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Timelike spherical curves according to equiform Bishop frame in 3-dimensional Minkowski space

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In this paper, we study the equiform Bishop formulae for the equiform timelike curves in 3-dimensional Minkowski space where the equiform timelike spherical curves are defined according to the equiform Bishop frame. We establish a necessary and sufficient condition for an equiform timelike curve to be an equiform timelike spherical curve. Furthermore, we give certain characterizations of equiform spherical curves in 3-dimensional Minkowski space, which are timelike with an equiform spacelike principal normal vector.

Key words and phrases: Minkowski space, Bishop frame, spherical curve, equiform curvature.

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Introduction

Curves in Minkowski space have been the focus of much research and development by several mathematicians over the last two decades such as [2, 9, 20, 22]. For each point along a differentiable curve, there are three unit vectors that are orthogonal to each other. These vectors are known as tangent vector, normal vector and binormal vector, respectively. However along the differentiable curve, there are some points, where the curvature may vanish, and because of this, the normal vectors are not determined at these points. In this case, a different frame is required to solve this problem in three-dimensional Minkowski space. In 1966, L.R. Bishop defined a parallel transport frame as an alternative frame for a differentiable curve, in which it is well-defined even if its curvature vanishes at some points in three-dimensional Minkowski space [4].

Also, in 1995, A.J. Hanson and H. Ma published a study that discussed the advantages of the Bishop frame and compared it to the Frenet frame in 3-dimensional Euclidean space [10]. M.K. Karacan and Y. Tunçer studied Bäcklund transformation according to Bishop frame in three-dimensional Minkowski space [14]. H. Kocayigit, A. Özdemir, M. Çetin and B. Arda according to Bishop's frame gave some characterization of timelike curves in 3-dimensional Minkowski space [16]. Bishop frame is also studied in E_1^3 by many mathematicians such as [5, 6, 13, 23].

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Equiform geometry is considered as extension of other geometries. Furthermore, an equiform frame is a generalization of the Frenet frame. The equiform geometry is of modest importance in comparison to the usual one, but the curves that emerge here may be considered as generalizations of well-known curves from other geometries and so could have been of scientific interest. The equiform geometry is defined in several spaces such as Galilean space [3,15], pseudo-Galilean space [1,8], Euclidean space [18] and Minkowski space [7,21].

In [11], timelike and null spherical curves in 3-dimensional Minkowski space are introduced. In [12], K. Iralslan studied spacelike normal curves in 3-dimensional Minkowski space E_1^3. Also, in [7], H.K. Elsayied et al. introduced the equiform spacelike and the timelike spherical curves according to the equiform frame in 3-dimensional Minkowski E_1^3.

This paper aims to study the spherical curves with regard to the equiform Bishop frame. The article is structured as follows. In Section 1, we provide context for the topic by introducing the Frenet frame and Frenet equations along a timelike unit-speed curve in 3-dimensional Minkowski space. In Section 2, for timelike curves in Minkowski space, we provide the equiform Bishop frame and its formulae. In Section 3, we define the equiform timelike spherical curve according to the Bishop frame in 3-dimensional Minkowski space and provide some characterization theorems for equiform timelike spherical curves.

1 Preliminaries

The three-dimensional Lorentz-Minkowski space E_1^3 is known as the 3-dimensional real space $\mathbb{R}^3$ with the metric g given by $g = -du_1^2 + du_2^2 + du_3^2$, where (u_1, u_2, u_3) is a system of coordinates in E_1^3. Let P be any vector in E_1^3. If $g(P,P) > 0$, or $P = 0$, then the vector P is said to be a spacelike vector. If $g(P,P) < 0$, then the vector P is said to be a timelike vector. If $g(P,P) = 0$, and $P \neq 0$, then the vector P is a lightlike (null) vector. In particular, the norm of a non-lightlike vector P is given by $\parallel P \parallel = \sqrt{|g(P,P)|}$, and P is called a unit vector if $\parallel P \parallel = 1$. Two vectors P and Q are said to be orthogonal if $g(P,Q) = 0$. For any $P, Q \in E_1^3$, Lorentzian vector product of P and Q is defined in [9,11,12,19,20] by

$$P \wedge Q = (p_3q_2 - p_2q_3, p_3q_1 - p_1q_3, p_1q_2 - p_2q_1),$$

where $P = (p_1, p_2, p_3), Q = (q_1, q_2, q_3)$ are any two vectors in E_1^3.

Overall, an arbitrary curve γ in E_1^3 is said to be a spacelike curve, timelike curve or null (lightlike) curve, if all of its velocity vectors γ' are spacelike, timelike, or null (lightlike), respectively. There are three and only three types of spacelike curves (spacelike with spacelike, timelike, or null principal normal). There is one and only one type of timelike curves and lightlike curves (timelike with spacelike principal normal and lightlike with spacelike principal normal). This classification follows from the close relationship between the casual characters of a vector subspace $U \subset E_1^3$ and its orthonormal complement $U^\perp$, i.e U is timelike, spacelike, or lightlike if and only if $U^\perp$ is spacelike, timelike, or lightlike [17].

A non-null curve γ is parameterized by arclength function s (i.e. unit speed curve) if $g(\gamma', \gamma') = \pm 1$. We say that $\{T_F, N_F, B_F\}$ are the moving Frenet frame along the curve $\gamma(s)$. Then, T_F, N_F, and B_F are the tangent, the principal normal, and the binormal vector fields for the curve $\gamma(s)$, respectively. Frenet formulae depend on the causal character of the curve $\gamma(s)$. For an arbitrary curve $\gamma(s)$ in the space E_1^3, the following Frenet formulae are given in [9,12].

If $\gamma(s)$ is a timelike curve in which the principal normal N_F is a spacelike vector, then the Frenet equations are

$$\frac{d}{ds}\begin{bmatrix} T_F \\ N_F \\ B_F \end{bmatrix} = \begin{bmatrix} 0 & h & 0 \\ h & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}\begin{bmatrix} T_F \\ N_F \\ B_F \end{bmatrix},$$

where $g(T_F,T_F)=-1$, $g(N_F,N_F)=1$, $g(B_F,B_F)=-1$, $g(T_F,N_F)=g(T_F,B_F)=g(N_F,B_F)=0$. If the curve γ is unit speed, then $h=\|\gamma''(s)\|$, $\tau=\|b'(s)\|$.

The pseudo-Riemannian sphere is defined by $S_1^2(p,r)=\{v\in E_1^3 : g(v-p,v-p)=r^2\}$, where p is a fixed point in E_1^3 and $r>0$ is constant. The pseudo-Riemannian hyperbolic space is defined by $H_0^2(p,r)=\{v\in E_1^3 : g(v-p,v-p)=-r^2\}$, where p is a fixed point in E_1^3 and $r>0$ is constant. The pseudo-Riemannian lightlike cone is defined by $C(p)=\{v\in E_1^3 : g(v-p,v-p)=0\}$, where p is a fixed point in E_1^3 [11,12].

The parallel transport frame or the Bishop frame is a way to define a moving frame that is also defined when the curve has a vanishing second derivative ($h=0$). Simply an orthonormal frame on a curve can be parallel transported by parallel transporting each component of the frame. For any curve, the Bishop frame is defined as we can find the tangent vector field for each point on the curve $t(s)$ and also we may choose any adequate arbitrary basis $(u(s),v(s))$ for the rest of the frame, so long as it lies in the normal plane orthogonal to $t(s)$ at each point. Moreover, suppose that the curve γ is parameterized by the arc length parameter, and u and v are C^1 unit vector fields that $v=t\wedge u$, along the curve γ so that $g(t,u)=g(t,v)=g(u,v)=0$, i.e. t, u, v will be an orthonormal frame as we move along the curve. If the curve has the property that $h\neq 0$ for any point of the curve, the Frenet frame is not defined at that point on the curve. Now, we want to put the extra condition $g(u',v)=0$. This means that the only change of u is in the direction of t. In this case, a Bishop frame will be defined even when a Frenet frame cannot (when there are points with vanishing its curvature). We call u and v the first and the second normal vectors, respectively. Therefore, we have the alternative frame equations [10,14].

If γ is a timelike curve with a spacelike normal vector u, then the Bishop formulae are

$$\frac{d}{ds}\begin{bmatrix} t \\ u \\ v \end{bmatrix} = \begin{bmatrix} 0 & h_1 & h_2 \\ h_1 & 0 & 0 \\ h_2 & 0 & 0 \end{bmatrix}\begin{bmatrix} t \\ u \\ v \end{bmatrix},$$

where $g(t,t)=-1$, $g(u,u)=1$, $g(v,v)=1$ and $h(s)=\sqrt{h_1^2+h_2^2}$, $\theta(s)=\arctan(h_2/h_1)$, $\tau(s)=d\theta/ds$ [14].

2 Equiform Bishop formulae for equiform timelike curves in E_1^3

In this section, the equiform geometry in three-dimensional Minkowski space E_1^3 is introduced. Equiform Bishop formulae for equiform timelike curves are investigated.

Definition 1. *Let $\gamma : I\to E_1^3$ be curve in three-dimensional Minkowski space E_1^3 parameterized by the the arc length s. Let $\eta(s)$ be the equiform parameter of $\gamma(s)$ defined as*

$$\eta=\int\frac{ds}{\omega}=\int h_1 ds,$$

where $\omega=1/h_1$ is the radius of curvature of the curve $\gamma(s)$.

Therefore,

$$\frac{ds}{d\eta} = \omega. \tag{1}$$

Define the equiform formulae of the curve γ parameterized by the invariant equiform parameter η in E_1^3. The equiform tangent, defined as $T = d\gamma/d\eta$, is called the equiform tangent vector field of the curve γ. From (1) we get

$$T = \frac{d\gamma}{d\eta} = \frac{d\gamma}{ds}\frac{ds}{d\eta} = \frac{d\gamma}{ds}\omega = \omega t. \tag{2}$$

Define the equiform principal normal vector field N_q and the equiform binormal vector field B_q according to Bishop frame by

$$N_q = \omega u, \quad B_q = \omega v. \tag{3}$$

Then, we can easily show that $\{T, N_q, B_q\}$ is an equiform invariant orthogonal frame, however, it is not an orthonormal frame of the curve γ.

Definition 2. *For the curve γ in E_1^3 the first equiform curvature is a function $H_1 : I \to R$ which is defined by $H_1 = \omega'$.*

Definition 3. *For the curve γ in E_1^3 the second equiform curvature is a function $H_2 : I \to R$ which is defined by $H_2 = h_2/h_1$.*

Theorem 1. *Let $\gamma(\eta)$ be an equiform timelike curve in which the principal normal N_q is the equiform spacelike vector, then the equiform Bishop formulae provided by*

$$\frac{d}{d\eta}\begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix} = \begin{bmatrix} H_1 & 1 & H_2 \\ 1 & H_1 & 0 \\ H_2 & 0 & H_1 \end{bmatrix}\begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix}, \tag{4}$$

where $g(T,T) = -\omega^2$, $g(N_q,N_q) = \omega^2$, $g(B_q,B_q) = \omega^2$, $g(T,N_q) = g(T,B_q) = g(N_q,B_q) = 0$.

Proof. Suppose that $\gamma(\eta)$ is an equiform timelike curve, then by using equations (1)–(3), we obtain

$$\begin{aligned}\frac{dT}{d\eta} = \frac{d(\omega t)}{d\eta} &= \frac{d\omega}{d\eta}t + \omega\frac{dt}{d\eta} = \frac{d\omega}{ds}\frac{ds}{d\eta}t + \omega\frac{dt}{ds}\frac{ds}{d\eta} = \omega'\omega t + \omega^2 t' \\ &= \omega' T + \omega^2(h_1 u + h_2 v) = \omega' T + N_q + \left(\frac{h_2}{h_1}\right)B_q = H_1 T + N_q + H_2 B_q.\end{aligned}$$

Similarly, we can deduce that

$$N_q' = T + H_1 N_q, \quad B_q' = H_2 T + H_1 B_q,$$

where the prime $'$ means $d/d\eta$. □

Corollary 1. *If γ is an equiform timelike curve according to the equiform Bishop frame, then we obtain the first equiform curvature and second equiform curvature, respectively, by*

$$H_1 = g(T,T') = g(N_q,N_q') = g(B_q,B_q'), \quad H_2 = g(B_q',T) = g(T',B_q).$$

3 Main Results

In this section, we define an equiform timelike spherical curve according to Bishop frame in 3-dimensional Minkowski space E_1^3 and provide some characterization theorems for equiform timelike spherical curves in E_1^3.

Definition 4. *The equiform timelike curve $\gamma(\eta)$ is said to be a spherical curve according to equiform Bishop frame in 3-dimensional Minkowski space if its position vector always lies in its normal plane spanned by $\{N_q, B_q\}$.*

Theorem 2. *Let $\gamma = \gamma(\eta)$ be an equiform timelike curve in E_1^3 with an equiform spacelike principal normal vector N_q and with an equiform curvatures $H_1(\eta) > 0$, $H_2(\eta) \neq 0$ for each $\eta \in I \subset R$. Then the curve γ is normal if and only if the equiform curvatures $H_1(\eta)$, $H_2(\eta)$ satisfy $CH_2 = \omega - A$ or $CH_2' = \omega H_1$, where A and C are constants.*

Proof. Let $\gamma(\eta)$ be an equiform timelike normal curve in E_1^3 with an equiform spacelike principal normal vector N_q. Then we can write $\gamma(\eta)$ as

$$\gamma(\eta) = \lambda(\eta)N_q(\eta) + \mu(\eta)B_q(\eta).$$

Differentiating the above equation with respect to η and using equations (4), we find

$$\begin{aligned} T &= \lambda' N_q + \lambda(T + H_1 N_q) + \mu' B_q + \mu(H_2 T + H_1 B_q) \\ &= (\lambda + \mu H_2)T + (\lambda' + \lambda H_1)N_q + (\mu' + \mu H_1)B_q. \end{aligned}$$

Then, we deduce that

$$\lambda + \mu H_2 = 1, \quad \lambda' + \lambda H_1 = 0, \quad \mu' + \mu H_1 = 0. \tag{5}$$

By solving equations (5) by separation of variables, we obtain

$$\lambda = \frac{A}{\omega}, \quad \mu = \frac{C}{\omega}, \tag{6}$$

where A and C are constants. Therefore,

$$\gamma(\eta) = \frac{A}{\omega}N_q + \frac{C}{\omega}B_q. \tag{7}$$

Substitution from equation (6) into the first equation in (5), we obtain

$$CH_2 = \omega - A. \tag{8}$$

By differentiating equation (8) with respect to η, we get

$$CH_2' = \omega H_1. \tag{9}$$

Conversely, suppose that equations (8) and (9) hold. Then

$$\begin{aligned} \frac{d}{d\eta}\Big(\gamma(\eta) - \frac{A}{\omega}N_q - \frac{C}{\omega}B_q\Big) &= T + \frac{A}{\omega^2}\omega H_1 N_q - \frac{A}{\omega}(T + H_1 N_q) + \frac{C}{\omega^2}\omega H_1 B_q \\ &\quad - \frac{C}{\omega}(H_2 T + H_1 B_q) = \Big(1 - \frac{A}{\omega} - \frac{CH_2}{\omega}\Big)T = 0. \end{aligned}$$

Consequently, γ is a spherical curve or congruent to a spherical curve, which completes the proof. □

Theorem 3. *Let $\gamma = \gamma(\eta)$ be an equiform timelike curve in E_1^3 in which the principal normal is equiform spacelike vector and the equiform curvatures $H_1(\eta) > 0$, $H_2(\eta) \neq 0$ for each $\eta \in I \subset R$. Then γ is a normal curve if and only if the equiform principal normal N_q and the equiform binormal B_q components with the equiform position vector are given respectively by $g(\gamma, N_q) = A\omega$, $g(\gamma, B_q) = C\omega$, where A and C are constants.*

Proof. Let $\gamma(\eta)$ be an equiform timelike normal curve in E_1^3 with an equiform spacelike principal normal vector N_q. Then equation (7) holds and we get

$$g(\gamma, N_q) = \frac{A}{\omega} g(N_q, N_q) + \frac{C}{\omega} g(B_q, N_q) = A\omega.$$

Similarly, $g(\gamma, B_q) = C\omega$.

Conversely, assume that the equations $g(\gamma, N_q) = A\omega$ and $g(\gamma, B_q) = C\omega$ hold. Differentiating the equation $g(\gamma, N_q) = A\omega$ with respect to η, we have

$$g(T, N_q) + g(\gamma, N_q') = A\omega'.$$

Applying equations (4), we get

$$g(\gamma, T) + H_1 g(\gamma, N_q) = A\omega H_1,$$

which implies $g(\gamma, T) = 0$.

Similarly, differentiating the equation $g(\gamma, B_q) = C\omega$ with respect to η, we have

$$g(T, B_q) + g(\gamma, B_q') = C\omega'$$

and

$$H_2 g(\gamma, T) + H_1 g(\gamma, B_q) = C\omega H_1,$$

which implies that $g(\gamma, T) = 0$. Thus, γ is a spherical curve and the proof is completed. $\square$

Theorem 4. *Let $\gamma = \gamma(\eta)$ be an equiform timelike spherical curve in E_1^3 with an equiform spacelike or an equiform timelike principal normal vector N_q in which the equiform curvatures $H_1(\eta) > 0$, $H_2(\eta) \neq 0$ for every $\eta \in I \subset R$. For the equiform spacelike curve, the position vector is an equiform spacelike vector if and only if the spherical curve γ is located on $S_1^2(p, r)$ and the second equiform curvature $H_2(\eta)$ satisfy $\pm\sqrt{r^2 - A^2} H_2 = \omega - A$, where A is constant.*

Proof. Let γ be an equiform timelike spherical curve and the position vector is also equiform spacelike. Then we have $g(\gamma, \gamma) = r^2$. Since equation (7) holds, then

$$g(\gamma, \gamma) = \frac{A^2}{\omega^2} g(N_q, N_q) + \frac{C^2}{\omega^2} g(B_q, B_q) = A^2 + C^2 = r^2.$$

Thus, the constant C is given by $C = \pm\sqrt{r^2 - A^2}$. Substituting from this equation into the equation (8), we obtain

$$\pm\sqrt{r^2 - A^2} H_2 = \omega - A. \tag{10}$$

On the other hand, let us consider the vector

$$p = \gamma(\eta) - \frac{A}{\omega} N_q - \frac{C}{\omega} B_q. \tag{11}$$

Calculating the differential of equation (11) with respect to η and using equations (4), we find $p' = 0$ and therefore $p =$ constant. Then

$$g(\gamma - p, \gamma - p) = \frac{A^2}{\omega^2}(\omega^2) + \frac{C^2}{\omega^2}(\omega^2) = A^2 + C^2 = r^2.$$

This means that γ is located on $S_1^2(p, r)$ with center p and radius r.

Conversely, assume that γ located on $S_1^2(p, r)$ with center p, p is constant, and the second equiform curvature $H_2(\eta)$ satisfies equation (10), then we can write $g(\gamma - p, \gamma - p) = r^2$. By differentiating this equation with respect to η three times, we get

$$g(\gamma - p, T) = 0, \quad g(\gamma - p, N_q) = \omega^2\Big(\frac{H_1 H_2}{H_2'} - 1\Big), \quad g(\gamma - p, B_q) = \omega^2 \frac{H_1}{H_2'}.$$

Therefore, we can write the vector $\gamma - p$ as

$$\gamma - p = 0T + \Big(1 - \frac{H_1 H_2}{H_2'}\Big)N_q + \frac{H_1}{H_2'}B_q.$$

This signifies that a curve is congruent to a spherical curve up to a translation for vector p. Let us put $p = 0$ and using $\pm\sqrt{r^2 - A^2}H_2 = \omega - A$, we can easily find $g(\gamma, \gamma) = r^2$, which completes the proof. $\square$

Conclusion

This paper provides a comprehensive study of equiform Bishop formulae for equiform timelike curves in 3-dimensional Minkowski space, specifically exploring equiform timelike spherical curves as defined by the equiform Bishop frame. Our findings highlight a necessary and sufficient condition for an equiform timelike curve to be an equiform timelike spherical curve. In addition, we have offered various characterizations of equiform spherical curves in 3-dimensional Minkowski space and demonstrated their timelike nature with an equiform spacelike principal normal vector.

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Ель Шаркаві Н., Чезарано К., Дмитришин Р., Ель Шаркаві А. *Часоподібні сферичні криві відповідно до еквіформного репера Бішопа у тривимірному просторі Мінковського* // Карпатські матем. публ. — 2023. — Т.15, №2. — C. 388–395.

У цій статті вивчаються еквіформні формули Бішопа для еквіформних часоподібних кривих у тривимірному просторі Мінковського, де еквіформні часоподібні сферичні криві визначаються відповідно до еквіформного реперу Бішопа. Встановлено необхідну і достатню умову того, щоб еквіформна часоподібна крива була еквіформною часоподібною сферичною кривою. Крім того, наведено деякі характеристики еквіформних сферичних кривих у тривимірному просторі Мінковського, які є часоподібними до еквіформного просторовоподібного головного вектора нормалі.

Ключові слова і фрази: простір Мінковського, репер Бішопа, сферична крива, еквіформна кривина.