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On the classes of n-Nadir's and n^*-Nadir's operators on Hilbert spaces

Smati A.[1], Guesba M.[2]

The main aim of this paper is to present some new results for Nadir's operator $N = AB^* - BA^*$, where A and B are two bounded linear operators. We introduce a generalization of this concept, that is, the n-Nadir's and n^*-Nadir's operators, where n is a positive integer. We present fundamental properties of these operators, including compactness and normality. Additionally, we establish relationships between these classes, we show that the adjoint of n-Nadir's operator is n^*-Nadir's operator. Furthermore, if T is an n-Nadir's operator, such that A and B are two bounded commuting normal operators, then T is an n^*-Nadir's operator. Moreover, we study these classes in the special case $n = 2$, with some interesting examples. Other related results are also established.

Key words and phrases: Hilbert space, bounded operator, n-Nadir's operator, n^*-Nadir's operator, 2-Nadir's operator.

[1] Laboratoire de Mathématiques et Physique Appliquées, Ecole Normale Supérieure de Bousaada, 28001 Bousaada, Algeria
[2] Department of Mathematics, University of El Oued, El Oued 39000, Algeria
E-mail: smati.abdellatif@ens-bousaada.dz (Smati A.), guesba-messaoud@univ-eloued.dz (Guesba M.)

Introduction

We denote by $B(H)$ the algebra of bounded linear operators on a Hilbert space H. Let T^* be the adjoint of an operator T. Recall that an operator $T \in B(H)$ is said to be normal if $TT^* = T^*T$, self-adjoint if $T^* = T$, skew-adjoint if $T^* = -T$, projection if $T^2 = T = T^*$, and unitary if $TT^* = T^*T = I$. An operator S is said to be unitarily equivalent to an operator T if there exists a unitaty operator U such that $T = U^*SU$.

In classical physics, measurements can be made on a system without actually disturbing the system. However, in quantum mechanics, the measurement process can seriously disturb the system (see, e.g., [8, 9, 16]). In the theory we can present a measuring device by an operator so that, after carrying out the measurement, the system will be in one of the eigenstates of the operator. Since the act of measurement generally perturbs the system, the result of measuring two observables A and B on a given system therefore depends on the order in which they are carried out. The act of measuring A first and then B is represented by the action of product BA of their corresponding operators on the state vector (see [8, 16]).

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In 2017, M. Nadir introduced and studied some basic properties of bounded Nadir's operator $N = AB^* - BA^*$, where A and B are two linear operators. The class of Nadir's operators is a central concept in quantum mechanics, since it quantifies how well the two observables described by these operators can be measured simultaneously. In 2018, M. Nadir and A. Smati present some sufficient conditions which ensure the closedness and skew-adjointness of the unbounded Nadir's operators (see [8, 9, 12–14, 16]).

A.A.S. Jibril introduced the class of n-power normal operators as a generalization of the class of normal operators and studied some properties of this class for different values of the parameter n (see [10, 11]). The properties of n-normal operators and operators related with them were extensively studied by many authors (see, e.g., [1, 3, 4, 7, 15]).

In this paper, as the class of n-normal operators we introduce a generalization of the class of Nadir's operator, which are n-Nadir's and n^*-Nadir's operators, and present some properties of these concepts.

First, we recall the Fuglede's theorem which will be needed to prove some of our results.

Theorem 1 ([6]). *If T is a bounded operator, and N is normal and not necessarily bounded, then*

$$TN \subseteq NT \implies TN^* \subseteq N^*T.$$

Now, we present some well-known definitions and lemmas that will be needed (see [2, 5]).

Definition 1. *The closed linear subspace M is said to be invariant under the operator T if*

$$T(M) \subset M.$$

Recall that if $T \in B(H)$ and M is an invariant subspace for T, then T/M is used to denote the restriction of T to M. That is, T/M is the operator on M defined by $(T/M)h = Th$ whenever $h \in M$.

Lemma 1. *Let M be a closed linear subspace of H and $\lambda \in \mathbb{C}$. If M is invariant under the operators S and T, then it is invariant under $S + T$, ST, and λT. Moreover,*

$$(S + T)/M = (S/M) + (T/M),$$
$$(ST)/M = (S/M)(T/M),$$
$$(\lambda T)/M = \lambda(T/M).$$

Definition 2. *A closed linear subspace M is said to reduce the operator T if both M and $M^\perp$ are invariant under T.*

Lemma 2. *If T is an operator, and M is a closed linear subspace, the following conditions are equivalent:*

(a) M reduces T;

(b) M reduces T^*;

(c) M is invariant under both T and T^*.

Lemma 3. *If M reduces T, then*

$$(T/M)^* = T^*/M.$$

Main Results

In this section, we investigate some properties of n-Nadir's and n^*-Nadir's operators. We will first start showing new results about Nadir's operator.

In what follows, we denote by $[Nr]$ the class of all Nadir's operators.

Proposition 1. *Let T be a Nadir's operator. If T is invertible, then T^*T^{-1} and $T^{-1}T^*$ are self-adjoint operators.*

Proof. We have

$$T \in [Nr] \implies T = AB^* - BA^* \implies T^* = BA^* - AB^* = -T,$$

therefore $T^*T^{-1} = -TT^{-1} = -I$ and $\left(T^*T^{-1}\right)^* = (-I)^* = -I$. Hence $\left(T^*T^{-1}\right)^* = T^*T^{-1}$, this means that T^*T^{-1} is a self-adjoint operator.

Similarly, we can prove that $T^{-1}T^*$ is also a self-adjoint operator. □

Proposition 2. *Let T be a Nadir's operator. Then T^n is self-adjoint if n is an even number and it is skew-adjoint if n is an odd number.*

Proof. We have

$$(T^n)^* = (T^*)^n = (-T)^n = \begin{cases} T^n, & \text{if } n \text{ is an even number,} \\ -T^n, & \text{if } n \text{ is an odd number.} \end{cases}$$

□

Proposition 3. *If T_1 and T_2 are two commuting Nadir's operators, then T_1T_2 is a self-adjoint operator.*

Proof. Since T_1 and T_2 are two Nadir's operators, we have $T_1^* = -T_1$ and $T_2^* = -T_2$. Thus,

$$(T_1T_2)^* = T_2^*T_1^* = T_2T_1.$$

By the commutativity of T_1 and T_2, the operator T_1T_2 is self-adjoint. □

Now, we introduce a generalization of Nadir's operator, which are n-Nadir's and n^*-Nadir's operators.

Definition 3. *An operator $T \in B(H)$ is called n-Nadir's operator for a positive integer n if*

$$T = A^nB^* - B^nA^*,$$

where A and B are two bounded operators.

The class of all n-Nadir's operators is denoted by $[n\text{-}Nr]$.

Definition 4. *An operator $T \in B(H)$ is called n^*-Nadir's operator for a positive integer n if*

$$T = A(B^*)^n - B(A^*)^n,$$

where A and B are two bounded operators.

The class of all n^*-Nadir's operators is denoted by $[n^*\text{-}Nr]$.

To simplify writing, we adopt coding $(A^*)^n = A^{*n}$, then we have

$$T \in [n^*\text{-}Nr\,] \implies T = AB^{*n} - BA^{*n}.$$

Remark 1. *If T is a Nadir's operator, then $T \in \left[1\text{-}Nr\right]$ and $T \in \left[1^*\text{-}Nr\right]$, i.e.*

$$\left[Nr\right] = \left[1\text{-}Nr\right] = \left[1^*\text{-}Nr\right].$$

Proposition 4. *Let $T = A^n B^* - B^n A^*$ be a bounded n-Nadir's operator. If A or B is a compact operator, then T is also compact one.*

Proof. Suppose that A is a compact operator and B is bounded. Since A is compact, both A^* and A^n are also compact ones. And since B is bounded, both $A^n B^*$ and $B^n A^*$ are compact operators. Hence $T = A^n B^* - B^n A^*$ is compact one.

Similarly, we can prove that the n-Nadir's operator T is compact if A is a bounded operator and B is compact. □

Theorem 2. *Let T be a bounded n-Nadir's operator on a Hilbert space H. For an arbitrary $\lambda \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$, the operator λT is bounded n-Nadir's operator, i.e.*

$$T \in \left[n\text{-}Nr\right] \implies \lambda T \in \left[n\text{-}Nr\right].$$

Proof. For $\lambda \geq 0$, we have

$$\begin{aligned} \lambda T &= \lambda \left(A^n B^* - B^n A^*\right) = \left(\sqrt[n+1]{\lambda}\right)^{n+1} \left(A^n B^* - B^n A^*\right) \\ &= \left(\sqrt[n+1]{\lambda} A\right)^n \left(\sqrt[n+1]{\lambda} B\right)^* - \left(\sqrt[n+1]{\lambda} B\right)^n \left(\sqrt[n+1]{\lambda} A\right)^* = S^n K^* - K^n S^*, \end{aligned}$$

where $S = \sqrt[n+1]{\lambda} A$ and $K = \sqrt[n+1]{\lambda} B$.

For $\lambda < 0$, we have

$$\lambda T = -\left|\lambda\right| \left(A^n B^* - B^n A^*\right) = \left|\lambda\right| \left(B^n A^* - A^n B^*\right).$$

Since $|\lambda| > 0$ and $(B^n A^* - A^n B^*) \in \left[n\text{-}Nr\right]$, it follows from the case proved above that $\lambda T \in \left[n\text{-}Nr\right]$. □

Corollary 1. *Let T be a bounded n-Nadir's operator on a Hilbert space H. If $\lambda \in \mathbb{C}$, then there exists an n-Nadir's operator K such that*

$$\lambda T = e^{i\theta} K,$$

where i is the imaginary unit, and θ is the argument of λ.

Proof. We have $\lambda T = |\lambda|\, e^{i\theta} T = e^{i\theta}\, |\lambda|\, T$. By Theorem 2, we obtain $K = |\lambda|\, T \in \left[n\text{-}Nr\right]$. □

Proposition 5. *The adjoint operator T^* of n-Nadir's operator T is n^*-Nadir's operator, i.e.*

$$T \in \left[n\text{-}Nr\right] \implies T^* \in \left[n^*\text{-}Nr\right].$$

Proof. We have

$$T^* = \left(A^n B^* - B^n A^*\right)^* = \left(A^n B^*\right)^* - \left(B^n A^*\right)^* = B\left(A^n\right)^* - A\left(B^n\right)^* = B\left(A^*\right)^n - A\left(B^*\right)^n.$$

Hence, T^* is an n^*-Nadir's operator. □

Recall that by Fuglede's theorem, if A and B are two bounded commuting operators and A is a normal operator, then A^* and B also commute.

Theorem 3. *Let $T = A^nB^* - B^nA^*$ be an n-Nadir's operator, such that A and B are two bounded commuting normal operators. Then*

(i) T is a normal operator;

(ii) T is an n^*-Nadir's operator, i.e. $T \in [n^*\text{-}Nr]$;

(iii) T^* is an n-Nadir's operator, i.e. $T^* \in [n\text{-}Nr]$.

Proof. Since A and B are two bounded commuting normal operators, each of A, A^*, B and B^* commutes with all the others.

(i) For the normality of T we have

$$\begin{aligned} TT^* &= (A^nB^* - B^nA^*)(A^nB^* - B^nA^*)^* \\ &= (A^nB^* - B^nA^*)(BA^{*n} - AB^{*n}) \\ &= A^nB^*BA^{*n} - A^nB^*AB^{*n} - B^nA^*BA^{*n} + B^nA^*AB^{*n} \end{aligned}$$

and

$$\begin{aligned} T^*T &= (A^nB^* - B^nA^*)^*(A^nB^* - B^nA^*) \\ &= (BA^{*n} - AB^{*n})(A^nB^* - B^nA^*) \\ &= BA^{*n}A^nB^* - BA^{*n}B^nA^* - AB^{*n}A^nB^* + AB^{*n}B^nA^* \\ &= A^nB^*BA^{*n} - A^nB^*AB^{*n} - B^nA^*BA^{*n} + B^nA^*AB^{*n}. \end{aligned}$$

Hence, $TT^* = T^*T$.

(ii) For $T \in [n^*\text{-}Nr]$, we have

$$T = A^nB^* - B^nA^* = B^*A^n - A^*B^n.$$

Put $M = B^*$ and $N^* = A$, then we obtain

$$T = M(N^*)^n - N(M^*)^n.$$

Hence, T is an n^*-Nadir's operator.

(iii) For $T^* \in [n\text{-}Nr]$, we have

$$T^* = (A^nB^* - B^nA^*)^* = B(A^n)^* - A(B^n)^* = (A^*)^nB - (B^*)^nA.$$

Put $S = A^*$ and $K = B^*$, then we obtain

$$T^* = S^nK^* - K^nS^*.$$

Hence, T^* is an n-Nadir's operator. □

Theorem 4. *Let A and B be two bounded self-adjoint operators and BA be a normal operator. Then*

$$T \in [2\text{-}Nr] \implies T = B(A-B)A.$$

Proof. We have

$$A(BA) = (AB)A \implies A(BA) = (B^*A^*)^* A.$$

Since A and B are two bounded self-adjoint operators, we get

$$A(BA) = (BA)^* A.$$

Note, that BA and $(BA)^*$ are both normal operators. So, by means of the Fuglede-Putnam theorem, we get

$$A(BA)^* = (BA)A \implies A^2B = BA^2.$$

Thus,

$$T \in [2\text{-}Nr] \implies T = A^2B^* - B^2A^* = A^2B - B^2A = BA^2 - B^2A = B(A-B)A.$$

□

Example 1. *Consider the matrices A and B defined as*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Obviously $BA = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a normal operator. Then, we have

$$A^2B^* - B^2A^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

and

$$B(A-B)A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

It is clear that if A and B are self-adjoint, then AB is self-adjoint if and only if $AB = BA$. Since a self-adjoint operator is normal, we can state the following assertion.

Corollary 2. *Let A and B be two commuting self-adjoint operators. Then*

$$A^2B^* - B^2A^* = B(A-B)A.$$

Remark 2. *The hypothesis that BA is normal operator cannot merely be dropped. As counter example, consider for instance the matrices A and B defined as*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

The operator $BA = \begin{pmatrix} 0 & -2 \\ 1 & -2 \end{pmatrix}$ is not normal. We have

$$T = A^2B^* - B^2A^* = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4 & 4 \end{pmatrix} - \begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 3 & 6 \end{pmatrix},$$

but

$$B(A-B)A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 6 \\ 0 & 8 \end{pmatrix}.$$

Hence,

$$T \in [2\text{-}Nr] \quad \not\Longrightarrow \quad T = B(A-B)A.$$

Theorem 5. *Let T, A, B be three bounded operators, and S be an operator unitarily equivalent to T, such that the unitary operator U is commute with both A^* and B^*. Then*

(i) if $T = A^n B^* - B^n A^*$, then there exists an n-Nadir's operator K_1 such that

$$S = U^{n-1} K_1;$$

(ii) if $T = A\,(B^*)^n - B\,(A^*)^n$, then there exists an n^*-Nadir's operator K_2 such that

$$S = K_2\,(U^*)^{n-1}.$$

Moreover, $K_2 = -K_1^*$.

Proof. *(i)* Suppose that $T = A^n B^* - B^n A^*$. Since S is unitarily equivalent to T, then there exists a unitary operator U such that

$$S = U^* T U = U^*\,(A^n B^* - B^n A^*)\,U = U^* A^n B^* U - U^* B^n A^* U.$$

On the other hand,

$$UU^* = U^* U = I \Rightarrow (UU^*)^{n-1} = U^{n-1}\,(U^*)^{n-1}.$$

It is clearly that if U commutes with both A^* and B^*, then U^* commutes with both A and B. Therefore

$$U^{*n} A^n = (U^* A)^n, \qquad U^{*n} B^n = (U^* B)^n.$$

This in turn shows that

$$\begin{aligned} S &= U^{n-1} U^{*n} A^n B^* U - U^{n-1} U^{*n} B^n A^* U \\ &= U^{n-1}\,(U^* A)^n\, B^* U - U^{n-1}\,(U^* B)^n\, A^* U \\ &= U^{n-1}\big[\,(U^* A)^n\,(B^* U) - (U^* B)^n\,(A^* U)\,\big] \\ &= U^{n-1}\,(D^n C^* - C^n D^*)\,, \end{aligned}$$

where $D = U^* A$ and $C = U^* B$.

Hence, $S = U^{n-1} K_1$ such that $K_1 = D^n C^* - C^n D^*$ is an n-Nadir's operator.

(ii) In the same way, suppose that $T = A\,(B^*)^n - B\,(A^*)^n$, then

$$\begin{aligned} S &= U^* T U = U^*\big(A\,(B^*)^n - B\,(A^*)^n\,\big)U \\ &= U^* A\,(B^*)^n\, U - U^* B\,(A^*)^n\, U \\ &= (U^* A)\,(B^*)^n\, U^n\,(U^*)^{n-1} - U^* B\,(A^*)^n\, U^n\,(U^*)^{n-1} \\ &= \big[\,(U^* A)\,(B^* U)^n - (U^* B)\,(A^* U)^n\,\big]\,(U^*)^{n-1} \\ &= \big[\,(U^* A)\,(U^* B)^{*n} - (U^* B)\,(U^* A)^{*n}\,\big]\,(U^*)^{n-1} \\ &= \big[D\,(C^*)^n - C\,(D^*)^n\,\big]\,(U^*)^{n-1}. \end{aligned}$$

Hence $S = K_2\,(U^*)^{n-1}$ such that $K_2 = D\,(C^*)^n - C\,(D^*)^n$ is an n^*-Nadir's operator. □

Theorem 6. *Let A and B be two bounded operators on a Hilbert space H. Let M be a closed linear subspace of H. If M reduces both A and B, then*

(i) M is invariant under the n-Nadir's operator $T_1 = A^n B^* - B^n A^*$ and

$$T_1/M \in \left[n\text{-}Nr\right];$$

(ii) M is invariant under the n^*-Nadir's operator $T_2 = A(B^*)^n - B\left(A^*\right)^n$ and

$$T_2/M \in \left[n^*\text{-}Nr\right].$$

Proof. *(i)* Since M reduces both A and B, we get that M is invariant under A, B, A^*, B^*, A^n and B^n. This means that M is invariant under both A^nB^* and B^nA^*. Thus M is invariant under $T = A^nB^* - B^nA^*$ and

$$T_1/M = \left(A^nB^* - B^nA^*\right)/M = \left(A^nB^*\right)/M - \left(B^nA^*\right)/M.$$

Since M reduces both A and B, we obtain

$$A^*/M = (A/M)^* \quad \text{and} \quad B^*/M = (B/M)^*.$$

Since M is invariant under A and B, we have

$$(A^n/M) = (A/M)^n \quad \text{and} \quad (A^n/M) = (A/M)^n.$$

Therefore,

$$T_1/M = (A^n/M)\,(B^*/M) - (B^n/M)\,(A^*/M) = (A/M)^n(B/M)^* - (B/M)^n(A/M)^*.$$

Hence T_1/M is an n-Nadir's operator.

(ii) In the same way, since M is invariant under both $A\left(B^*\right)^n$ and $B\left(A^*\right)^n$, we get that M is invariant under $T_2 = A\left(B^*\right)^n - B\left(A^*\right)^n$ and

$$\begin{aligned} T_2/M &= \left[A\left(B^*\right)^n - B\left(A^*\right)^n\right]/M \\ &= \left[A\left(B^*\right)^n\right]/M - \left[\left(B\left(A^*\right)^n\right)\right]/M \\ &= (A/M)\left(B^{*n}/M\right) - (B/M)\left(A^{*n}/M\right) \\ &= (A/M)\left(B^*/M\right)^n - (B/M)\left(A^*/M\right)^n \\ &= (A/M)(B/M)^{*n} - (B/M)(A/M)^{*n}. \end{aligned}$$

Hence, T_2/M is an n^*-Nadir's operator. □

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Сматі А., Гесба М. *Про класи n-операторів Надіра та n^*-операторів Надіра в гільбертових просторах* // Карпатські матем. публ. — 2025. — Т.17, №1. — С. 137–145.

Основною метою цієї статті є представлення нових результатів щодо оператора Надіра $N = AB^* - BA^*$, де A і B — два обмежені лінійні оператори. Ми вводимо узагальнення цього поняття, а саме n-оператори Надіра та n^*-оператори Надіра, де n — додатне ціле число. У роботі наведено основні властивості цих операторів, зокрема компактність і нормальність. Додатково встановлено зв'язки між цими класами: показано, що спряжений до n-оператора Надіра є n^*-оператором Надіра. Більше того, якщо T — такий n-оператор Надіра, що A і B — два обмежені комутуючі нормальні оператори, то T також є n^*-оператором Надіра. Окремо розглянуто випадок $n = 2$ з наведенням деяких цікавих прикладів. Також встановлено інші пов'язані результати.

Ключові слова і фрази: гільбертовий простір, обмежений оператор, n-оператор Надіра, n^*-оператор Надіра, 2-оператор Надіра.