. Then there exist a function $f \in \mathcal{E}_\zeta$ and a function $\alpha \in L$ such that* (4) *holds as $r \to +\infty$ outside an exceptional set $E \subset (1, +\infty)$ satisfying*

$$\int_E r^{\alpha(r)} dr < +\infty. \tag{6}$$

Theorem F ([1]). *Let $\psi \in L$ be an arbitrary function such that*

$$\liminf_{x \to +\infty} \frac{\psi(x)}{x} = 0. \tag{7}$$

Then there exist a sequence $\zeta \in \mathcal{Z}$ and sequences (s_k) and (r_k) satisfying (3) *such that for any function $f \in \mathcal{E}_\zeta$ we have* (5) *as $r \to +\infty$ through the set $F = \bigcup_{k=1}^{\infty} (s_k; r_k)$.*

In connection with the above results, the following question arises: *is it possible to find a function $h \in L$ such that for an arbitrary sequence $\zeta \in \mathcal{Z}$ and every unbounded set $E \subset (1, +\infty)$ there exists a function $f \in \mathcal{E}_\zeta$ such that*

$$\liminf_{r \in E,\ r \to +\infty} \frac{\log\log M_f(r)}{h\big(N_\zeta(r)\big)} = 0?$$

The answer to this question remains open, but it will be positive under additional (in some sense even minimal) assumptions about the size of the set E. This fact is confirmed by the following theorem, which directly follows from Theorem E.

Theorem G. *Let $\zeta \in \mathcal{Z}$ be an arbitrary sequence. Then it is possible to find a function $\alpha \in L$ such that for an arbitrary set $E \subset (1, +\infty)$ satisfying*

$$\int_E r^{\alpha(r)} dr = +\infty \tag{8}$$

there exists a function $f \in \mathcal{E}_\zeta$ for which

$$\liminf_{r \in E,\ r \to +\infty} \frac{\log\log M_f(r)}{N_\zeta(r)} = 0. \tag{9}$$

Let $\psi \in L$. If, for the function ψ, (7) is not satisfied, then Theorem G will remain correct when $N_\zeta(r)$ in (9) is replaced by $\psi\big(N_\zeta(r)\big)$. It follows from Theorem F that such a replacement is impossible if (7) is satisfied. Therefore, (9) is best possible in the whole class $\mathcal{Z}$. We will show below that in wide subclasses of the class $\mathcal{Z}$ this relation can be refined.

Denote by Ω the class of all non-negative, convex functions Φ on $\mathbb{R}$ such that $\sigma = o\big(\Phi(\sigma)\big)$ as $\sigma \to +\infty$. We remark that $N_\zeta\left(e^\sigma\right)$ is a function from the class Ω for each sequence $\zeta \in \mathcal{Z}$.

Let $\Phi \in \Omega$. Then, as it is easy to see, the function $\varphi(\sigma) = \Phi(\sigma)/\sigma$ is continuous, increasing to $+\infty$ on $(\sigma_0, +\infty)$ for some $\sigma_0 > 0$.

Theorem 1. *Let $\Phi \in \Omega$, and $\alpha \in L$. If*

$$\alpha(r) = o\big(\log \varphi(\log r)\big), \qquad r \to +\infty, \tag{10}$$

then, for any sequence $\zeta \in \mathcal{Z}$ such that $N_\zeta(r) \geq \Phi(\log r)$ for all sufficiently large r and for an arbitrary set $E \subset (1, +\infty)$ satisfying (8)*, there exists a function $f \in \mathcal{E}_\zeta$ such that*

$$\liminf_{r \in E,\, r \to +\infty} \frac{\log\log M_f(r)}{\Phi^{-1}\big(N_\zeta(r)\big) \log \varphi\Big(\Phi^{-1}\big(N_\zeta(r)\big)\Big)} = 0. \tag{11}$$

If $x \to +\infty$, then, taking $\sigma = \Phi^{-1}(x)$, we get

$$\Phi^{-1}(x) \log \varphi\left(\Phi^{-1}(x)\right) = \sigma \log \varphi(\sigma) = \frac{\log \varphi(\sigma)}{\varphi(\sigma)} \Phi(\sigma) = \frac{\log \varphi(\sigma)}{\varphi(\sigma)} x = o(x).$$

Therefore, (11) is stronger than (9). Moreover, as the following theorem shows, (11) is exact in the sense that in it $\log \varphi(\Phi^{-1}(N_\zeta(r)))$ cannot be replaced by $\psi\Big(\log \varphi(\Phi^{-1}(N_\zeta(r)))\Big)$, where $\psi \in L$ is an arbitrary function satisfying (2). It is clear that such a replacement is possible for every $\psi \in L$ that does not satisfy (2).

Theorem 2. *Let $\Phi \in \Omega$, and let $\psi \in L$ be an arbitrary function satisfying* (7)*. Then there exist a sequence $\zeta \in \mathcal{Z}$ such that $N_\zeta(r) \geq \Phi(\log r)$ for all sufficiently large r and $N_\zeta(r) = \Phi(\log r)$ on an unbounded from above set of values of r and sequences (s_k) and (r_k) satisfying* (3) *such that for any function $f \in \mathcal{E}_\zeta$ we have*

$$\Phi^{-1}\big(N_\zeta(r)\big) \psi\Big(\log \varphi\big(\Phi^{-1}(N_\zeta(r))\big)\Big) = o\big(\log T_f(r)\big) \tag{12}$$

as $r \to +\infty$ through the set $F = \bigcup_{k=1}^{\infty} (s_k; r_k)$.

Theorems 1 and 2 are easy to obtain from the following two theorems.

Theorem 3. *Let $\zeta \in \mathcal{Z}$, and let $\alpha \in L$. If*

$$\alpha(r) = o\big(\log(N_\zeta(r)/\log r)\big), \qquad r \to +\infty, \tag{13}$$

then for every set $E \subset (1, +\infty)$ satisfying (8) *there exists a function $f \in \mathcal{E}_\zeta$ such that*

$$\liminf_{r \in E,\, r \to +\infty} \frac{\log\log M_f(r)}{\log r \log\left(N_\zeta(r)/\log r\right)} = 0. \tag{14}$$

Theorem 4. *Let $\Phi \in \Omega$, and let $\psi \in L$ be an arbitrary function satisfying* (7)*. Then there exist a sequence $\zeta \in \mathcal{Z}$ such that $N_\zeta(r) \geq \Phi(\log r)$ for all sufficiently large r and sequences (s_k) and (r_k) satisfying* (3) *such that $N_\zeta(r_k) = \Phi(\log r_k)$ for all integers $k \geq 1$ and for any function $f \in \mathcal{E}_\zeta$ we have*

$$\log r_k\, \psi\big(\log(N_\zeta(r_k)/\log r_k)\big) = o\big(\log T_f(s_k)\big), \quad k \to \infty. \tag{15}$$

Note that we can rewrite (15) in the form

$$\Phi^{-1}\big(N_\zeta(r_k)\big) \psi\Big(\log \varphi\big(\Phi^{-1}(N_\zeta(r_k))\big)\Big) = o\big(\log T_f(s_k)\big), \quad k \to \infty.$$

It follows that (12) holds as $r \to +\infty$ through the set $F = \bigcup_{k=1}^{\infty} (s_k; r_k)$. Therefore, Theorem 2 is an immediate consequence of Theorem 4.

To prove Theorems 1, 3 and 4 we will need some auxiliary results, which are given in the next section.

2 Auxiliary results

Let $z \in \mathbb{C}$, and let $p \geq 0$ be an integer. By $E(z,p)$ we denote the usual Weierstrass primary factor, i.e.

$$E(z,p) = \begin{cases} 1-z, & \text{if } p = 0, \\ (1-z)\exp\left(\sum\limits_{n=1}^{p} \dfrac{z^n}{n}\right), & \text{if } p \geq 1. \end{cases}$$

Lemma 1 ([2]). *Let $\zeta = (\zeta_n)$ be a sequence from the class $\mathcal{Z}$. Then there exists a non-negative sequence (λ_n) with the following properties:*

(i) $\lambda_n \sim \log n / \log|\zeta_n|$ as $n \to \infty$;

(ii) for every sequence (p_n) of non-negative integers such that $p_n \geq [\lambda_n]$ for all sufficiently large n, the series

$$\sum_{n=1}^{\infty} \left(\frac{r}{|\zeta_n|}\right)^{p_n+1} \tag{16}$$

converges for every $r \geq 0$, and the product

$$\prod_{n=1}^{\infty} E\left(\frac{z}{\zeta_n}, p_n\right) \tag{17}$$

converges uniformly and absolutely to an entire function $f \in \mathcal{E}_\zeta$ on any compact subset of $\mathbb{C}$, and for all $r \geq 0$ we have $\log M_f(r) \leq G(r)$, where $G(r)$ is the sum of series (16).

Let f be an entire function, $r > 0$, and let $c_p(r)$ be the pth Fourier coefficient of the function $\log|f(re^{i\theta})|$, that is

$$c_p(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ip\theta} \log|f(re^{i\theta})| d\theta, \quad p \in \mathbb{Z}.$$

If $f(0) \neq 0$, then in a neighborhood $\mathbb{D}$ of the point $z = 0$ the function f has no zeros, and therefore there exists an analytic function

$$g(z) = \sum_{p=0}^{\infty} a_p z^p \tag{18}$$

in $\mathbb{D}$ such that $f(z) = e^{g(z)}$ for all $z \in \mathbb{D}$. Then, for every integer $p \geq 1$ and all $r > 0$, by the Poisson-Jensen formula (see, for example, [14, p. 16–17]) we have

$$c_p(r) = \frac{1}{2}a_p r^p + \frac{1}{2p}\sum_{|\zeta_n|<r}\left(\left(\frac{r}{\zeta_n}\right)^p - \left(\frac{\overline{\zeta_n}}{r}\right)^p\right), \tag{19}$$

where ζ_n are the zeros of the function f. In addition, the following statement is well known (see, for example, [14, p. 62]).

Lemma 2. *Let f be an entire function. Then for every integer p and all $r > 0$ the inequality $|c_p(r)| \leq 2T_f(r)$ holds.*

3 Proof of Theorems

Proof of Theorem 3. Let $\zeta = (\zeta_n)$ be a sequence from the class $\mathcal{Z}$, and let $\alpha \in L$ be a function that satisfies (13). We suppose that $E \subset (1, +\infty)$ is a set satisfying (8) and prove that there exists a function $f \in \mathcal{E}_\zeta$ such that (14) holds.

We set $\gamma(r) = N_\zeta(r)/\log r$ for all $r > 1$ and note that

$$\log N_\zeta(r) = \log \gamma(r) + \log\log r = o\big(\log r \log \gamma(r)\big) = o\big(\log r \log\big(N_\zeta(r)/\log r\big)\big) \tag{20}$$

as $r \to +\infty$.

Let $r_0 > |\zeta_1|$ be a fixed number. Consider the set $E_1 = \big\{r > r_0 : n_\zeta(r) > r^{\alpha(r)+1}N_\zeta^2(r)\big\}$. Since $n_\zeta(r) = r\big(N_\zeta(r)\big)'_+$ for all $r \ge 0$, setting $y_0 = N_\zeta(r_0)$, we have

$$\int_{E_1} r^{\alpha(r)}dr \le \int_{E_1} \frac{n_\zeta(r)}{rN_\zeta^2(r)}dr \le \int_{r_0}^{+\infty} \frac{n_\zeta(r)}{rN_\zeta^2(r)}dr = \int_{r_0}^{+\infty} \frac{dN_\zeta(r)}{N_\zeta^2(r)} = \int_{y_0}^{+\infty} \frac{dy}{y^2} < +\infty.$$

Thus, (8) implies that the set $E_2 = E \backslash E_1$ is unbounded. In addition, if $r > r_0$ and $r \in E_2$, then $\log n_\zeta(r) \le \big(\alpha(r)+1\big)\log r + 2\log N_\zeta(r)$, and therefore, using (13) and (20), we obtain

$$\log n_\zeta(r) = o\big(\log r \log\big(N_\zeta(r)/\log r\big)\big) \tag{21}$$

as $r \to +\infty$ through the set E_2.

For the sequence ζ, let $\lambda = (\lambda_n)$ be a sequence whose existence is asserted by Lemma 1. For every integer $n \ge 1$, we put $q_n = \Big[\frac{2\log n}{\log|\zeta_n|}\Big]$. Note that $q_n \ge [\lambda_n]$ for all sufficiently large n.

Consider the series

$$\sum_{n=0}^{\infty} \left(\frac{r}{|\zeta_n|}\right)^{q_n+1},$$

that converges for all $r \ge 0$. For every $r \ge 0$, we put

$$m_\zeta(r) = \min\left\{m \ge n_\zeta(r) + 2 : \sum_{n=m}^{\infty} \left(\frac{r}{|\zeta_n|}\right)^{q_n+1} \le 1\right\}, \tag{22}$$

and let $p_\zeta(r)$ be the smallest integer such that $p_\zeta(r) \ge q_n$ for all integers $n \in \big(n_\zeta(r), m_\zeta(r)\big)$ and, in addition,

$$\sum_{n_\zeta(r)<n<m_\zeta(r)} \left(\frac{r}{|\zeta_n|}\right)^{p_\zeta(r)+1} \le 1. \tag{23}$$

We note further that for any polynomial $Q(r) = \sum_{n=1}^{m} b_n r^{x_n}$, where $b_1, \dots, b_n$ and $x_1, \dots, x_n$ are positive real numbers, the relation $\log Q(r) = O(\log r)$ holds as $r \to +\infty$. Thus, since the set E_2 is unbounded, in this set we can choose a sequence (r_k) increasing to $+\infty$ such that for all integers $k \ge 2$ we have simultaneously

$$\log \sum_{n \le m_\zeta(r_{k-1})} \left(\frac{r_k}{|\zeta_n|}\right)^{q_n+1} \le \frac{\log r_k}{k} \log \frac{N_\zeta(r_k)}{\log r_k}, \tag{24}$$

$$\log \sum_{j=1}^{k-1} \sum_{n_\zeta(r_j)<n<m_\zeta(r_j)} \left(\frac{r_k}{|\zeta_n|}\right)^{p_\zeta(r_j)+1} \le \frac{\log r_k}{k} \log \frac{N_\zeta(r_k)}{\log r_k}, \tag{25}$$

$$n_\zeta\left(\sqrt{r_k}\right) \ge m_\zeta(r_{k-1}) + 2.$$

Let $k \geq 1$ be an integer. Put $l_k = n_\zeta\left(\sqrt{r_k}\right)$, $n_k = n_\zeta(r_k)$ and $m_k = m_\zeta(r_k)$. Note that $l_k \leq n_k < m_k \leq l_{k+1} - 2$.

For any integer $n \geq 1$, we set $p_n = p_\zeta(r_k)$ in the case when $n \in (n_k, m_k)$ for some integer $k \geq 1$, and let $p_n = q_n$ in the opposite case. Note that $p_n \geq q_n$ for every integer $n \geq 1$, and therefore series (16) converges for all $r \geq 0$. Then, by Lemma 1, product (17) converges uniformly and absolutely to an entire function $f \in \mathcal{E}_\zeta$ on any compact subset of $\mathbb{C}$, and for all $r \geq 0$ we have $\log M_f(r) \leq G(r)$, where $G(r)$ is the sum of series (16). Let us prove that for this function (14) holds.

Assume that $k \geq 2$ is an integer, and put

$$A_k = \sum_{n \leq m_{k-1}} \left(\frac{r_k}{|\zeta_n|}\right)^{p_n+1}, \quad B_k = \sum_{m_{k-1} < n < l_k} \left(\frac{r_k}{|\zeta_n|}\right)^{p_n+1}, \quad C_k = \sum_{l_k \leq n \leq n_k} \left(\frac{r_k}{|\zeta_n|}\right)^{p_n+1}, \quad D_k = \sum_{n > n_k} \left(\frac{r_k}{|\zeta_n|}\right)^{p_n+1}.$$

We next estimate each of these sums.

From (24) and (25) it follows that

$$\log A_k \leq \frac{\log r_k}{k} \log \frac{N_\zeta(r_k)}{\log r_k} + \log 2, \tag{26}$$

and by (23) and (22) we have

$$D_k \leq \sum_{n_k < n < m_k} \left(\frac{r_k}{|\zeta_n|}\right)^{p_\zeta(r_k)+1} + \sum_{n \geq m_k} \left(\frac{r_k}{|\zeta_n|}\right)^{q_n+1} \leq 2. \tag{27}$$

Now we put

$$\mu_k = \max\left\{r_k^{p_n+1} : m_{k-1} < n < l_k\right\}.$$

Recalling that for every integer $n \in (m_{k-1}, l_k)$ the equality $p_n = q_n$ holds, we obtain

$$\log \mu_k \leq \left(\frac{2\log n_\zeta(\sqrt{r_k})}{\log|\zeta_{m_{k-1}}|} + 1\right) \log r_k.$$

Since $B_k \leq \mu_k G(1)$, and

$$N_\zeta(r) \geq N_\zeta(r) - N_\zeta\left(\sqrt{r}\right) = \int_{\sqrt{r}}^{r} \frac{n_\zeta(t)}{t} dt \geq n_\zeta\left(\sqrt{r}\right) \log \sqrt{r}$$

for each $r > 0$, we have

$$\log B_k \leq \log \mu_k + \log G(1) \leq \frac{2\log r_k}{\log|\zeta_{m_{k-1}}|} \log \frac{2N_\zeta(r_k)}{\log r_k} + \log r_k + \log G(1). \tag{28}$$

Next, we put

$$\nu_k = \max\left\{\left(\frac{r_k}{|\zeta_n|}\right)^{p_n+1} : l_k \leq n \leq n_k\right\}.$$

Since, for every integer $n \in [l_k, n_k]$, the equality $p_n = q_n$ and the inequality $|\zeta_n| \geq \sqrt{r_k}$ hold, we obtain

$$\log \nu_k \leq \left(\frac{2\log n_\zeta(r_k)}{\log|\zeta_{l_k}|} + 1\right) \log \sqrt{r_k} \leq 2\log n_\zeta(r_k) + \frac{1}{2}\log r_k.$$

Noting that $C_k \le \nu_k n_k = \nu_k n_\zeta(r_k)$, we have

$$\log C_k \le \log \nu_k + \log n_\zeta(r_k) \le 3\log n_\zeta(r_k) + \log r_k. \tag{29}$$

Since $\log M_f(r_k) \le G(r_k) = A_k + B_k + C_k + D_k$, from (26), (27), (28), (29), and (21) we see that

$$\log\log M(r_k) = o\Big(\log r_k \, \log\big(N_\zeta(r_k)/\log r_k\big)\Big)$$

as $k \to \infty$. This implies (14), because (r_k) is a sequence of points in the set E. Theorem 3 is proved. □

Proof of Theorem 1. First of all, we note that if a, b, and c are positive numbers such that $a < b \le c/e$, then the inequality $a\log(c/a) < b\log(c/b)$ holds. In fact, considering the function $y = x\log(c/x)$ for a fixed $c > 0$, we see that this function is increasing on $(0, c/e]$. This implies the required inequality.

Now let $\Phi \in \Omega$, $\alpha \in L$ be a function satisfying (10), $\zeta \in \mathcal{Z}$ be a sequence such that $N_\zeta(r) \ge \Phi(\log r)$ for all $r \ge r_0$, and let $E \subset (1, +\infty)$ be a set for which (8) holds. Clearly we can assume that $r_0 > 1$ and $\varphi(\log r_0) \ge e$.

Since $N_\zeta(r)/\log r \ge \varphi(\log r)$ for all $r \ge r_0$, (10) implies (13). According to Theorem 3, there exists a function $f \in \mathcal{E}_\zeta$ such that (14) holds.

Suppose that $r \ge r_0$ is a fixed number, and set $s = \exp\big(\Phi^{-1}(N_\zeta(r))\big)$. Then

$$\log r \le \Phi^{-1}\big(N_\zeta(r)\big) = \log s \le \log s\,\frac{\varphi(\log s)}{e} \le \frac{N_\zeta(r)}{e}.$$

Recalling the remark formulated at the beginning of the proof, we get

$$\log r \log\frac{N_\zeta(r)}{\log r} \le \log s \log\frac{N_\zeta(r)}{\log s} = \log s\, \log\varphi(\log s) = \Phi^{-1}\big(N_\zeta(r)\big)\log\varphi\Big(\Phi^{-1}\big(N_\zeta(r)\big)\Big).$$

Therefore, (11) follows from (14). Theorem 1 is proved. □

Proof of Theorem 4. Let $\Phi \in \Omega$, and let $\psi \in L$ be an arbitrary function satisfying (7). We prove that there exist a sequence $\zeta \in \mathcal{Z}$ such that $N_\zeta(r) \ge \Phi(\log r)$ for all sufficiently large r and sequences (s_k) and (r_k) satisfying (3) such that $N_\zeta(r_k) = \Phi(\log r_k)$ for all integers $k \ge 1$ and for any function $f \in \mathcal{E}_\zeta$ we have (12).

We may suppose without loss of generality that there exists a number $\sigma_0 > 1$ such that $\Phi(\sigma) = 0$ for all $\sigma \le \sigma_0$. Note also that for a fixed number $a \in \mathbb{R}$ the function

$$h(\sigma) = \frac{\Phi(\sigma) - \Phi(a)}{\sigma - a}$$

is continuous, increasing to $+\infty$ on the interval $(a, +\infty)$. Furthermore, according to (7), there exists a function $\gamma \in L$ such that the set S of all positive integers n satisfying the inequality

$$\gamma(\log n)\psi(\log n) \le \log n \tag{30}$$

is infinite. From what has been said it follows that there exists a positive, increasing to $+\infty$ sequence (r_k) with $r_1 = \exp(\sigma_0)$ such that it together with the sequence

$$n_k = \frac{\Phi(\log r_{k+1}) - \Phi(\log r_k)}{\log r_{k+1} - \log r_k}, \quad k = 1, 2, \ldots,$$

have the following properties:

(i) $n_k \in S$ for all integers $k \geq 1$, and $n_k = o(n_{k+1})$ as $k \to +\infty$;

(ii) $\log^3 r_k < \log r_{k+1}$ for all integers $k \geq 1$;

(iii) $\log^2 r_k = o\big(\gamma(\log n_k)\big)$ as $k \to \infty$.

Put $s_1 = e$ and let $s_{k+1} = \exp(\log r_{k+1}/\log r_k)$ for an arbitrary integer $k \geq 1$. It is clear that the sequences (s_k) and (r_k) satisfy (3) according to (ii).

Let $m_1 = n_1$, and let $m_k = n_k - n_{k-1}$ for each integer $k \geq 2$. Note that $\sum_{j=1}^{k} m_j = n_k$ for any integer $k \geq 1$.

We form the required sequence $\zeta = (\zeta_n)$ as follows

$$\underbrace{r_1,\ldots,r_1}_{m_1 \text{ times}},\underbrace{r_2,\ldots,r_2}_{m_2 \text{ times}},\ldots,\underbrace{r_k,\ldots,r_k}_{m_k \text{ times}},\ldots,$$

that is, we set $\zeta_n = r_k$ for all integers $n \in (n_k - m_k, n_k]$ and $k \geq 1$. Then $n_\zeta(r) = 0$ if $r \in [0, r_1)$, and $n_\zeta(r) = n_k$ if $r \in [r_k, r_{k+1})$ for some integer $k \geq 1$.

Since $N_\zeta(r) = \Phi(\log r) = 0$ for all $r \in (0, r_1]$, and, for every integer $k \geq 1$,

$$N_\zeta(r_{k+1}) - N_\zeta(r_k) = \int_{r_k}^{r_{k+1}} \frac{n_k}{t} dt = n_k(\log r_{k+1} - \log r_k) = \Phi(\log r_{k+1}) - \Phi(\log r_k),$$

we see that $N_\zeta(r_k) = \Phi(\log r_k)$ for all integers $k \geq 1$. In addition, since $N_\zeta(r)$ is a linear function of $\log r$ and $\Phi(\log r)$ is a convex function of $\log r$ on each of the segments $[r_k, r_{k+1}]$, we have $N_\zeta(r) \geq \Phi(\log r)$ for all $r > 0$.

Suppose that $f \in \mathcal{E}(\zeta)$ is an arbitrary function. We prove that the function f satisfies (15).

For every $r > 0$, let $c_p(r)$ be the pth Fourier coefficient of the function $\log|f(re^{i\theta})|$. The function f has no zeros in the circle $\mathbb{D} = \{z \in \mathbb{C} : |z| < r_1\}$, and therefore for every integer $p \geq 1$ and all $r > 0$ we have (19), where the numbers a_p are Maclaurin coefficients of an analytic function in $\mathbb{D}$ of the form (18). Since $r_1 > e$, there exists a constant $C > 0$ such that $|a_p| \leq 2C$ for every integer $p \geq 1$. Therefore, according to (19),

$$|c_p(r)| \geq \frac{1}{2p} \sum_{|\zeta_n|<r} \left(\left(\frac{r}{\zeta_n}\right)^p - \left(\frac{\zeta_n}{r}\right)^p \right) - Cr^p. \tag{31}$$

Let $k \geq 1$ be an integer. We put $p_k = \left[\frac{\log n_k}{2\log r_k}\right]$. Note that by (iii) and by (30) with $n = n_k$ we have $p_k \to +\infty$ as $k \to \infty$. Since by (i) we have $m_k \sim n_k$ as $k \to +\infty$, we get

$$\log\left(\frac{m_k}{p_k}\left(\frac{s_{k+1}}{r_k}\right)^{p_k}\right) = p_k \log s_{k+1} + \log m_k - p_k \log r_k - \log p_k \geq p_k \log s_{k+1} + \frac{1}{3}\log n_k, \tag{32}$$

if $k \geq k_1$. Therefore, using (31) with $r = s_{k+1}$ and $p = p_k$, and taking into account that by (ii) we have $\log r_k = o(\log s_{k+1})$ as $k \to \infty$, we obtain

$$|c_{p_k}(s_{k+1})| \geq \frac{m_k}{2p_k}\left(\left(\frac{s_{k+1}}{r_k}\right)^p - \left(\frac{r_k}{s_{k+1}}\right)^p\right) - Cs_{k+1}^{p_k} \geq \frac{m_k}{3p_k}\left(\frac{s_{k+1}}{r_k}\right)^{p_k} - Cs_{k+1}^{p_k} \geq \frac{m_k}{4p_k}\left(\frac{s_{k+1}}{r_k}\right)^{p_k},$$

if $k \geq k_2$. Therefore, according to Lemma 2 and (32),

$$\log T_f(s_{k+1}) \geq \log |c_{p_k}(s_{k+1})| - \log 2 \geq p_k \log s_{k+1}, \quad k \geq k_3.$$

Further, taking into account that for each integer $k \geq 1$ we have

$$N_\zeta(r_{k+1}) = \int_{r_1}^{r_{k+1}} \frac{n_\zeta(t)}{t} dt \leq \int_{r_1}^{r_{k+1}} \frac{n_k}{t} dt \leq n_k \log r_{k+1},$$

and using (30) with $n = n_k$, for all integers $k \geq k_4$ we obtain

$$\frac{\log T_f(s_{k+1})}{\log r_{k+1}\, \psi\big(\log\big(N_\zeta(r_{k+1})/\log r_{k+1}\big)\big)} \geq \frac{p_k \log s_{k+1}}{\log r_{k+1}\, \psi(\log n_k)} = \frac{p_k}{\log r_k} \frac{1}{\psi(\log n_k)}$$
$$\geq \frac{\log n_k}{3\log^2 r_k} \frac{\gamma(\log n_k)}{\log n_k} = \frac{\gamma(\log n_k)}{3\log^2 r_k}.$$

This and (iii) imply (15). Theorem 4 is proved. □

4 Some open problems

In connection with Theorems E and G, the following two problems arise.

Problem 1. *Is it possible to find a function $\alpha \in L$ such that for every sequence $\zeta \in \mathcal{Z}$ there exist a function $f \in \mathcal{E}_\zeta$ for which* (4) *holds as $r \to +\infty$ outside an exceptional set $E \subset (1, +\infty)$ satisfying* (6)*?*

Problem 2. *Is it possible to find a function $\alpha \in L$ such that for every sequence $\zeta \in \mathcal{Z}$ and an arbitrary set $E \subset (1, +\infty)$ satisfying* (8) *there exists a function $f \in \mathcal{E}_\zeta$ for which* (9) *holds?*

In other words, does α in Theorem E (in Theorem G) necessarily depend on ζ?

It is clear that if the answer to Problem 1 is positive, then the answer to Problem 2 is also positive. A negative answer to Problem 1 will mean that the estimate given in Theorem E for the size of the exceptional set E is best possible. In this regard, we note that questions about the sizes of exceptional sets in various asymptotic relations between characteristics of entire functions were investigated, for example, in [6, 9–12, 17–19].

The following problem arises in connection with Theorem 1.

Problem 3. *Let $\Phi \in \Omega$, and let $\alpha \in L$ be a function such that* (10) *is not satisfied. Is it true that there exist a sequence $\zeta \in \mathcal{Z}$ such that $N_\zeta(r) \geq \Phi(\ln r)$ for all sufficiently large r and a set $E \subset (1, +\infty)$ satisfying* (8) *such that* (11) *is false for every function $f \in \mathcal{E}_\zeta$?*

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Нехай (ζ_n) — комплексна послідовність така, що $0 < |\zeta_1| \leq |\zeta_2| \leq \ldots$ і $\zeta_n \to \infty$, $n \to \infty$, $N(r)$ — усереднена лічильна функція цієї послідовності, а α — додатна, неперервна, зростаюча до $+\infty$ на $\mathbb{R}$ функція, для якої $\alpha(r) = o(\ln(N(r)/\ln r))$, $r \to +\infty$. Доведено, що для кожної множини $E \subset (1, +\infty)$, яка задовольняє оцінку $\int_E r^{\alpha(r)} dr = +\infty$, існує ціла функція f з нулями в точках ζ_n і лише в них (з урахуванням кратності), для якої правильне співвідношення

$$\varliminf_{r\in E,\, r\to+\infty} \frac{\ln\ln M(r)}{\ln r \ln(N(r)/\ln r)} = 0,$$

де $M(r)$ — максимум модуля функції f. Показано також, що наведене співвідношення є в певному сенсі остаточним.

Ключові слова і фрази: ціла функція, максимум модуля, характеристика Неванлінни, нуль, лічильна функція, усереднена лічильна функція.