ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2023, **15** (2), 411–419
doi:10.15330/cmp.15.2.411-419

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2023, Т.15, №2, С.411–419

Algebras of weakly symmetric functions on spaces of Lebesgue measurable functions

Burtnyak I.V., Chopyuk Yu.Yu., Vasylyshyn S.I., Vasylyshyn T.V.

In this work, we investigate algebras of block-symmetric and weakly symmetric polynomials and analytic functions on complex Banach spaces of Lebesgue measurable functions, for which the pth power of the absolute value is Lebesgue integrable, where $p \in [1, +\infty)$, and Lebesgue measurable essentially bounded functions on $[0, 1]$. We construct generating systems of algebras of all weakly symmetric continuous complex-valued polynomials on these spaces. Also we establish conditions under which sets of weakly symmetric analytic functions are algebras.

Key words and phrases: symmetric function, weakly symmetric function, analytic function on an infinite dimensional space, space of Lebesgue measurable functions.

Vasyl Stefanyk Precarpathian National University, 57 Shevchenka str., 76018, Ivano-Frankivsk, Ukraine
E-mail: ivan.burtnyak@pnu.edu.ua (Burtnyak I.V.), ur.chopiuk@gmail.com (Chopyuk Yu.Yu.), sv.halushchak@ukr.net (Vasylyshyn S.I.), taras.vasylyshyn@pnu.edu.ua (Vasylyshyn T.V.)

Introduction

The notion of symmetry is very useful in the study of analytic functions on infinite dimensional spaces (see [2, 3, 5, 6, 8, 10–12, 14, 21, 26]), especially, in the investigations of spectra of topological algebras of these functions (see [1, 4, 7, 9, 15]). A large number of algebras of symmetric analytic functions are relatively easy to study because they are finitely or countably generated.

In [24], the authors introduced the notion of weak symmetry. The main advantage of this notion is following. On the one hand, in contrast to the case of symmetric analytic functions, the completions of algebras of weakly symmetric analytic functions can contain functions that are not weakly symmetric. On the other hand, these algebras remain to be finitely or countably generated. So, methods developed for algebras of symmetric analytic functions can be applied to much wider algebras of analytic functions.

In this work, we investigate algebras of block-symmetric and weakly symmetric polynomials and analytic functions on complex Banach spaces $L_p[0, 1]$ and $L_\infty[0, 1]$.

1 Preliminaries

Let $\mathbb{N}$ be the set of all positive integers. Let $\mathbb{Z}_+$ be the set of all nonnegative integers. Let μ be the Lebesgue measure on $[0, 1]$.

УДК 517.98
2020 *Mathematics Subject Classification:* 46G20, 46J20.

This research was funded by the National Research Foundation of Ukraine, 2020.02/0025, 0120U103996.

Symmetric and weakly symmetric mappings. Let A, B be arbitrary nonempty sets. Let S be an arbitrary fixed set of mappings that act from A to itself. A mapping $f : A \to B$ is called S-symmetric if $f(s(a)) = f(a)$ for every $a \in A$ and $s \in S$.

Let Λ be some index set. For every $\alpha \in \Lambda$, let S_α be a set of mappings that act from A to itself. Let $\mathcal{S} = \{S_\alpha : \alpha \in \Lambda\}$ be a family of all sets S_α. A mapping $f : A \to B$ is called $\mathcal{S}$-weakly symmetric if there exists $\alpha \in \Lambda$ such that f is S_α-symmetric.

In this work, we are interested in families $\mathcal{S}$ that satisfy the following property.

Property 1. For every $\alpha, \beta \in \Lambda$ there exists $\gamma \in \Lambda$ such that $S_\gamma \subset S_\alpha \cap S_\beta$.

The algebra $H_b(X)$. Let X be a complex Banach space. Let $H_b(X)$ be the Fréchet algebra of all entire functions $f : X \to \mathbb{C}$ which are bounded on bounded sets endowed with the topology of uniform convergence on bounded sets.

Let

$$\|f\|_r = \sup_{\|x\| \le r} |f(x)|$$

for $f \in H_b(X)$ and $r > 0$. The topology of $H_b(X)$ can be generated by an arbitrary set of norms $\{\|\cdot\|_r : r \in \Gamma\}$, where Γ is any unbounded subset of $(0, +\infty)$.

The algebra $H_{b,S}(X)$. Let X be a complex Banach space. Let S be a set of operators on X. Let $H_{b,S}(X)$ be the subalgebra of all S-symmetric elements of $H_b(X)$. By [19, Lemma 3], $H_{b,S}(X)$ is closed in $H_b(X)$. So, $H_{b,S}(X)$ is a Fréchet algebra.

The group of bijections $\Xi_{[0,1]}$. Let $\Xi_{[0,1]}$ be the set of all bijections $\sigma : [0,1] \to [0,1]$ such that both σ and σ^{-1} are measurable and preserve the Lebesgue measure, i.e. for every Lebesgue measurable set $E \subset [0,1]$, both sets $\sigma(E)$ and $\sigma^{-1}(E)$ are Lebesgue measurable and $\mu(\sigma(E)) = \mu(\sigma^{-1}(E)) = \mu(E)$. Note that $\Xi_{[0,1]}$ is a group with respect to the operation of composition.

The group of bijections $\Xi^{(n)}_{[0,1]}$. Let $n \in \mathbb{N}$. Let $\Xi^{(n)}_{[0,1]}$ be the set of all bijections $\sigma \in \Xi_{[0,1]}$ such that

$$\sigma(t + 1/n) = \sigma(t) + 1/n \tag{1}$$

for every $t \in [0, 1 - 1/n]$. By [25, Proposition 2], $\Xi^{(n)}_{[0,1]}$ is a subgroup of $\Xi_{[0,1]}$.

The group of operators $S(\Xi, X^n)$. Let Ξ be an arbitrary subgroup of $\Xi_{[0,1]}$. Let X be an arbitrary linear space of equivalence classes with respect to the equivalence relation $x \sim y \Leftrightarrow x \overset{\text{a.e.}}{=} y$ of Lebesgue measurable functions on $[0,1]$ such that $x \circ \sigma$ belongs to X for every $x \in X$ and $\sigma \in \Xi$. Let X^n be the nth Cartesian power of X, where $n \in \mathbb{N}$. For $\sigma \in \Xi$, let the operator s_σ be defined by

$$s_\sigma : X^n \ni (x_1, \ldots, x_n) \mapsto (x_1 \circ \sigma, \ldots, x_n \circ \sigma) \in X^n.$$

Let

$$S(\Xi, X^n) = \{s_\sigma : \sigma \in \Xi\}.$$

It can be verified that $S(\Xi, X^n)$ is a group of operators on X^n. If the context is clear, we shall write $S(\Xi)$ instead of $S(\Xi, X^n)$.

The Cartesian power of $L_p[0,1]$. Let $L_p[0,1]$, where $p \in [1,+\infty)$, be the complex Banach space of all Lebesgue measurable functions $x : [0,1] \to \mathbb{C}$ for which the pth power of the absolute value is Lebesgue integrable with norm

$$\|x\|_p = \Big(\int_{[0,1]} |x(t)|^p \, dt\Big)^{1/p}.$$

Let $\big(L_p[0,1]\big)^n$, where $n \in \mathbb{N}$, be the nth Cartesian power of $L_p[0,1]$ with norm

$$\|x\|_{p,n} = \Big(\sum_{s=1}^{n} \int_{[0,1]} |x_s(t)|^p \, dt\Big)^{1/p},$$

where $x = (x_1, \ldots, x_n) \in \big(L_p[0,1]\big)^n$.

The Cartesian power of $L_\infty[0,1]$. Let $L_\infty[0,1]$ be the complex Banach space of all Lebesgue measurable essentially bounded functions $x : [0,1] \to \mathbb{C}$ with norm

$$\|x\|_\infty = \operatorname*{ess\,sup}_{t\in[0,1]} |x(t)|.$$

Let $\big(L_\infty[0,1]\big)^n$, where $n \in \mathbb{N}$, be the nth Cartesian power of $L_\infty[0,1]$ with norm

$$\|x\|_{\infty,n} = \max_{1\le j\le n} \|x_j\|_\infty,$$

where $x = (x_1, \ldots, x_n) \in \big(L_\infty[0,1]\big)^n$.

Symmetric functions on Cartesian powers of $L_p[0,1]$ and $L_\infty[0,1]$. Let X be an element of the set

$$\mathcal{L} = \big\{L_p[0,1] : \ p \in [1,+\infty)\big\} \cup \big\{L_\infty[0,1]\big\}, \tag{2}$$

i.e. X is equal to $L_p[0,1]$ or $L_\infty[0,1]$. By the definition of the group of operators $S(\Xi_{[0,1]}, X^n)$, a function f on X^n is $S(\Xi_{[0,1]}, X^n)$-symmetric if

$$f(x_1 \circ \sigma, \ldots, x_n \circ \sigma) = f(x_1, \ldots, x_n)$$

for every $(x_1, \ldots, x_n) \in X^n$ and $\sigma \in \Xi_{[0,1]}$.

Note, that in works [16–20,22,23], $S(\Xi_{[0,1]}, X^n)$-symmetric functions on Cartesian powers of $L_p[0,1]$ and $L_\infty[0,1]$ are called symmetric. In this work, we also call these functions symmetric if the context is clear.

Let

$$M_{X,n} = \begin{cases} \big\{k \in \mathbb{Z}_+^n : \ 1 \le |k| \le p\big\}, & \text{if } X = L_p[0,1], \\ \{k \in \mathbb{Z}_+^n : \ |k| \ge 1\}, & \text{if } X = L_\infty[0,1], \end{cases} \tag{3}$$

where $|k| = k_1 + \ldots + k_n$ for $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$.

For every multi-index $k \in M_{X,n}$, let us define the mapping $R_{k,X^n} : X^n \to \mathbb{C}$ by

$$R_{k,X^n}(y) = \int_{[0,1]} \prod_{\substack{s=1 \\ k_s>0}}^{n} (y_s(t))^{k_s} \, dt, \tag{4}$$

where $y = (y_1, \ldots, y_n) \in X^n$. Note that R_{k,X^n} is a symmetric continuous $|k|$-homogeneous polynomial.

For every nonempty finite set $M \subset \mathbb{Z}_+^n$ and for every mapping $l : M \to \mathbb{Z}_+$, let

$$\varkappa(l, M) = \sum_{k \in M} |k| l(k). \tag{5}$$

For $N \in \mathbb{N}$, let

$$M_{X,n,N} = \{k \in M_{X,n} : \ |k| \leq N\}, \tag{6}$$

where $M_{X,n}$ is defined by (3).

Theorem 1 ([16, Theorem 2.10] and [18, Theorem 2]). *Let $n, N \in \mathbb{N}$. Let $X \in \mathcal{L}$, where $\mathcal{L}$ is defined by (2). Every symmetric continuous N-homogeneous polynomial $P : X^n \to \mathbb{C}$ can be uniquely represented as*

$$P(y) = \sum_{\substack{l : M_{X,n,N} \to \mathbb{Z}_+ \\ \varkappa(l, M_{X,n,N}) = N}} \alpha_l \prod_{\substack{k \in M_{X,n,N} \\ l(k) > 0}} \left(R_{k,X^n}(y)\right)^{l(k)},$$

where $y \in X^n$, $\alpha_l \in \mathbb{C}$, R_{k,X^n} is defined by (4)*, $\varkappa$ is defined by* (5) *and $M_{X,n,N}$ is defined by* (6).

Block-symmetric functions on $L_p[0,1]$ and $L_\infty[0,1]$. Let $X \in \mathcal{L}$, where $\mathcal{L}$ is defined by (2). A function f on X, which is $S\big(\Xi_{[0,1]}^{(n)}, X\big)$-symmetric, we call n-block-symmetric.

The isomorphism of Fréchet algebras $H_{b,S(\Xi_{[0,1]})}(X^n)$ and $H_{b,S(\Xi_{[0,1]}^{(n)})}(X)$. Let $X \in \mathcal{L}$, where $\mathcal{L}$ is defined by (2). Let $n \in \mathbb{N}$. Let us define the function $\lambda_{[a,b]} : [a,b] \to [0,1]$ by

$$\lambda_{[a,b]}(t) = \frac{t-a}{b-a}. \tag{7}$$

Note that $\lambda_{[a,b]}$ is a bijection.

For $x = (x_1, \ldots, x_n) \in X^n$, let us define the function $\iota_{X,n}(x) : [0,1] \to \mathbb{C}$ by

$$\iota_{X,n}(x)(t) = \begin{cases} \big(x_j \circ \lambda_{[(j-1)/n, j/n]}\big)(t), & \text{if } t \in \big[(j-1)/n, j/n\big), \text{ where } j \in \{1, \ldots, n\}, \\ 0, & \text{if } t = 1, \end{cases} \tag{8}$$

where $\lambda_{[(j-1)/n, j/n]}$ is defined by (7).

Let us define the mapping $\iota_{X,n} : X^n \to X$ by

$$\iota_{X,n} : X^n \ni x \mapsto \iota_{X,n}(x) \in X, \tag{9}$$

where $\iota_{X,n}(x)$ is defined by (8).

By [25, Proposition 5], the mapping $\iota_{X,n}$, defined by (9), is an isomorphism.

Theorem 2 ([25, Theorem 5]). *Let $n \in \mathbb{N}$ and X be equal to $L_p[0,1]$ or $L_\infty[0,1]$, where $p \in [1, +\infty)$. The mapping*

$$I_{X,n} : H_{b,S\left(\Xi_{[0,1]}^{(n)}\right)}(X) \ni g \mapsto g \circ \iota_{X,n} \in H_{b,S\left(\Xi_{[0,1]}\right)}(X^n) \tag{10}$$

is an isomorphism of Fréchet algebras $H_{b,S\left(\Xi_{[0,1]}^{(n)}\right)}(X)$ and $H_{b,S\left(\Xi_{[0,1]}\right)}(X^n)$, where $\iota_{X,n}$ is defined by (9).

Note that, by (10),

$$I_{X,n}(f) = f \circ \iota_{p,n}^{-1} \tag{11}$$

for every $f \in H_{b,S\left(\Xi_{[0,1]}\right)}(X^n)$.

2 Block-symmetric polynomials on $L_p[0,1]$ and $L_\infty[0,1]$

Lemma 1. *Let $n, N \in \mathbb{N}$. Let $X \in \mathcal{L}$, where $\mathcal{L}$ is defined by* (2)*. Then*

a) for every continuous N-homogeneous n-block-symmetric polynomial $P : X \to \mathbb{C}$, the function $I_{X,n}(P)$, where $I_{X,n}$ is defined by (10)*, is a continuous N-homogeneous symmetric polynomial on X^n;*

b) for every continuous N-homogeneous symmetric polynomial $P : X^n \to \mathbb{C}$, the function $I_{X,n}^{-1}(P)$ is a continuous N-homogeneous n-block-symmetric polynomial on X.

Proof. a) Note that $P \in H_{b,\mathcal{S}\left(\Xi_{[0,1]}^{(n)}\right)}(X)$. Therefore $I_{X,n}(P) \in H_{b,\mathcal{S}\left(\Xi_{[0,1]}\right)}(X^n)$. Consequently, $I_{X,n}(P)$ is a continuous $\mathcal{S}\left(\Xi_{[0,1]}, X^n\right)$-symmetric function on X^n. By (10), $I_{X,n}(P) = P \circ \iota_{p,n}$. Consequently, taking into account that P is an N-homogeneous polynomial and $\iota_{p,n}$ is a continuous linear operator, $I_{X,n}(P)$ is an N-homogeneous polynomial.

b) The proof follows a similar approach as the proof of the previous item. □

Let us define some "elementary" n-block-symmetric polynomials. Let $n \in \mathbb{N}$ and $X \in \mathcal{L}$. Let $k \in M_{X,n}$, where $M_{X,n}$ is defined by (3). Note that $R_{k,X^n} \in H_{b,\mathcal{S}\left(\Xi_{[0,1]}\right)}(X^n)$, where R_{k,X^n} is defined by (4). Let

$$G_{k,n,X} = I_{X,n}^{-1}\left(R_{k,X^n}\right), \tag{12}$$

where $I_{X,n}$ is defined by (10). By Lemma 1, $G_{k,n,X}$ is a continuous n-block-symmetric $|k|$-homogeneous polynomial. By (11),

$$G_{k,n,X} = R_{k,X^n} \circ \iota_{X,n}^{-1}, \tag{13}$$

where $\iota_{X,n}$ is defined by (9).

By (13), (8) and (4), taking into account (7), we have

$$G_{k,n,X}(x) = \int_{[0,1]} \prod_{\substack{s=1\\ k_s>0}}^{n} \left(\left(x \circ \lambda^{-1}_{[(s-1)/n, s/n]}\right)(t)\right)^{k_s} dt = \int_{[0,1]} \prod_{\substack{s=1\\ k_s>0}}^{n} \left(x\left(\frac{s-1+t}{n}\right)\right)^{k_s} dt.$$

Theorem 3. *Let $N \in \mathbb{N}$. Let $X \in \mathcal{L}$, where $\mathcal{L}$ is defined by* (2)*. Every n-block-symmetric continuous N-homogeneous polynomial $P : X \to \mathbb{C}$ can be uniquely represented as*

$$P(x) = \sum_{\substack{l: M_{X,n,N} \to \mathbb{Z}_+\\ \varkappa(l, M_{X,n,N}) = N}} \alpha_l \prod_{\substack{k \in M_{X,n,N}\\ l(k)>0}} \left(G_{k,n,X}(x)\right)^{l(k)},$$

where $x \in X$, $\alpha_l \in \mathbb{C}$, $G_{k,n,X}$ is defined by (12)*, $\varkappa$ is defined by* (5) *and $M_{X,n,N}$ is defined by* (6)*.*

Proof. By Lemma 1, $I_{X,n}(P)$ is a continuous N-homogeneous symmetric polynomial on X^n. Therefore, by Theorem 1, $I_{X,n}(P)$ can be uniquely represented in the form

$$I_{X,n}(P) = \sum_{\substack{l: M_{X,n,N} \to \mathbb{Z}_+\\ \varkappa(l, M_{X,n,N}) = N}} \alpha_l \prod_{\substack{k \in M_{X,n,N}\\ l(k)>0}} \left(R_{k,X^n}\right)^{l(k)}, \tag{14}$$

where $\alpha_l \in \mathbb{C}$. Consequently, taking into account that $I_{X,n}$ is an isomorphism, we obtain

$$P = I_{X,n}^{-1}\left(\sum_{\substack{l:M_{X,n,N}\to\mathbb{Z}_+\\ \varkappa(l,M_{X,n,N})=N}} \alpha_l \prod_{\substack{k\in M_{X,n,N}\\ l(k)>0}} \left(R_{k,X^n}\right)^{l(k)}\right) = \sum_{\substack{l:M_{X,n,N}\to\mathbb{Z}_+\\ \varkappa(l,M_{X,n,N})=N}} \alpha_l \prod_{\substack{k\in M_{X,n,N}\\ l(k)>0}} \left(I_{X,n}^{-1}(R_{k,X^n})\right)^{l(k)}$$
$$= \sum_{\substack{l:M_{X,n,N}\to\mathbb{Z}_+\\ \varkappa(l,M_{X,n,N})=N}} \alpha_l \prod_{\substack{k\in M_{X,n,N}\\ l(k)>0}} \left(G_{k,n,X}\right)^{l(k)}. \tag{15}$$

Let us show that the representation (15) is unique. Suppose there exists another representation

$$P = \sum_{\substack{l:M_{X,n,N}\to\mathbb{Z}_+\\ \varkappa(l,M_{X,n,N})=N}} \beta_l \prod_{\substack{k\in M_{X,n,N}\\ l(k)>0}} \left(G_{k,n,X}\right)^{l(k)},$$

where $\beta_l \in \mathbb{C}$. Then

$$I_{X,n}(P) = I_{X,n}\left(\sum_{\substack{l:M_{X,n,N}\to\mathbb{Z}_+\\ \varkappa(l,M_{X,n,N})=N}} \beta_l \prod_{\substack{k\in M_{X,n,N}\\ l(k)>0}} \left(G_{k,n,X}\right)^{l(k)}\right) = \sum_{\substack{l:M_{X,n,N}\to\mathbb{Z}_+\\ \varkappa(l,M_{X,n,N})=N}} \beta_l \prod_{\substack{k\in M_{X,n,N}\\ l(k)>0}} \left(I_{X,n}(G_{k,n,X})\right)^{l(k)}$$
$$= \sum_{\substack{l:M_{X,n,N}\to\mathbb{Z}_+\\ \varkappa(l,M_{X,n,N})=N}} \beta_l \prod_{\substack{k\in M_{X,n,N}\\ l(k)>0}} \left(R_{k,X^n}\right)^{l(k)}.$$

But, the representation (14) is unique. So, $a_l = \beta_l$. Consequently, the representation (15) is unique. This completes the proof. □

Let us construct linear dependencies between block-symmetric polynomials. For $n, q \in \mathbb{N}$, $s \in \{1, \ldots, q\}$ and $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$, let us define $k^{(n,q,s)} \in \mathbb{Z}_p^{nq}$ by

$$k^{(n,q,s)} = \big(\underbrace{0,\ldots,0}_{s-1}, k_1, \underbrace{0,\ldots,0}_{q-s}, \underbrace{0,\ldots,0}_{s-1}, k_2, \underbrace{0,\ldots,0}_{q-s}, \ldots, \underbrace{0,\ldots,0}_{s-1}, k_n, \underbrace{0,\ldots,0}_{q-s}\big).$$

Note that $\left|k^{(n,q,s)}\right| = |k|$. Consequently, if $k \in M_{X,n,N}$ for some $X \in \mathcal{L}$ and $N \in \mathbb{N}$, then, taking into account (3) and (6), we have $k^{(n,q,s)} \in M_{X,nq,N}$.

It can be checked that the following dependency

$$G_{k,n,X} = \sum_{s=1}^{q} G_{k^{(n,q,s)},nq,X} \tag{16}$$

holds for every $X \in \mathcal{L}$ and $n, q \in \mathbb{N}$.

3 Weakly symmetric functions on $L_p[0,1]$ and $L_\infty[0,1]$

Let $X \in \mathcal{L}$, where L is defined by (2). For $\mathcal{N} \subset \mathbb{N}$, let

$$\mathcal{S}_{\mathcal{N},X} = \left\{S\left(\Xi_{[0,1]}^{(n)}, X\right) : n \in \mathcal{N}\right\}. \tag{17}$$

We consider $\mathcal{S}_{\mathcal{N},X}$-weakly symmetric functions on X.

Let us establish some auxiliary results.

Lemma 2. *Let $m, n \in \mathbb{N}$ be such that n is a divisor of m. Then $\Xi^{(n)}_{[0,1]} \supset \Xi^{(m)}_{[0,1]}$.*

Proof. Let $\sigma \in \Xi^{(m)}_{[0,1]}$. Let us show that $\sigma \in \Xi^{(n)}_{[0,1]}$. Since $\sigma \in \Xi^{(m)}_{[0,1]}$, it follows that $\sigma \in \Xi_{[0,1]}$ and, by (1),

$$\sigma(t + 1/m) = \sigma(t) + 1/m \tag{18}$$

for every $t \in [0, 1 - 1/m]$. Since n is a divisor of m, there exists $k \in \mathbb{N}$ such that $n = m/k$. Consequently, taking into account (18), we obtain

$$\sigma(t + 1/n) = \sigma(t + k/m) = \sigma\big(t + (k-1)/m + 1/m\big) = \sigma\big(t + (k-1)/m\big) = \ldots = \sigma(t)$$

for every $t \in [0, 1 - 1/n]$. Thus, $\sigma \in \Xi^{(n)}_{[0,1]}$. This completes the proof. □

Lemma 2 implies the following assertion.

Corollary 1. *Let $m, n \in \mathbb{N}$ be such that n is a divisor of m. Then $S\big(\Xi^{(n)}_{[0,1]}, X\big) \supset S\big(\Xi^{(m)}_{[0,1]}, X\big)$, where $X \in \mathcal{L}$.*

Let us consider subsets $\mathcal{N} \subset \mathbb{N}$ with the following property.

Property 2. For every $k, l \in \mathcal{N}$ there exists $m \in \mathcal{N}$ such that m is a common multiplier of k and l.

Corollary 1 implies the following assertion.

Corollary 2. *Let $X \in \mathcal{L}$, where $\mathcal{L}$ is defined by* (2)*. Let $\mathcal{N}$ be a subset of $\mathbb{N}$ that satisfies Property 2. Then the family $\mathcal{S}_{\mathcal{N},X}$, defined by* (17)*, satisfies Property 1.*

Theorem 4. *Let $X \in \mathcal{L}$, where $\mathcal{L}$ is defined by* (2)*. Let $\mathcal{N}$ be a subset of $\mathbb{N}$ that satisfies Property 2. Then the set of all $\mathcal{S}_{\mathcal{N},X}$-weakly symmetric elements of some algebra of functions on X is an algebra.*

Proof. By [24, Theorem 1], the subset of all $\mathcal{S}$-weakly symmetric elements of some algebra of functions on some nonempty set is an algebra if the family $\mathcal{S}$ satisfies Property 1. Consequently, taking into account Corollary 2, if $\mathcal{N} \subset \mathbb{N}$ satisfies Property 2, then the set of all $\mathcal{S}_{\mathcal{N},X}$-weakly symmetric elements of some algebra of functions on X is an algebra. □

Theorem 4 implies the following corollary.

Corollary 3. *Let $X \in \mathcal{L}$, where $\mathcal{L}$ is defined by* (2)*. Let $\mathcal{N}$ be a subset of $\mathbb{N}$ that satisfies Property 2. Then*

a) the set $\mathcal{P}_{\mathcal{S}_{\mathcal{N},X}-w.s.}(X)$ of all $\mathcal{S}_{\mathcal{N},X}$-weakly symmetric continuous complex-valued polynomials on X is an algebra;

b) the set $H_{b,\mathcal{S}_{\mathcal{N},X}-w.s.}(X)$ of all $\mathcal{S}_{\mathcal{N},X}$-weakly symmetric elements of $H_b(X)$ is an algebra.

Theorem 5. *Let $X \in \mathcal{L}$, where $\mathcal{L}$ is defined by* (2)*. Let $\mathcal{N}$ be a subset of $\mathbb{N}$ that satisfies Property 2. Then the set*

$$\big\{G_{k,n,X} : \ n \in \mathcal{N}, k \in M_{X,n}\big\} \tag{19}$$

is a generating system of the algebra $\mathcal{P}_{\mathcal{S}_{\mathcal{N},X}-w.s.}(X)$ of all $\mathcal{S}_{\mathcal{N},X}$-weakly symmetric continuous complex-valued polynomials on X.

Proof. Let $P \in \mathcal{P}_{\mathcal{S}_{\mathcal{N},X}-w.s.}(X)$. Then

$$P = P_0 + P_1 + \ldots + P_m, \tag{20}$$

where $m \in \mathbb{N}$, $P_0 \in \mathbb{C}$ and P_j is a continuous j-homogeneous polynomial for every $j \in \{1, \ldots, m\}$. Since P is $\mathcal{S}_{\mathcal{N},X}$-weakly symmetric, there exists $n \in \mathcal{N}$ such that P is $S\big(\Xi^{(n)}_{[0,1]}, X\big)$-symmetric, i.e. P is n-block-symmetric. Therefore, taking into account the Cauchy integral formula (see [13, Corollary 7.3]), P_j is n-block-symmetric for every $j \in \{1, \ldots, m\}$. Thus, P_j is n-block-symmetric continuous j-homogeneous polynomial for every $j \in \{1, \ldots, m\}$. Consequently, by (20) and by Theorem 3, P can be represented in the form

$$P = P_0 + \sum_{j=1}^{m} \sum_{\substack{l: M_{X,n,j} \to \mathbb{Z}_+ \\ \varkappa(l, M_{X,n,j}) = j}} \alpha_l^{(j)} \prod_{\substack{k \in M_{X,n,j} \\ l(k) > 0}} \big(G_{k,n,X}(x)\big)^{l(k)},$$

where $x \in X$ and $\alpha_l^{(j)} \in \mathbb{C}$. Thus, the set (19) is a generating system of the algebra $\mathcal{P}_{\mathcal{S}_{\mathcal{N},X}-w.s.}(X)$. □

Note that (16) is an algebraic dependency between elements of (19). Thus, the set (19) is algebraically dependent. Consequently, the set (19) is not an algebraic basis of the algebra $\mathcal{P}_{\mathcal{S}_{\mathcal{N},X}-w.s.}(X)$.

References

[1] Alencar R., Aron R., Galindo P., Zagorodnyuk A. *Algebras of symmetric holomorphic functions on ℓ_p*. Bull. Lond. Math. Soc. 2003, **35** (1), 55–64. doi:10.1112/S0024609302001431

[2] Aron R., Galindo P., Pinasco D., Zalduendo I. *Group-symmetric holomorphic functions on a Banach space*. Bull. Lond. Math. Soc. 2016, **48** (5), 779–796. doi:10.1112/blms/bdw043

[3] Bandura A., Kravtsiv V., Vasylyshyn T. *Algebraic basis of the algebra of all symmetric continuous polynomials on the Cartesian product of ℓ_p-spaces*. Axioms 2022, **11** (2). doi:10.3390/axioms11020041

[4] Chernega I., Galindo P., Zagorodnyuk A. *Some algebras of symmetric analytic functions and their spectra*. Proc. Edinb. Math. Soc. 2012, **55** (1), 125–142. doi:10.1017/S0013091509001655

[5] Chernega I., Holubchak O., Novosad Z., Zagorodnyuk A. *Continuity and hypercyclicity of composition operators on algebras of symmetric analytic functions on Banach spaces*. Eur. J. Math. 2020, **6** (1), 153–163. doi:10.1007/s40879-019-00390-z

[6] Chopyuk Yu., Vasylyshyn T., Zagorodnyuk A. *Rings of multisets and integer multinumbers*. Mathematics 2022, **10** (5). doi:10.3390/math10050778

[7] Galindo P., Vasylyshyn T., Zagorodnyuk A. *The algebra of symmetric analytic functions on L_∞*. Proc. Roy. Soc. Edinburgh Sect. A 2017, **147** (4), 743–761. doi:10.1017/S0308210516000287

[8] Galindo P., Vasylyshyn T., Zagorodnyuk A. *Symmetric and finitely symmetric polynomials on the spaces ℓ_∞ and $L_\infty[0, +\infty)$*. Math. Nachr. 2018, **291** (11–12), 1712–1726. doi:10.1002/mana.201700314

[9] Galindo P., Vasylyshyn T., Zagorodnyuk A. *Analytic structure on the spectrum of the algebra of symmetric analytic functions on L_∞*. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 2020, **114**, article number 56. doi:10.1007/s13398-020-00791-w

[10] González M., Gonzalo R., Jaramillo J. A. *Symmetric polynomials on rearrangement invariant function spaces*. J. Lond. Math. Soc. 1999, **59** (2), 681–697. doi:10.1112/S0024610799007164

[11] Halushchak S.I. *Isomorphisms of some algebras of analytic functions of bounded type on Banach spaces*. Mat. Stud. 2021, **56** (1), 107–112. doi:10.30970/MS.56.1.106-112

[12] Kravtsiv V., Vasylyshyn T., Zagorodnyuk A. *On algebraic basis of the algebra of symmetric polynomials on* $\ell_p(\mathbb{C}^n)$. J. Funct. Spaces 2017, **2017**. doi:10.1155/2017/4947925

[13] Mujica J. Complex Analysis in Banach Spaces. North Holland, 1986.

[14] Nemirovskii A.S., Semenov S.M. *On polynomial approximation of functions on Hilbert space*. Mat. USSR Sbornik 1973, **21** (2), 255–277. doi:10.1070/SM1973v021n02ABEH002016

[15] Vasylyshyn T. *Algebras of entire symmetric functions on spaces of Lebesgue measurable essentially bounded functions*. J. Math. Sci. (N.Y.) 2020, **246** (2), 264–276. doi:10.1007/s10958-020-04736-x

[16] Vasylyshyn T. *Symmetric polynomials on* $(L_p)^n$. Eur. J. Math. 2020, **6** (1), 164–178. doi:10.1007/s40879-018-0268-3

[17] Vasylyshyn T.V. *Symmetric polynomials on the Cartesian power of* L_p *on the semi-axis*. Mat. Stud. 2018, **50** (1), 93–104. doi:10.15330/ms.50.1.93-104

[18] Vasylyshyn T.V. *The algebra of symmetric polynomials on* $(L_\infty)^n$. Mat. Stud. 2019, **52** (1), 71–85. doi:10.30970/ms.52.1.71-85

[19] Vasylyshyn T. *Algebras of symmetric analytic functions on Cartesian powers of Lebesgue integrable in a power* $p \in [1, +\infty)$ *functions*. Carpathian Math. Publ. 2021, **13** (2), 340–351. doi:10.15330/cmp.13.2.340-351

[20] Vasylyshyn T. *Symmetric analytic functions on the Cartesian power of the complex Banach space of Lebesgue measurable essentially bounded functions on* $[0, 1]$. J. Math. Anal. Appl. 2022, **509** (2), 125977. doi:10.1016/j.jmaa.2021.125977

[21] Vasylyshyn T.V., Strutinskii M.M. *Algebras of symmetric* $*$*-polynomials in the space* $\mathbb{C}^2$. J. Math. Sci. (N.Y.) 2021, **253** (1), 40–53. doi:10.1007/s10958-021-05211-x

[22] Vasylyshyn T.V., Zagorodnyuk A.V. *Symmetric polynomials on the Cartesian power of the real Banach space* $L_\infty[0, 1]$. Mat. Stud. 2020, **53** (2), 192–205. doi:10.30970/ms.53.2.192-205

[23] Vasylyshyn T., Zagorodnyuk A. *Continuous symmetric 3-homogeneous polynomials on spaces of Lebesgue measurable essentially bounded functions*. Methods Funct. Anal. Topology. 2018, **24** (4), 381–398.

[24] Vasylyshyn T., Zahorodniuk V. *Weakly symmetric functions on spaces of Lebesgue integrable functions*. Carpatian Math. Publ. 2022, **14** (2), 437–441. doi:10.15330/cmp.14.2.437-441

[25] Vasylyshyn T., Zahorodniuk V. *On isomorphisms of algebras of entire symmetric functions on Banach spaces*. J. Math. Anal. Appl. 2024, **529** (2), 127370. doi:10.1016/j.jmaa.2023.127370

[26] Vasylyshyn T., Zhyhallo K. *Entire symmetric functions on the space of essentially bounded integrable functions on the union of Lebesgue-Rohlin spaces*. Axioms 2022, **11** (9), 460. doi:10.3390/axioms11090460

Received 05.08.2023

Буртняк І.В., Чоп'юк Ю.Ю., Василишин С.І., Василишин Т.В. *Алгебри слабко симетричних функцій на просторах вимірних за Лебегом функцій* // Карпатські матем. публ. — 2023. — Т.15, №2. — C. 411–419.

В даній роботі досліджено алгебри блочно-симетричних і слабко симетричних поліномів і аналітичних функцій на комплексних банахових просторах вимірних за Лебегом функцій, для яких p-тий степінь абсолютного значення є інтегровний за Лебегом, де $p \in [1, +\infty)$, і вимірних за Лебегом суттєво обмежених функцій на відрізку $[0, 1]$. Побудовано системи твірних елементів алгебр всіх слабко симетричних неперервних комплекснозначних поліномів на цих просторах. Також встановлено умови, за виконання яких множини слабко симетричних аналітичних функцій є алгебрами.

Ключові слова і фрази: симетрична функція, слабко симетрична функція, аналітична функція на нескінченновимірному просторі, простір вимірних за Лебегом функцій.