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The Cauchy problem for a double nonlinear time-dependent parabolic equation with absorption in a non-homogeneous medium

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In this paper, our primary focus is on studying the properties of self-similar solutions to the Cauchy problem. We specifically examine the behavior of these solutions in a double nonlinear time-dependent parabolic equation and their absorption in a non-homogeneous medium. Through the research of the topic matter, our aim is to deliver a more thorough comprehension of the finite speed perturbations propagation in the solution of the Cauchy problem for a nonlinear parabolic equation. This establishment of a property is essential to understand the dynamic nature of these equations. Furthermore, we delve into the self-similar analysis of the solution, which allows us to ascertain the condition of Fujita type global solvability for the Cauchy problem in a double nonlinear degenerate-type parabolic equation within a non-homogeneous medium. This analysis provides valuable insights into the behavior and potential solvability of these equations on a global mean. Additionally, we establish estimates for weak solutions depending on the growing density and the value of numerical parameters. By establishing these estimates, we provide a more comprehensive understanding of the behavior of the solutions in different scenarios.

Key words and phrases: finite speed, perturbation, global solution, estimate solution, critical case, asymptotic behavior, numerical analysis.

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1 Introduction

Consider in $Q = \{(t,x):\ t > t_0,\ x \in \mathbb{R}^N,\ t_0 = \text{const} > 0\}$ the following Cauchy problem:

$$\rho_2(x)\,\partial_t u = u^q\, div\left(\rho_1(x)\, u^{m-1}\left|\nabla u^k\right|^{p-2}\nabla u\right) - \rho_3(x)\, t^l u^\beta, \tag{1}$$

$$u(t_0,x) = u_0(x), \quad x \in \mathbb{R}^N, \tag{2}$$

where $\rho_1(x) = |x|^n$, $\rho_{i+1}(x) = |x|^{-l_i}$, $q \in (0,1)$, $m,k \geq 1$, $p \geq 2$, n, l_i, l, β, $i = 1,2$, are the numerical parameters, and the function $u = u(t,x) \geq 0$ is the solution.

It is clear that the equation (1) is degenerate. Therefore, in the domain Q, where $u = 0, \nabla u = 0$, it is a degenerate type. Therefore, in this case, we need to consider a weak solution from having a physical sense class.

Equation (1) plays an important role in modeling a wide range of physical phenomena. For instance, equation (1) is frequently employed to describe and analyze diffusive processes found in biological species, diffusion phenomena in force-free magnetic fields, curve shortening flow,

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and the spreading of infectious diseases, among others (see [7,13,14,17,21] for further details). By utilizing mathematical equations, researchers and scientists are able to delve deep into the intricate mechanisms underlying these phenomena, allowing for a better understanding and prediction of their behavior.

Weak solutions play an essential role in modelling real-world phenomena, since many differential equations are unable to admit sufficiently smooth solutions, and the weak formulation is the only method available for solving them. It often proves advantageous to prove the existence of weak solutions first and then show that those solutions are sufficiently smooth, even in cases when an equation has differentiable solutions. This elegant and beautiful work [10] revealed a new phenomenon of nonlinear PDEs and stimulated the study of similar phenomena for various parabolic, hyperbolic, and nonlinear Schrodinger-type equations. Many significant discoveries have emerged since then, but we cannot provide a thorough summary in such a short paper. The interested reader is referred to the paper by F.B. Weissler [26] and the references therein for a good account of related works. A brief review of some related results on parabolic equations will be given below.

The problem (1)–(2) intensively studied by many authors [19,20,27]. Zh. Xiang et. al. [28] studied the properties of solutions to the following problem (1)–(2) in one-dimension case:

$$\begin{aligned} &\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{m-1}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right) - \gamma u^{\beta}, \quad x \in \mathbb{R}, \ t > 0, \\ &u(x,0) = u_0(x), \quad x \in \mathbb{R}, \end{aligned}$$

where $m > 1$, $p \geq 2$, $\beta > 0$ and $\gamma > 0$ are constants, the nontrivial initial data $u_0(x)$ is nonnegative continuous and compactly supported, while the variable function $\rho(x)$ is positive, bounded, and smooth enough.

Qualitative properties of solutions of doubly nonlinear reaction-diffusion, self-similar profiles of solutions, global existence, and blow-up solutions studied in [8, 18]. Asymptotic behavior of solutions of the nonlinear diffusion equation with absorption at a critical exponent considered in the works [1, 5, 22, 25]. In [15, 16], the Cauchy problem for the following two equations with variable coefficients is studied:

$$\rho(x)\frac{\partial u}{\partial t} = div\left(u^{m-1}|\nabla u|^{p-2}\nabla u\right) + \rho(x)u^{\beta}$$

and

$$\rho(x)\frac{\partial u}{\partial t} = div\left(u^{m-1}|\nabla u|^{p-2}\nabla u\right) + u^{\beta}, \quad x \in \mathbb{R}^N, \ t > 0,$$

where $p > 1$, $m + p > 3$, $\beta > m + p - 2$, $\rho(x) = |x|^{-n}$ or $\rho(x) = \left(1 + |x|\right)^{-n}$. The authors showed that under some conditions for the parameters and for the initial data, any nontrivial solution of the Cauchy problem blows up in a finite time. Moreover, the authors established a sharp universal estimate of the solution near the blow-up point.

D. Andreucci and A.F. Tedeev [2,3] studied the Neumann problem for the following equations

$$\left(w^{\beta}\right)_t = div\left(|Dw|^{m-1}Dw\right) + w^{\nu}, \quad |Dw|^{m-1}Dw \cdot n = 0, \quad w^{\beta}(x,0) = w_0^{\beta}(x),$$

where $0 < \beta \leq 1$, $m > 1$, $\nu > 1$, $w_0(x) \geq 0$, n denotes the outer normal. They established a sharp universal estimate of the solution near the blow-up point and showed the condition for the solutions to exist and to which class they belonged.

Definition 1. *We will call a weak solution to the problem* (1)–(2) *in* (t_0, T) *with* $0 < T \leq +\infty$ *the function* $u = u(t, x) \geq 0$ *with the property* $\rho_1(x)\, u^{m-1}\left|\nabla u^k\right|^{p-2}\nabla u \in C(Q)$, *and satisfying to the equation* (1) *in distribution sense*

$$\int_{t_0}^{T}\int_{R^N}\left(\rho_2(x)u\frac{\partial\psi}{\partial t} - \rho_1(x)\, u^{m-1}\left|\nabla u^k\right|^{p-2}\nabla u\cdot\nabla\left(\psi u^q\right) + \rho_3(x)\, t^l u^\beta\right)dx\,dt = 0$$

for any $\psi \in C_0^\infty\left(\mathbb{R}^N \times (t_0, T)\right)$.

The existence of a weak solution is proven in [6, 19], where doubly nonlinear parabolic equations are considered.

2 The self-similar equation

We introduce the notation $u = v^{\frac{1}{1-q}}$ and put this into the equation (1)–(2). We get

$$\rho_2(x)\,\partial_t v = div\left(\rho_1(x)\, v^{m_2-1}\left|\nabla v^{k_2}\right|^{p-2}\nabla v\right) - (1-q)\,\rho_3(x)\, t^l v^{\beta_2}, \tag{3}$$

$$v|_{t=t_0} = v_0(x). \tag{4}$$

Also, consider the radial symmetric case

$$\rho_2(r)\,\partial_t v = r^{1-N}\frac{\partial}{\partial r}\left(\rho_1(r)\, r^{N-1} v^{m_2-1}\left|\frac{\partial v^{k_2}}{\partial r}\right|^{p-2}\frac{\partial v}{\partial r}\right) - (1-q)\,\rho_3(r)\, t^l v^{\beta_2}, \tag{5}$$

where $r = |x|$, $m_2 = \dfrac{m-q}{1-q}$, $k_2 = \dfrac{k}{1-q}$, $\beta_2 = \dfrac{\beta-q}{1-q}$ and N is the dimension of space.

We look for the solution of (5) in the form $v(t, r) = t^{-\alpha} f(\xi)$, $\xi = rt^{-\mu}$, where

$$\alpha = \frac{(l+1)(p-n-l_1) + l_1 - l_2}{d}, \qquad \mu = \frac{\beta_2 - 1 - (l+1)(m_2 + k_2(p-2) - 1)}{d},$$
$$d = (\beta_2 - 1)(p-n-l_1) + (l_1 - l_2)(m_2 + k_2(p-2) - 1) \neq 0.$$

It is easy to check that for an unknown function f the following equation

$$L(\xi) = \xi^{1-N}\frac{d}{d\xi}\left(\xi^{N-1+n} f^{m_2-1}\left|\frac{df^{k_2}}{d\xi}\right|^{p-2}\frac{df}{d\xi}\right) + \mu\xi^{1-l_1}\frac{df}{d\xi} + \alpha\xi^{-l_1} f - (1-q)\,\xi^{-l_2} f^{\beta_2} = 0 \tag{6}$$

is obtained. According to the original problem statement, we will be looking for a non-trivial non-negative solution of the equation (6) that fulfills the following condition

$$f'(0) = 0, \quad f(c) = 0, \quad 0 < c < +\infty.$$

We look for the solution (6) the WKB solution [6] in the Hardy form [23]:

$$f(\xi) = z(\varphi(\xi)). \tag{7}$$

The function $z(\varphi(\xi))$ is the solution to the equation

$$\frac{d}{d\varphi}\left(z^{m_2-1}\left|\frac{dz^{k_2}}{d\varphi}\right|^{p-2}\frac{dz}{d\varphi}\right) = \frac{dz}{d\varphi}, \tag{8}$$

where the function $\varphi(\xi)$ is to be chosen. Then after substituting (7) into (6), and choosing $\varphi(\xi)$ as the solution of the equation $(\varphi_\xi)^{p-1} = -\mu\xi^{1-n-l_1}$, we get

$$\varphi(\xi) = \begin{cases} c + \frac{(-\mu)^{\frac{1}{p-1}}}{\gamma_1}\xi^{\gamma_1}, & \text{if } p \neq n + l_1, \ \gamma_1 = \frac{p-n-l_1}{p-1}, \ c = \text{const} \geq 0, \\ (-\mu)^{\frac{1}{p-1}} \ln|\xi|, & \text{otherwise.} \end{cases} \tag{9}$$

Note, that equation (8) have particular solution

$$z(\varphi) = \begin{cases} (a - b\varphi)_+^{\gamma}, & \gamma = \frac{p-1}{m_2-1+k_2(p-2)} > 0, \\ (a + b\varphi)^{\gamma}, & \gamma < 0, \\ \exp\left(-\frac{\varphi}{k_2^{\frac{p-2}{p-1}}}\right), & m_2 - 1 + k_2(p-2) = 0, \end{cases}$$

where $b = \dfrac{m_2 + k_2(p-2) - 1}{(p-1)k_2^{\frac{p-2}{p-1}}}$, $a = \text{const} \geq 0$, $(d)_+ = \max(d, 0)$.

Therefore, the function $\overline{f}(\xi) \cong f(\xi)$, can be considered as the congruence of function $f(\xi)$ (see [6, Paragraph 4.3] and [23, Paragraph 2, topic 13]). So, we obtain $\overline{f}(\xi)$ in the following form

$$\overline{f}(\xi) = \begin{cases} A(a - \xi^{\gamma_1})_+^{\gamma}, & \gamma > 0, \\ A(a + \xi^{\gamma_1})^{\gamma}, & \gamma < 0, \\ \exp\left(-\frac{\varphi}{k_2^{\frac{p-2}{p-1}}}\right), & \varphi(\xi) = \frac{|\mu|^{\frac{1}{p-1}}}{\gamma_1}\xi^{\gamma_1}, \ m_2 - 1 + k_2(p-2) = 0, \end{cases} \tag{10}$$

where $A = \left[\frac{b|\mu|^{\frac{1}{p-1}}}{\gamma_1}\right]^{\gamma}$, $p \neq n + l_1$.

Then we need to prove that all solutions satisfying equation (6) have an asymptotic in the form (10).

3 Main results

3.1 The slow diffusion case $(\gamma > 0)$

Since problem (1)–(2) is equivalent to problem (3)–(4), it is sufficient to solve (3)–(4). Using constructed above function $\overline{f}(\xi)$ and the solution comparison method [6, 27], we are in a position to establish explicit estimates of the solution of an estimate of the Cauchy problem (3)–(4).

Consider the function $v_+(t,x) = t^{-\alpha}\overline{f}(\xi)$, $\overline{f}(\xi) = A(a - \xi^{\gamma_1})_+^{\gamma}$, where $\alpha, \xi, a, \gamma_1, \gamma$ are defined above.

Theorem 1. *Let $v(t,x)$ be a weak solution of the problem* (3)–(4), $\gamma > 0$,

$$\beta_2 \geq 1 + (l+1)(m_2 + k_2(p-2) - 1) + \frac{(l+1)(p-n-l_1) + l_1 - l_2}{N - l_1}, \quad p \neq l_1 + n,$$

$$l_1 < \min\left(N, \frac{(l+1)(p-n) - l_2}{l}\right), \quad l \neq 0$$

and $v_0(x) \le v_+(t_0,x)$, $x\in\mathbb{R}^N$. *Then for the solution of the problem* (3)–(4) *in* Q *the estimate*

$$v(t,x)\le v_+(t,x)$$

and for the front the following estimate

$$|x|\ge x_f(t)=a^{\frac{1}{\gamma_1}}t^{\mu}$$

hold.

Proof. The proof is based on the principle of comparison [11, 26]. For comparison function we take the function $v_+(t,x)$ defined above. We note that according definition the function has the property $v_+(t,x)\equiv 0$, when $|x|\ge a^{\frac{1}{\gamma_1}}t^{\mu}$. Now to prove the theorem we need to calculate $L(v_+(t,x))$ and show that $L(v_+(t,x))<0$. Indeed, we will show that $L(\overline{f}(\xi))<0$ in $|\xi|<a^{\frac{1}{\gamma_1}}$.

After calculating, we get

$$L(\overline{f}(\xi))\Big|_{\overline{f}(\xi)=z(\varphi)}=\xi^{1-N}\frac{d}{d\xi}\left((\varphi_\xi)^{p-1}\xi^{N-1+n}z^{m_2-1}\left|\frac{dz^{k_2}}{d\varphi}\right|^{p-2}\frac{dz}{d\varphi}\right)+\mu\xi^{1-l_1}\varphi_\xi\frac{dz}{d\varphi}+\alpha\xi^{-l_1}z$$
$$+(1-q)\xi^{-l_2}z^{\beta_2}=\xi^{-l_1}z\left[\alpha-\mu(N-l_1)-(1-q)\,\xi^{l_1-l_2}z^{\beta_2-1}\right].$$

It easy to see that $L(\overline{f}(\xi))\le 0$ in $a-\xi^{\gamma_1}>0$, i.e. $|\xi|<a^{\frac{1}{\gamma_1}}$ if $\alpha-\mu(N-l_1)\le 0$, i.e.

$$\frac{(l+1)(p-n-l_1)+l_1-l_2-(N-l_1)\left[\beta_2-1-(l+1)(m_2+k_2(p-2)-1)\right]}{(\beta_2-1)(p-n-l_1)+(l_1-l_2)(m_2+k_2(p-2)-1)}\le 0,$$
$$\beta_2\ge\beta_{2c}=1+(l+1)(m_2+k_2(p-2)-1)+\frac{(l+1)(p-n-l_1)+l_1-l_2}{N-l_1},$$
$$l_1<\min\left(N,\ \frac{(l+1)(p-n)-l_2}{l}\right),\quad l\ne 0.$$

Applying the comparison principle, we have $v(t,x)\le v_+(t,x)$ in Q. □

Remark. *The number* $\beta_{2c}=1+(l+1)(m_2+k_2(p-2)-1)+\dfrac{(l+1)(p-n-l_1)+l_1-l_2}{N-l_1}$ *is called the critical Fujita exponent [10], which separates the nonexistence and existence of global in-time solutions of* (1)–(2).

3.2 Fast diffusion case ($\gamma<0$)

Suppose that for equation (6) the following conditions are fulfilled

$$f'(0)=0,\qquad f(\infty)=0.$$

Note that the function $\bar{f}(\xi)=A(a+\xi^{\gamma_1})^{\gamma}$ with $a=\mathrm{const}>0$ satisfies the above conditions.

Theorem 2. *Let* $\gamma<0$, $\beta_2>\beta_{2c}$, $v_0(x)\ge v_-(t_0,x)$, $x\in\mathbb{R}^N$. *Then for the problem* (3)-(4), *there exist a global solution in* Q *and it satisfies the estimation*

$$v(t,x)\ge v_-(t,x),\quad x\in\mathbb{R}^N,$$

where

$$v_-(t,x)=At^{-\alpha}(a+\xi^{\gamma_1})^{\gamma}.$$

Proof. The proof of the Theorem 2 is similar to the proof of the Theorem 1. □

3.3 The critical case $(m_2 - 1 + k_2(p-2) = 0)$

Theorem 3. *Let* $v(t,x)$ *be a weak solution of the problem* (3)–(4)*,* $m_2 - 1 + k_2(p-2) = 0$*,*

$$\beta_2 \geq 1 + \frac{(l+1)(p-n-l_1) + l_1 - l_2}{N - l_1}, \quad p \neq l_1 + n,$$
$$l_1 < \min\left(N, \frac{(l+1)(p-n) - l_2}{l}\right), \quad l \neq 0,$$
$$v_0(x) \leq v_1(t_0, x), \quad x \in \mathbb{R}^N.$$

Then for the solution of the problem (3)–(4) *in* Q *the estimate*

$$v(t,x) \leq v_1(t,x) = t^{-\alpha} \exp\left(-\frac{|\mu|^{\frac{1}{p-1}} \xi^{\frac{p-n-l_1}{p-1}}}{\gamma_1 k_2^{\frac{p-2}{p-1}}}\right), \quad \xi = |x| t^{-\mu},$$

holds.

Theorem 3 contains the Fujita result (see [9,12]) on global solvability of the problem (1)–(2) for the case $m + k(p-2) - 1 = 0$, $k = 1$, $p = 2$.

3.4 Double critical case

The case $m_2 + k_2(p-2) - 1 = 0$, $p = l_1 + n$ we will call a double critical case.

Theorem 4. *Let* $v(t,x)$ *be a weak solution of the problem* (3)–(4)*,* $m_2 - 1 + k_2(p-2) = 0$*,*

$$\beta_2 \geq 1 + \frac{(l+1)(p-n-l_1) + l_1 - l_2}{N - l_1}, \quad p = l_1 + n,$$
$$l_1 < \min\left(N, \frac{(l+1)(p-n) - l_2}{l}\right), \quad l \neq 0,$$
$$v_0(x) \leq v_2(t_0, x), \quad x \in \mathbb{R}^N \setminus \{0\}.$$

Then for the solution of the problem (3)–(4) *in* Q *the estimate*

$$v(t,x) \leq v_1(t,x) = t^{-\alpha} \exp\left(\frac{|\mu|^{\frac{1}{p-1}}}{k_2^{\frac{p-2}{p-1}}} \ln|\xi|\right), \quad \xi = |x| t^{-\mu},$$

holds.

Proof. Theorems 3, 4 can be proved using the same approach as in proving of Theorem 1. As functions for comparison the corresponding functions $v_1(t,x)$, $v_2(t,x)$ are taken. After calculating $L(v_1(t,x))$, $L(v_2(t,x))$, it easy to check conditions $L(v_1(t,x)) < 0$, $L(v_2(t,x)) < 0$. □

In [8], authors established a large time an asymptotic solution of the problem (1)–(2) for the case $q = l = 0$, $l_1 = l_2$, $\beta = \beta_c$. They established the following asymptotic solution

$$u(t,x) \sim (t \ln t)^{-\frac{1}{\beta_c - 1}} \exp\left(-|x|^2/t\right), \quad \beta_c = 1 = 2/N \quad \text{for } t \sim \infty.$$

Generalization large time asymptotic and behavior of the front of this result for to a degenerate nonlinear parabolic equation considered in the works [4, 5, 24]. Authors of these works showed the following large time asymptotic of solution of the problem (1)–(2)

$$u(t,x) \sim \left((T+t) \ln (T+t) \right)^{-\frac{1}{\beta_c - 1}} \exp\left(- |x|^2 / t \right)$$

for the following critical exponent case

$$\beta_c = 1 + (l+1) \left[m_2 + k_2 (p-2) - 1 + \frac{p - l_1 - n}{N - n} \right], \quad n < N, \;\; p - l_1 - n > 0,$$

and behavior of a free boundary for a particular value of the numerical parameters ($l = 1$, $l_1 = l_2, q = 0$).

3.5 Asymptotic of compactly supported weak solutions

Next, the asymptotic behavior of the self-similar solutions of the equation (6) is studied.

Theorem 5. *Let* $\gamma (1 - \beta_2) < 1$. *Then a finite solution of the problem* (3)–(4) *has an asymptotic representation*

$$f(\xi) = \bar{f}(\xi) (1 + o(1)).$$

Proof. We will search for a solution to the equation (6) in the following form

$$f(\xi) = \bar{f}(\xi) w(\eta). \tag{11}$$

Here $\eta = -\ln (a - \xi^{\gamma_1})$, $w(\eta)$ is to be determined.

Now, let us investigate the asymptotic behavior of the solution of the equation (6) at $\xi_1 = \xi \to a^{\frac{1}{\gamma_1}}$. It is to check that substituting (11) into (6), for $w(\eta)$ we obtain the equation

$$\frac{d}{d\eta}(Kw) + a_0(\eta) Kw + a_1(\eta) \left(\frac{dw}{d\eta} - \gamma w \right) + a_2(\eta) w + a_3(\eta) w^{\beta_2} = 0, \tag{12}$$

where

$$a_0(\eta) = \frac{N - l_1}{\gamma_1} \frac{e^{-\eta}}{a - e^{-\eta}} - \gamma, \quad a_1(\eta) = \mu \left(\gamma_1 A^{\frac{1}{\gamma}} \right)^{1-p}, \quad a_2(\eta) = \frac{\alpha}{\mu \gamma_1} a_1(\eta) \left(\frac{e^{-\eta}}{a - e^{-\eta}} \right),$$

$$a_3(\eta) = -(1-q) \gamma_1^{-p} A^{\beta_2 - m_2 - k_2(p-2)} \frac{e^{(\gamma(1-\beta_2)-1)\eta}}{(a - e^{-\eta})^{1 + \frac{l_2 - l_1}{\gamma_1}}} w^{\beta_2},$$

$$Kw = w^{m_2 - 1} \left| \frac{dw^{k_2}}{d\eta} - k_2 \gamma w^{k_2} \right|^{p-2} \left(\frac{dw}{d\eta} - \gamma w \right).$$

It is supposed that $\xi \in [\xi_0, \xi_1)$, $0 < \xi_0 < \xi_1$. Therefore, the function $\eta(\xi)$ has the properties

$$\eta(\xi) > 0 \;\text{ at }\; \xi \in [\xi_0, \xi_1), \quad \eta_0 = \eta(\xi_0) > 0, \quad \lim_{\xi \to \xi_1} \eta(\xi) = +\infty.$$

Note that the study of the solutions of the (12) is equivalent to the study of those solutions of the equation (3), each of which in some interval $[\eta_0, +\infty)$ satisfies $w(\eta) > 0$, $Kw \neq 0$.

First, we show that the solutions $w(\eta)$ of equation (12) has a finite limit w_0 at $\eta \to \infty$. Assume

$$y(\eta) = Kw, \tag{13}$$

$$y'(\eta) = -a_0(\eta)\,y(\eta) - a_1(\eta)\left(\frac{dw}{d\eta} - \gamma w\right) - a_2(\eta)\,w - a_3(\eta)\,w^{\beta_2}. \tag{14}$$

Now, we consider the function

$$g(\lambda, \eta) = -a_0(\eta)\,\lambda - a_1(\eta)\left(\frac{dw}{d\eta} - \gamma w\right) - a_2(\eta)\,w - a_3(\eta)\,w^{\beta_2}, \tag{15}$$

where λ is the real number. From here it is easy to see that for each value λ, the function $g(\lambda, \eta)$ preserves sign on some interval $[\eta_1, +\infty) \subset [\eta_0, +\infty)$ and for every fixed value λ, different from the value satisfying equation

$$-\overset{0}{a}_0(\eta)\,\lambda - \overset{0}{a}_1(\eta)\,(-\gamma w_0) - \overset{0}{a}_2(\eta)\,w_0 - \overset{0}{a}_3(\eta)\,w_0^{\beta_2} = 0.$$

Here $\overset{0}{a}_0 = \lim\limits_{\eta\to+\infty} a_0(\eta) = -\gamma$, $\overset{0}{a}_1 = \lim\limits_{\eta\to+\infty} a_1(\eta) = \mu\left(\gamma_1 A^{\frac{1}{\gamma}}\right)^{1-p}$, $\overset{0}{a}_2 = \lim\limits_{\eta\to+\infty} a_2(\eta) = 0$,

$$\overset{0}{a}_3 = \lim_{\eta\to+\infty} a_3(\eta) = \begin{cases} 0, & \gamma(1-\beta_2) < 1, \\ -(1-q)\gamma_1^{-p} a^{\frac{l_1-l_2}{\gamma_1}-1} A^{\beta_2-m_2-k_2(p-2)}, & \gamma(1-\beta_2) = 1, \\ \infty, & \gamma(1-\beta_2) > 1. \end{cases}$$

This implies that the function $g(\lambda, \eta)$ preserves sign on the interval $[\eta_2, +\infty) \subset [\eta_0, +\infty)$, where $\lambda \neq 0$. Thus, the function $g(\lambda, \eta)$ for all $\eta \in [\eta_2, +\infty)$ satisfy one of the inequalities:

$$g(\lambda, \eta) > 0 \quad \text{or} \quad g(\lambda, \eta) < 0. \tag{16}$$

Suppose now that for the function $y(\eta)$ limit at $\eta \in [\eta_2, +\infty)$ does not exist. Consider that case when one of the inequalities (16) is satisfied. As $y(\eta)$ is an oscillating function around a straight line $\bar{y} = \lambda$, its graph intersects this straight line infinitely many times in $[\eta_2, +\infty)$. However, this is impossible, since on the interval $[\eta_2, +\infty)$ just one of the inequalities (16) is valid, and therefore, from (15), it follows that the graph of the function $y(\eta)$ intersects the straight lines $\bar{y} = \lambda$ only once in an interval $[\eta_2, +\infty)$. Accordingly, the function $y(\eta)$ has a limit at $\eta \to +\infty$.

It is obvious that

$$\lim_{\eta\to+\infty} \frac{e^{-\eta}}{a - e^{-\eta}} = 0, \qquad \lim_{\eta\to+\infty} w' = 0.$$

By assumption, $f(\xi) = \bar{f}(\xi)\,w(\eta)$ and the function $y(\eta)$ defined in (12), has a limit at $\eta \to +\infty$. Then $g'(\eta)$ has a limit at $\eta \to +\infty$, and this limit is zero. Then

$$\lim_{\eta\to+\infty} y(\eta) = \lim_{\eta\to+\infty} w^{m_2-1}\left|\frac{dw^{k_2}}{d\eta} - k_2\gamma w^{k_2}\right|^{p-2}\left(\frac{dw}{d\eta} - \gamma w\right) = -\gamma^{p-1} k_2^{p-2} w_0^{m_2+k_2(p-2)} + o(1).$$

And by (14), the derivative of function $y(\eta)$ has a limit at $\eta \to +\infty$, which is obviously equal to zero. Consequently, the following

$$\lim_{\eta\to+\infty}\left[a_0(\eta)\,y(\eta) + a_1(\eta)\left(\frac{dw}{d\eta} - \gamma w\right) + a_2(\eta)\,w + a_3(\eta)\,w^{\beta_2}\right] = 0$$

is necessary. Then from (12)–(16) we obtain the following algebraic equation

$$\gamma^p k_2^{p-2} w_0^{\frac{p-1}{\gamma}} + \overset{0}{a}_3 w_0{}^{\beta_2 - 1} - \mu\gamma\gamma_1^{1-p} A^{\frac{1-p}{\gamma}} = 0.$$

From these expressions, it is easy to see that the equation (13) has a solution $y(\eta)$ with a finite non-zero limit at $\eta \to +\infty$. On behalf of (12), we have proved that $f(\xi) = C\overline{f}(\xi)(1+o(1))$, where C is the root of the above algebraic equation. The proof is completed. □

3.6 Fundamental solution

If in (3) the initial condition is equivalent to the condition $v(x,0) = E_0\delta(x)$, then $\beta_{2c} = 1 + (l+1)(m_2 + k_2(p-2) - 1) + \dfrac{(l+1)(p-n-l_1) + l_1 - l_2}{N - l_1}$ can be calculated as the value of a, where $\delta(x)$ denotes the Dirac delta function and $E_0 = \int_{\mathbb{R}^N} \rho_2(x)\, v(t,x)\, dx$.

Let the constant a satisfy the Dirac condition, then we say that the function $v(t,x)$ is the generalized (weak) fundamental solution of the problem (3)–(4).

In the particular case $q = n = l = 0$, $k = 1$, $p = 2$ and sourceless form of (1)–(2), A.A. Samarskii et. al. [23] determined the value of a.

Consider the following integrals

$$I_N(\gamma,\gamma_1) = \int_{\mathbb{R}^N} \left(a_0^{\gamma_1} - \xi^{\gamma_1}\right)_+^{\gamma} d\xi, \tag{17}$$

$$J_N(\gamma,\gamma_1,\gamma_2) = \int_{\mathbb{R}^N} \left(b_0^{\gamma_1} - \xi^{\gamma_1}\right)_+^{\gamma} \xi^{\gamma_2}\, d\xi. \tag{18}$$

We introduce the notation $\eta = \xi^{1+\gamma_2}$, $b_0 = a_0^{\frac{1}{\gamma_2+1}}$ and put them into the integral (18), we get

$$\begin{aligned} J_N(\gamma,\gamma_1,\gamma_2) &= \int_{\mathbb{R}^N} \left(b_0^{\gamma_1} - \xi^{\gamma_1}\right)_+^{\gamma} \xi^{\gamma_2}\, d\xi \\ &= \int_{\mathbb{R}^N} \left(a_0^{\frac{\gamma_1}{\gamma_2+1}} - \eta^{\frac{\gamma_1}{\gamma_2+1}}\right)_+^{\gamma} \eta^{\frac{\gamma_2}{\gamma_2+1}}\, d\eta^{\frac{1}{\gamma_2+1}} = \left(\frac{1}{\gamma_2+1}\right)^N \int_{\mathbb{R}^N} \left(a_0^{\frac{\gamma_1}{\gamma_2+1}} - \eta^{\frac{\gamma_1}{\gamma_2+1}}\right)_+^{\gamma} d\eta. \end{aligned}$$

Consequently, we have $J_N(\gamma,\gamma_1,\gamma_2) = \left(\frac{1}{\gamma_2+1}\right)^N I_N\left(\gamma, \frac{\gamma_1}{\gamma_2+1}\right)$.

Therefore, it is enough to consider the integral (17). Introduce the following notations

$$\theta_i = (\xi_1,\dots,\xi_i), \qquad \chi_i = \left(a_0^{\gamma_1} - \sum_{j=1}^{i} \xi_j^{\gamma_1}\right)^{\frac{1}{\gamma_1}}, \quad i = \overline{N-1,1},$$

we can rewrite (17) as follows

$$\begin{aligned} I_N(\gamma,\gamma_1) &= \int_{\mathbb{R}^N} \left(a_0^{\gamma_1} - \xi^{\gamma_1}\right)_+^{\gamma} d\xi = \int_{\mathbb{R}^{N-1}} d\theta_{N-1} \int_{\mathbb{R}} \left(\chi_{N-1}^{\gamma_1} - \xi_N^{\gamma_1}\right)_+^{\gamma} d\xi_N \\ &= \int_{\mathbb{R}^{N-1}} d\theta_{N-1} \int_{-\chi_{N-1}}^{\chi_{N-1}} \left(\chi_{N-1}^{\gamma_1} - \xi_N^{\gamma_1}\right)_+^{\gamma} d\xi_N = \left|\xi_N = \chi_{N-1} z_N^{\frac{1}{\gamma_1}}\right| \\ &= \frac{2}{\gamma_1} \int_{\mathbb{R}^{N-1}} \chi_{N-1}^{\gamma\gamma_1+1} d\theta_{N-1} \int_0^1 (1-z_N)_+^{\gamma} z_N^{\frac{1}{\gamma_1}-1} dz_N = \frac{2}{\gamma_1} B\left(\gamma+1, \frac{1}{\gamma_1}\right) I_{N-1}\left(\gamma + \frac{1}{\gamma_1}\right) \\ &= \dots = I_1\left(\gamma + \frac{N-1}{\gamma_1}\right) \prod_{i=1}^{N-1} \left(\frac{2}{\gamma_1} B\left(\gamma + 1 + \frac{i-1}{\gamma_1}, \frac{1}{\gamma_1}\right)\right) \\ &= a_0^{N+\gamma\gamma_1} \prod_{i=1}^{N} \left[\frac{2\Gamma\left(\frac{1}{\gamma_1}\right)}{\gamma_1} \cdot \frac{\Gamma\left(\gamma+1+\frac{i-1}{\gamma_1}\right)}{\Gamma\left(\gamma+1+\frac{i}{\gamma_1}\right)}\right] = a_0^{N+\gamma\gamma_1} \frac{\Gamma(\gamma+1)}{\Gamma\left(\gamma+1+\frac{N}{\gamma_1}\right)} \left(\frac{2}{\gamma_1}\Gamma\left(\frac{1}{\gamma_1}\right)\right)^N. \end{aligned}$$

Given that $a = a_0^{\gamma_1}$, $\gamma_2 = -l_1$ and put that to the above equation, we obtain the following equation

$$\frac{E_0}{A} = J(\gamma, \gamma_1, \gamma_2) = \left(\frac{1}{\gamma_2+1}\right)^N I_N\left(\gamma, \frac{\gamma_1}{\gamma_2+1}\right) = \frac{a^{\frac{N}{\gamma_1}+\gamma}\Gamma(\gamma+1)}{\Gamma\left(\gamma+1+N\left(\frac{1-l_1}{\gamma_1}\right)\right)}\left(\frac{2}{\gamma_1}\Gamma\left(\frac{1-l_1}{\gamma_1}\right)\right)^N.$$

Hence,

$$a = a(E_0) = \left[\frac{E_0}{A}\frac{\Gamma\left(\gamma+1+N\left(\frac{1-l_1}{\gamma_1}\right)\right)}{\Gamma(\gamma+1)}\left(\frac{\gamma_1}{2\Gamma\left(\frac{1-l_1}{\gamma_1}\right)}\right)\right]^{\frac{\gamma_1}{N+\gamma\gamma_1}}.$$

4 The Numerical Scheme

We consider the numerical solution of the problem (1)–(2) and their approximations for one-dimensional space.

In the domain $\bar{D} = \{(t,x) : 0 < t_0 \le t \le T,\ 0 \le x \le b\}$, $T, t_0, b = \text{const} > 0$ we construct a uniform grid $\bar{\omega}_h$ for x in steps of h_x:

$$\bar{\omega}_{h_x} = \left\{x_i = ih_x : h_x > 0,\ i = \overline{0,n},\ h_x n = b\right\},$$

and a temporary grid

$$\bar{\omega}_{h_t} = \left\{t_j = jh_t : h_t > 0,\ j = \overline{0,m},\ h_t m = T - t_0\right\}.$$

Let us replace problem (1)–(2), by using the balance method, implicit difference scheme [1,6] and obtain the following difference problem with an error $O(h^2+\tau)$

$$\frac{y_i^{j+1} - y_i^j}{h_t} = \frac{c_i\left(y^{j+1}\right)}{h_x^2}\left[a_{i+1}(y^{j+1})(y_{i+1}^{j+1} - y_i^{j+1}) - a_i(y^{j+1})(y_i^{j+1} - y_{i-1}^{j+1})\right] - \varphi_i^j, \quad i = \overline{1,n-1},\ j = \overline{1,m-1}, \tag{19}$$

with boundary conditions

$$y_i^0 = u_0(x_i),\ i = \overline{1,n}, \qquad y_0^j = \theta_1(t_j),\ j = \overline{1,m}, \qquad y_n^j = \theta_2(t_j),\ j = \overline{1,m}. \tag{20}$$

For calculating $a(y)$, $c(y)$, $\theta_1(t)$, $\theta_2(t)$ the following formulas are used

$$c_i(y) = |x_i|^{l_1}\left(y_i^j\right)^q, \quad a_i(y^j) = |x_i|^n\left(y_i^{j+1}\right)^{m-1}\left|\frac{\left(y_i^{j+1}\right)^k - \left(y_{i-1}^{j+1}\right)^k}{h_x}\right|^{p-2},$$

$$\theta_1(t_j) = \left(Aw_0 t^{-\alpha}\right)^{\frac{1}{1-q}}, \quad \theta_2(t_j) = \left(Aw_0 t^{-\alpha}\left(a - b^{\gamma_1}t^{\mu\gamma_1}\right)_+^{\gamma}\right)^{\frac{1}{1-q}}, \quad \varphi_i^j = |x_i|^{l_1-l_2}\left(t_j\right)^l\left(y_i^j\right)^{\beta}.$$

The system of algebraic equations (19) is nonlinear relative to y^{j+1}. To solve the system of nonlinear equations, apply iterative methods, we obtain

$$\frac{\overset{s+1}{y}{}_i^{j+1} - \overset{s}{y}{}_i^{j}}{h_t} = \frac{c_i\left(\overset{s}{y}{}^{j+1}\right)}{h_x^2}\left[a_{i+1}\left(\overset{s}{y}{}^{j+1}\right)\left(\overset{s+1}{y}{}_{i+1}^{j+1} - \overset{s+1}{y}{}_i^{j+1}\right) - a_i\left(\overset{s}{y}{}^{j+1}\right)\left(\overset{s+1}{y}{}_i^{j+1} - \overset{s+1}{y}{}_{i-1}^{j+1}\right)\right] - \overset{s}{\varphi}{}_i^j, \quad i = \overline{1,n-1},\ j = \overline{1,m-1},\ s = 0,1,2,\dots. \tag{21}$$

The difference scheme (21) is linear with respect to $\overset{s+1}{y}$. As an initial iteration for $\overset{s+1}{y}$ it is taken the value from the previous time step and $\overset{0}{y^{j+1}} = y^j$. With the score on the iterative scheme set accuracy of iteration and process continues for as long while the condition $\max\limits_{0\le i\le n}\left|\overset{s+1}{y_i} - \overset{s}{y_i}\right| < \varepsilon$ is not satisfied.

In (21) we introduce the following notations

$$\overset{s}{A_i} = \frac{\tau}{h_x^2} c_i\left(\overset{s}{y^{j+1}}\right) a_{i+1}\left(\overset{s}{y^{j+1}}\right), \quad \overset{s}{B_i} = \frac{\tau}{h_x^2} c_i\left(\overset{s}{y^{j+1}}\right) a_i\left(\overset{s}{y^{j+1}}\right),$$
$$\overset{s}{C_i} = \overset{s}{A_i} + \overset{s}{B_i} + 1, \quad \overset{s}{F_i^j} = h_t\, \overset{s}{\varphi_i^j} - \overset{s}{y_i^j}.$$

Now difference equation (21) could be written as

$$\overset{s}{A_i}\overset{s+1}{y_{i+1}^{j+1}} - \overset{s}{C_i}\overset{s+1}{y_i^{j+1}} + \overset{s}{B_i}\overset{s+1}{y_{i-1}^{j+1}} = -\overset{s}{F_i^j}, \quad i = \overline{1, n-1}. \tag{22}$$

For the numerical solution of equation (22) it is used the Thomas (tridiagonal) algorithm [1]. From the boundary conditions we have $y_0^{j+1} = \theta_1\left(t_{j+1}\right)$, $y_n^{j+1} = \theta_2\left(t_{j+1}\right)$.

Let

$$y_i^{j+1} = \lambda_i y_{i+1}^{j+1} + \chi_i, \tag{23}$$

where $\lambda_i = \dfrac{A_i}{C_i - \lambda_{i-1}B_i}$, $\chi_i = \dfrac{B_i\chi_{i-1} + F_i}{C_i - \lambda_{i-1}B_i}$, $i = \overline{2, n}$.

The values

$$\lambda_1 = \frac{A_1}{C_1}, \qquad \chi_1 = \frac{B_1\theta_1 + F_1}{C_1}$$

could be found from (22), (23) at $i = 1$. The solution of system (19) is determined by the recursive formula (23). The values y_n^{j+1} we obtain from the boundary conditions (20), namely

$$\begin{cases} y_n^{j+1} = \theta_2, \\ y_{n-1}^{j+1} = \lambda_{n-1} y_n^{j+1} + \chi_{n-1}, \\ \dots \\ y_i^{j+1} = \lambda_i y_{i+1}^{j+1} + \chi_i, \\ \dots \end{cases}$$

To apply the tridiagonal matrix algorithm sufficient to require that the coefficients in (21) satisfy the conditions (diagonally dominant conditions)

$$A_i, B_i \neq 0, \quad |C_i| \geq |A_i| + |B_i|, \quad i = \overline{1, n}.$$

For the iterative scheme (22), the Thomas algorithm is stable and gives a unique solution [6].

4.1 Numerical analysis of solutions

For numerical solutions, it is important to choose an appropriate initial approximation that preserves its nonlinearity properties. In the slow diffusion case as an initial approximation it is used $u_0(t, x) = \left(Aw_0 t^{-\alpha}\overline{f}(\xi)\right)^{\frac{1}{1-q}}$. Based on the qualitative studies above, numerical calculations were carried out. The numerical results show quick convergence of the iterative process to the solution of the Cauchy problem (1)–(2), due to the successful choice of the initial approximation. For the numerical solution, the generalization of the Samarskii-Sobol scheme [6] for the numerical calculation of wave type temperature is used. In the same place running animation can trace the evolution of the process in time. The figures below illustrate an evolution of the diffusion process with a finite speed of perturbation in the one- and two-dimensional cases.

5 Conclusion

The property of the finite speed of perturbation and a space localization of the Cauchy problem for a nonlinear parabolic equation is established. This property highlights the significance of understanding the behavior of solutions about the speed at which perturbations propagate, as well as their spatial localization. Through self-similar analysis, a condition known as Fujita type global solvability has been derived for the Cauchy problem in the context of double nonlinear degenerate-type parabolic equations with variable density. By considering the estimates of weak solutions and fronts (also known as free boundaries), this study provides valuable insights into the behavior of solutions and the free boundary. It has been observed that the behavior of these solutions and free boundaries is closely related to the exponential growth rate of the density and initial data. Such analysis allows researchers to gain a deeper understanding of the intricate relationship between the properties of the equation and the resulting solutions. In all cases, it is important to consider the property of a finite speed of perturbation solution. This property ensures that any changes or disturbances in the system propagate at a limited speed, allowing for more stable and predictable behavior. An interesting aspect to explore is the concept of self-similar solutions. The asymptotic behavior of these self-similar solutions is of great interest in many scientific fields, as it provides valuable insights into the underlying dynamics of the system. When studying the asymptotic behavior of equations, it has been established that the coefficients of the main terms of the asymptotic solution satisfy a system of nonlinear algebraic equations.

The results obtained from the computational tests provide valuable information on the effectiveness and suitability of self-similar solutions as an approximation method. By utilizing both the solutions derived from the nonlinear splitting approach and the standard equation method as initial approximations, we can effectively employ an iterative method based on Picard's method to solve these nonlinear problems while also retaining the essential nonlinear effects [6].

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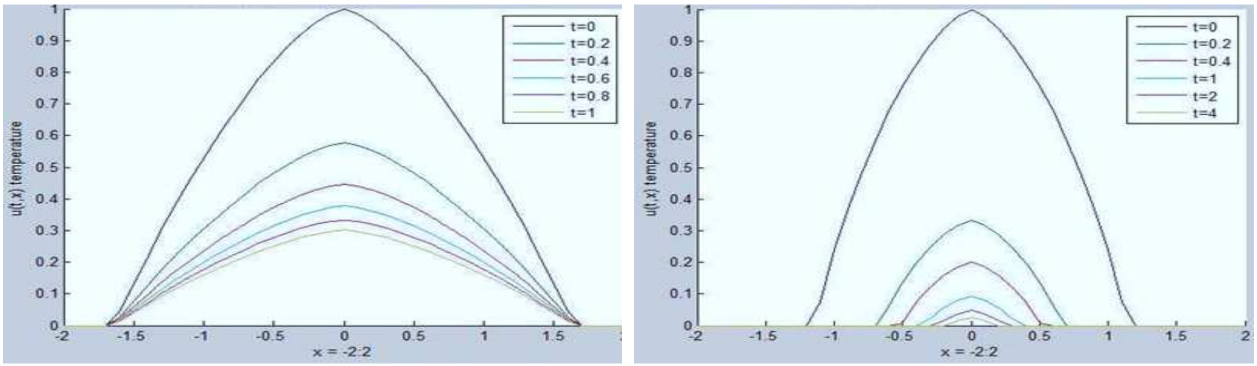
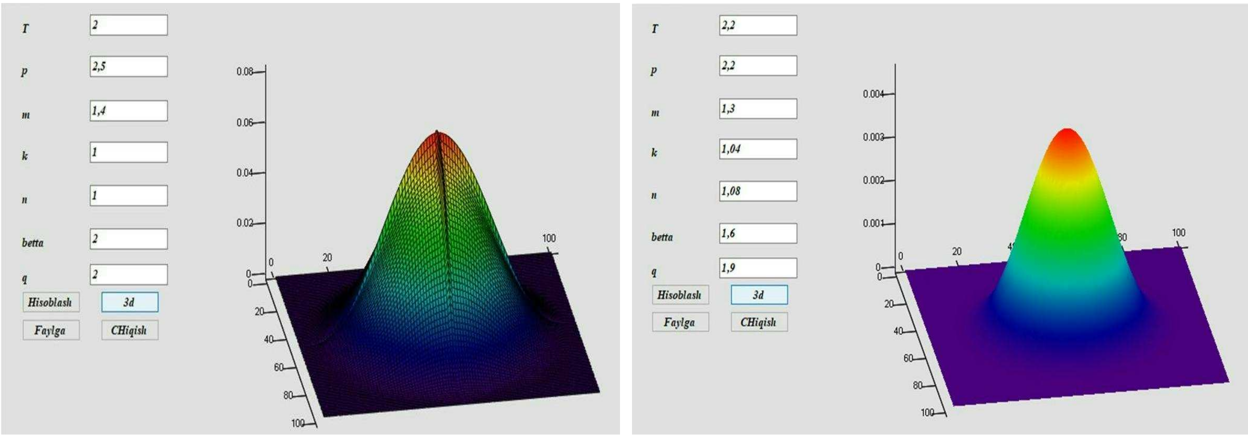


(a) $q = 0.8$, $m = k = 1$, $p = 3$, $n = 0$, $l_1 = l_2 = -1$, $\beta_2 = 1$

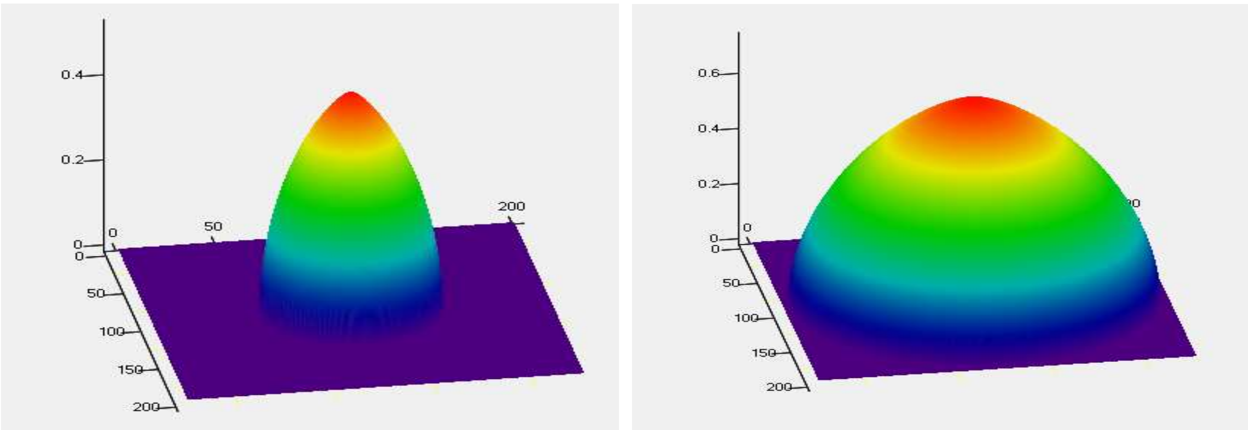
(b) $q = 0.9$, $m = k = 1$, $p = 2$, $n = l_1 = l_2 = 0$, $\beta_2 = 1$

Figure 1. One-dimensional case



(a) $q = 0.2$, $m = 1.4$, $k = 1$, $p = 2.5$, $n = 1$, $l_1 = l_2 = -1$, $l = 0$, $\beta = 2$

(b) $q = 0.19$, $m = 1.3$, $k = 1.04$, $p = 2.5$, $n = 1.08$, $l_1 = l_2 = 0$, $\beta = 1.6$



(c) $q = 0$, $m = 3$, $k = 3.2$, $p = 4$, $n = 0.2$, $l_1 = l_2 = -1$, $l = 0$, $\beta = 1.3$

(d) $q = 0$, $m = 3.1$, $k = 3$, $p = 3.4$, $n = 0.3$, $l_1 = l_2 = -1$, $\gamma = 1$, $l = 0$, $\beta = 3$

Figure 2. Two-dimensional case

References

[1] Aripov M., Matyakubov A., Bobokandov M. *Cauchy problem for the heat dissipation equation in non-homogeneous medium*. AIP Conference Proceedings 2023, **2781** (1), 020027. doi:10.1063/5.0144807

[2] Andreucci D., Tedeev A.F. *A Fujita type result for a degenerate Neumann problem in domains with noncompact boundary*. J. Math. Anal. Appl. 1999, **231** (2), 543–567. doi:10.1006/jmaa.1998.6253

[3] Andreucci D., Tedeev A.F. *Sharp estimates and finite speed of propagation for a Neumann problem in domains narrowing at infinity*. Adv. Differential Equations 2000, **5** (7–9), 833–860. doi:10.57262/ade/1356651289

[4] Aripov M., Bobokandov M. *Analysis of a double nonlinear parabolic equation with a source in an inhomogeneous medium*. Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. 2024, **50** (2), 285–306. doi:10.30546/2409-4994.2024.50.2.285

[5] Aripov M., Bobokandov M. *The Cauchy problem for a doubly nonlinear parabolic equation with variable density and nonlinear time-dependent absorption*. J. Math. Sci. 2023, **277** (3), 355–365. doi:10.1007/s10958-023-06840-0

[6] Aripov M., Sadullaeva S. Computer simulation of nonlinear diffusion processes. University Press, Tashkent, 2020. (in Russian)

[7] Aripov M., Bobokandov M., Mamatkulova M. *Analysis of a double nonlinear diffusion equation in inhomogeneous medium*. J. Math. Sci. 2024, 1–13. doi:10.1007/s10958-024-07384-7

[8] Aripov M., Mukimov A., Mirzayev B. *To asymptotic of the solution of the heat conduction problem with double nonlinearity with absorption at a critical parameter*. Math. Stat. 2019, **7** (5), 205–217. doi:10.13189/ms.2019.070507

[9] Jin C., Yin J. *Asymptotic behavior of solutions for a doubly degenerate parabolic non-divergence form equation*. Rocky Mountain J. Math. 2017, **47** (2), 479–510. doi:10.1216/RMJ-2017-47-2-479

[10] Fujita H. *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$. J. Fac. Sci. Univ. Tokyo, Sect. 1 A, Mathematics. 1996, **13** (2), 109–124. doi:10.15083/00039873

[11] Galaktionov V.A., Levine H.A. *A general approach to critical Fujita exponents in nonlinear parabolic problems*. Nonlinear Anal. 1998, **34** (7), 1005–1027. doi:10.1016/S0362-546X(97)00716-5

[12] Huang Q., Mochizuki K., Mukai K. *Life span and asymptotic behavior for a semilinear parabolic system with slowly decaying initial values*. Hokkaido Math. J. 1998, **27** (2), 393–407.

[13] Shao J., Guo Z., Shan X., Zhang C., Wu B. *A new non-divergence diffusion equation with variable exponent for multiplicative noise removal*. Nonlinear Anal. 2020, **56**, 103166. doi:10.1016/j.nonrwa.2020.103166

[14] Kamin S., Vazquez J.L. *Fundamental solutions and asymptotic behaviour for the p-Laplacian equation*. Rev. Mat. Iberoam. 1988, **4** (2), 339–354.

[15] Martynenko A.V., Tedeev A.F. *The Cauchy problem for a quasilinear parabolic equation with a source and nonhomogeneous density*. Comput. Math. Math. Phys. 2007, **47** (2), 238–248.

[16] Martynenko A.V., Tedeev A.F., Shramenko V.N. *Cauchy problem for a degenerate parabolic equation with inhomogeneous density and source in class slowly tending to zero initial functions*. Izv. RAN. Ser. Mat. 2012, **76** (3), 139–156.

[17] Winkler M., Stinner C. *Boundedness vs. blow-up in a degenerate diffusion equation with gradient nonlinearity*. Indiana Univ. Math. J. 2007, **56** (5), 2233–2264.

[18] Winkler M. *Infinite-time gradient blow-up in a degenerate parabolic equation*. Int. J. Dyn. Syst. Differ. Equ. 2012, **4** (1-2), 126–153. doi:10.1504/IJDSDE.2012.046001

[19] Mochizuki K., Mukai K. *Existence and nonexistence of global solutions to fast diffusions with source*. Methods Appl. Anal. 1995, **2** (1), 92–102. doi:10.4310/MAA.1995.v2.n1.a6

[20] Zheng P., Mu C., Liu D., Yao X., Zhou S. *Blow-up analysis for a quasilinear degenerate parabolic equation with strongly nonlinear source*. Abstr. Appl. Anal. 2012, **2012**, 109546. doi:10.1155/2012/109546

[21] Gianni R., Tedeev A., Vespri V. *Asymptotic expansion of solutions to the Cauchy problem for doubly degenerate parabolic equations with measurable coefficients*. Nonlinear Anal. 2016, **138**, 111–126. doi:10.1016/j.na.2015.09.006

[22] Sacks P.E. *Global behavior for a class of nonlinear evolution equations*. SIAM J. Math. Anal. 1985, **16**, 233–250.

[23] Samarskii A.A., Galaktionov V.A., Kurdyumov S.P., Mikhailov A.P. Blow-up in quasilinear parabolic equations. Walter de Grueter, Berlin, 1995. doi:10.1515/9783110889864

[24] Zhou S., Tang X., Yang C. *A doubly degenerate diffusion equation not in divergence form with gradient term*. Bound. Value Probl. 2016, **2016**, 126. doi:10.1186/s13661-016-0633-4

[25] Wang C.P., Yin J.X. *Shrinking self-similar solutions of a nonlinear diffusion equation with nondivergence form*. J. Math. Anal. Appl. 2004, **289** (2), 387–404. doi:10.1016/j.jmaa.2003.08.021

[26] Weissler F.B. *Existence and non-existence of global solutions for a semilinear heat equation*. Israel J. Math. 1981, **38** (1-2), 29–40.

[27] Wiegner M. *A degenerate diffusion equation with a nonlinear source term*. Nonlinear Anal. 1997, **28** (12), 1977–1995. doi:10.1016/S0362-546X(96)00027-2

[28] Xiang Zh., Mu Ch., Hu X. *Support properties of solutions to a degenerate equation with absorption and variable density*. Nonlinear Anal. 2008, **68**, 1940–1953. doi:10.1016/j.na.2007.01.021

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Аріпов М., Бобокандов М. *Задача Коші для подвійного нелінійного залежного від часу параболічного рівняння з поглинанням у неоднорідному середовищі* // Карпатські матем. публ. — 2025. — Т.17, №1. — С. 277–291.

У цій статті основну увагу приділено дослідженню властивостей самоподібних розв'язків задачі Коші. Ми зокрема аналізуємо поведінку цих розв'язків для подвійного нелінійного залежного від часу параболічного рівняння та їхнє поглинання в неоднорідному середовищі. Досліджуючи цю тему, ми ставимо за мету глибше зрозуміти властивість поширення збурень зі скінченною швидкістю при розв'язанні задачі Коші для нелінійного параболічного рівняння. Встановлення цієї властивості є ключовим для розуміння динамічної природи таких рівнянь. Крім того, ми здійснюємо самоподібний аналіз розв'язку, що дозволяє встановити умову глобальної розв'язності типу Фудзіти задачі Коші для подвійного нелінійного виродженого параболічного рівняння у неоднорідному середовищі. Такий аналіз надає цінну інформацію про поведінку та потенційну глобальну розв'язність цих рівнянь. Додатково ми встановлюємо оцінки для слабких розв'язків залежно від зростаючої густини та значень числових параметрів. Ці оцінки сприяють глибшому розумінню поведінки розв'язків у різних ситуаціях.

Ключові слова і фрази: скінченна швидкість, збурення, глобальний розв'язок, оцінка розв'язку, критичний випадок, асимптотична поведінка, числовий аналіз.