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Extended local convergence analysis of optimal eighth order method for solving equations in Banach space

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A local convergence analysis is developed for an eight-order method to solve Banach space defined nonlinear equation under ω-continuity. Earlier efforts require the existence of the ninth derivative to show the convergence on the finite Euclidean space $\mathbb{R}^k$. However, high order derivatives do not appear in the method. Moreover, no error estimates are available. Therefore, the previous efforts cannot assure the convergence if these derivatives do not exist although the method may converge. The present article addresses these problems. In particular, the new convergence conditions require only the existence of the first derivative appearing in the method. Moreover, error estimates become available. Furthermore, a region is determined containing only one solution of the equation. The novelty of the developed process allows its usage on other methods, since it is independent of the method. The numerical example complements the theory.

Key words and phrases: local convergence, eighth convergence order, convergence ball, Banach space.

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Introduction

A plethora of problems in Computational Mathematics reduce to solving the nonlinear equation [1, 10]

$$F(x) = 0, \tag{1}$$

where $F : \Omega \subset X \to Y$ is a differentiable operator according to Fréchet, X and Y denote Banach spaces, and $\Omega \subset X$ is an open and convex set. An exact solution x^* of the problem (1) can be found very rarely. Therefore, the approximate solution is usually found numerically by iterative methods [1, 5–10].

The most used method for solving nonlinear equation (1) is Newton's [1, 10]. Also, many scientists are developing and researching multi-step algorithms [2–6, 8, 9]. These methods have a higher order of convergence than Newton's method. The convergence order is affected by the number of method steps and the number of function computations. The convergence analysis of these methods was provided using derivatives or divided differences of order higher than one.

Considering these disadvantages we study multi-step methods using only the first derivative as well as the divided difference of order one that are in the iterative formulas of these

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methods. Moreover, error bounds on $\|x_n - x^*\|$ and uniqueness of the solution results are presented. We provide convergence analysis under ω-type conditions.

In this article, we demonstrate our technique on the eighth order method

$$\begin{aligned} y_n &= x_n - \alpha F'(x_n)^{-1}F(x_n), \\ z_n &= d_4(x_n, y_n), \\ x_{n+1} &= d_8(x_n, y_n, z_n), \end{aligned} \tag{2}$$

where α is a real or complex parameter, and $d_4 : \Omega \times \Omega \to X$, $d_8 : \Omega \times \Omega \times \Omega \to X$ are continuous operators.

If $X = Y = \mathbb{R}^k$, $\alpha = 1$, and d_4, d_8 are iterative functions of orders four and eight, respectively, then method (2) was shown in [9] to be of order eight.

1 Local Convergence

The local convergence analysis uses some scalar functions and positive parameters.

Set $S = [0, \infty)$. Suppose that the following assumptions hold.

(i) There exists a continuous and nondecreasing function $\omega_0 : S \to \mathbb{R}$ such that the equation

$$\omega_0(t) - 1 = 0$$

has a smallest solution $\rho_0 \in S \setminus \{0\}$. Set $S_0 = [0, \rho_0)$.

(ii) There exists a continuous and nondecreasing function $\omega : S_0 \to \mathbb{R}$ such that the equation

$$g_1(t) - 1 = 0$$

has a smallest solution $r_1 \in S_0 \setminus \{0\}$, where the function $g_1 : S_0 \to \mathbb{R}$ is defined by

$$g_1(t) = \frac{1}{1 - \omega_0(t)} \int_0^1 \omega(|1-\theta|t)d\theta + |1-\alpha| \left(1 + \int_0^1 \omega_0(|\theta|t)d\theta\right).$$

(iii) There exist continuous and nondecreasing functions $g_2, g_3 : S_0 \to \mathbb{R}$ such that the equations

$$g_2(t) - 1 = 0 \qquad \text{and} \qquad g_3(t) - 1 = 0$$

have smallest solutions $r_2, r_3 \in S_0 \setminus \{0\}$, respectively.

The parameter r, defined by

$$r = \min\{r_m\}, \quad m = 1, 2, 3, \tag{3}$$

will be shown to be a radius of convergence for method (2). Set $S_1 = [0, r)$. It follows by this definition and (3) that for all $t \in S_1$ we have

$$0 \le \omega_0(t) < 1$$

and

$$0 \le g_m(t) < 1, \quad m = 1, 2, 3. \tag{4}$$

Let $U(x, \lambda), U[x, \lambda]$ stand for the open and closed balls with center $x \in X$ and radius $\lambda > 0$, respectively.

The aforementioned functions and parameters are connected to the following conditions.

(h1) There exists a simple solution $x^* \in \Omega$ of an equation $F(x) = 0$.
Set $U_0 = U(x^*, \rho_0) \cap \Omega$.

(h2) $\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq \omega_0(\|x - x^*\|)$ for all $x \in \Omega$.

(h3) $\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq \omega(\|x - y\|)$ for all $x, y \in U_0$.

(h4) $\|d_4(x, y) - x^*\| \leq g_2(\|x - x^*\|)\|x - x^*\|$ for all $x \in U_0$ and $y = x - \alpha F'(x)^{-1}F(x)$.

(h5) $\|d_8(x, y, z) - x^*\| \leq g_3(\|x - x^*\|)\|x - x^*\|$ for all $x \in U_0$, $y = x - \alpha F'(x)^{-1}F(x)$ and $z = d_4(x, x - \alpha F'(x)^{-1}F(x))$.

(h6) $U[x^*, r] \subset \Omega$.

Next, the main local convergence result is presented using the developed terminology and the "h" conditions.

Theorem 1. *Suppose conditions* (h1)–(h6) *hold and choose* $x_0 \in U(x^*, r) \setminus \{x^*\}$. *Then, sequence* $\{x_n\}$ *generated by method* (2) *is well defined, remains in* $U(x^*, r)$ *and converges to* x^*. *Moreover, the following estimates*

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r, \tag{5}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{6}$$

and

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| \tag{7}$$

hold for all $n = 0, 1, 2, \ldots$, *where the functions* g_m *are previously defined and parameter* r *is given by formula* (3).

Proof. Let $v \in U(x^*, r) \setminus \{x^*\}$. It follows by applying (h1), (h2) and using (3) that

$$\|F'(x^*)^{-1}(F'(v) - F'(x^*))\| \leq \omega_0(\|v - x^*\|) \leq \omega_0(r) < 1.$$

Thus $F'(v)^{-1} \in L(Y, X)$ and

$$\|F'(v)^{-1}F'(x^*)\| \leq \frac{1}{1 - \omega_0(\|v - x^*\|)} \tag{8}$$

by the Banach lemma on linear operators with inverses [1]. If $v = x_0$, then (8) implies that $F'(x_0)^{-1}$ is invertible. Hence, the iterate y_0 is well defined by method (2) for $n = 0$. We can also write

$$\begin{aligned}
y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) + (1 - \alpha)F'(x_0)^{-1}F(x_0) \\
&= x_0 - x^* - F'(x_0)^{-1}(F(x_0) - F(x^*)) \\
&\quad + (1 - \alpha)F'(x_0)^{-1}(F(x_0) - F(x^*) - F'(x^*)(x_0 - x^*) + F'(x^*)(x_0 - x^*)) \\
&= F'(x_0)^{-1}F'(x^*)F(x^*)^{-1}\left(F'(x_0) - \int_0^1 F'(x_0 + (1 - \theta)(x^* - x_0))d\theta\right)(x_0 - x^*) \\
&\quad + (1 - \alpha)F'(x_0)^{-1}F'(x^*)F(x^*)^{-1}\left(\int_0^1 F'(x_0 + (1 - \theta)(x^* - x_0))d\theta\right. \\
&\qquad\qquad\qquad\qquad \left. - F'(x^*) + F'(x^*)\right)(x_0 - x^*).
\end{aligned}$$

By using (3), (4) for $m = 1$, (8) for $v = x_0$, and (h1) – (h3), we get

$$\|y_0 - x^*\| \le \frac{\left[\int_0^1 \omega(|1-\theta|\|x_0 - x^*\|)d\theta + |1-\alpha|\left(1 + \int_0^1 \omega_0(|\theta|\|x_0 - x^*\|)d\theta\right)\right]\|x_0 - x^*\|}{1 - \omega(\|x_0 - x^*\|)}$$
$$= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \le \|x_0 - x^*\| < r$$

showing (5) for $n = 0$ and $y_0 \in U(x^*, r)$. Similarly, using (2), (h4) and (h5), we obtain

$$\|z_0 - x^*\| \le g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \le \|x_0 - x^*\|$$

and

$$\|x_1 - x^*\| \le g_3(\|x_0 - x^*\|)\|x_0 - x^*\| \le \|x_0 - x^*\|$$

showing (6), (7), respectively, and $z_0, x_1 \in U(x^*, r)$.

By simply replacing x_0, y_0, z_0, x_1 by x_i, y_i, z_i, x_{i+1} in the preceding calculations we complete the induction for estimates (5) – (7). Then, by the estimate

$$\|x_{i+1} - x^*\| \le b(\|x_i - x^*\|)\|x_i - x^*\| \le \|x_i - x^*\|,$$

where $b(\|x_i - x^*\|) = g_3(\|x_i - x^*\|) \in [0, 1)$, we conclude

$$\lim_{i\to\infty} x_i = x^* \qquad \text{and} \qquad x_{i+1} \in U(x^*, r).$$

□

Next, a uniqueness of the solution result for equation $F(x) = 0$ is presented.

Proposition 1. *Assume that there exists a simple solution $x^* \in U(x^*, \rho)$ of equation $F(x) = 0$ for some $\rho > 0$. Let condition* (h2) *holds and there exists $\rho_1 \ge \rho$ such that*

$$\int_0^1 \omega_0(\theta\rho_1)d\theta < 1. \tag{9}$$

Then the point x^* is the only solution of equation $F(x) = 0$ in the set $U_1 := U[x^*, \rho_1] \cap \Omega$.

Proof. Let $y^* \in U_1$ with $F(y^*) = 0$. Define the linear operator

$$T = \int_0^1 F'(x^* + \theta(y^* - x^*))d\theta.$$

Then, by using (h2) and (9), we obtain

$$\|F'(x^*)^{-1}(T - F'(x^*))\| \le \int_0^1 \omega_0(\theta\|x^* - y^*\|)d\theta \le \int_0^1 \omega_0(\theta\rho_1)d\theta < 1.$$

Hence, $x^* = y^*$ by the invertability of T and the identity

$$T(y^* - x^*) = F(y^*) - F(x^*) = 0.$$

□

Remark 1. *The uniqueness of the solution x^* was shown in Proposition 1 by using only condition* (h2)*. However, if all "h" conditions are used, then $r = \rho$.*

2 Special Cases

In this Section we specialize method (2) to determine functions g_2 and g_3.

Case 1. Choose $d_4(x_n, y_n) = y_n - A_nF(y_n)$ and $d_8(x_n, y_n, z_n) = z_n - B_nF(z_n)$, where

$$A_n = \left[2[y_n, x_n; F] - F'(x_n)\right]^{-1}, \qquad B_n = [z_n, x_n; F]^{-1}[z_n, y_n; F]C_n^{-1}$$

and

$$C_n = 2[z_n, y_n; F] - [z_n, x_n; F].$$

Let functions ω_1, ω_2, ω_3, $\omega_5 : S_0 \times S_0 \to \mathbb{R}$, $\omega_4 : S_0 \times S_0 \times S_0 \to \mathbb{R}$ be continuous and non-decreasing. Moreover, define scalar sequences and functions, provided that iterates x_n, y_n, z_n exist and $\|x_n - x^*\| \le t$, by

$$\begin{aligned}
p_n &= \omega_1(\|y_n - x^*\|, \|z_n - x^*\|) + \omega_4(\|x_n - x^*\|, \|y_n - x^*\|, \|z_n - x^*\|)\\
&\le \omega_1(g_1(t)t, g_2(t)t) + \omega_4(t, g_1(t)t, g_2(t)t) = p(t) < 1,\\
q_n &= \frac{1}{1-p_n} \le \frac{1}{1-p(t)} = q(t),\\
s_n &= \omega_4(\|x_n - x^*\|, \|y_n - x^*\|, \|z_n - x^*\|) + \omega_3(\|y_n - x^*\|, \|z_n - x^*\|)\\
&\le \omega_4(t, g_1(t)t, g_2(t)t) + \omega_3(g_1(t)t, g_2(t)t) = s(t).
\end{aligned}$$

Define the functions g_2 and g_3 by

$$g_2(t) = \frac{(\omega_2(t, g_1(t)t) + \omega_3(t, g_1(t)t))g_1(t)}{1 - (\omega_1(t, g_1(t)t) + \omega_2(t, g_1(t)t))}$$

and

$$g_3(t) = \frac{(\omega_4(t, g_1(t)t, g_2(t)t) + q(t)s(t)\omega_5(g_2(t)t, g_1(t)t))g_2(t)}{1 - \omega_1(t, g_2(t)t)}.$$

Notice that functions g_2 and g_3 are well defined on $S_1 = [0, \gamma)$ provided that equations

$$\omega_1(t, g_1(t)t) + \omega_2(t, g_1(t)t) - 1 = 0 \qquad \text{and} \qquad \omega_1(t, g_2(t)t) - 1 = 0$$

have smallest solutions γ_1 and γ_2, respectively in $S_0 \setminus \{0\}$ and $\gamma = \min\{\rho_0, \gamma_1, \gamma_2\}$.

Then, consider the following conditions (h7), replacing (h4) and (h5):

(h7)

$$\begin{aligned}
\|F'(x^*)^{-1}([y, x; F] - F'(x^*))\| &\le \omega_1(\|x - x^*\|, \|y - x^*\|),\\
\|F'(x^*)^{-1}([y, x; F] - F'(x))\| &\le \omega_2(\|x - x^*\|, \|y - x^*\|),\\
\|F'(x^*)^{-1}([y, x; F] - [y, x^*; F])\| &\le \omega_3(\|x - x^*\|, \|y - x^*\|),\\
\|F'(x^*)^{-1}([z, x; F] - [z, y; F])\| &\le \omega_4(\|x - x^*\|, \|y - x^*\|, \|z - x^*\|),\\
\|F'(x^*)^{-1}[z, y; F]\| &\le \omega_5(\|z - x^*\|, \|y - x^*\|),\\
\|F'(x^*)^{-1}F'(x)\| &\le \omega_6(\|x - x^*\|),\\
\|F'(x^*)^{-1}([x, x^*; F] - F'(y))\| &\le \omega_7(\|x - x^*\|, \|y - x^*\|),\\
\|F'(x^*)^{-1}[x, x^*; F]\| &\le \omega_8(\|x - x^*\|).
\end{aligned}$$

Then we use the estimates

$$\|F'(x^*)^{-1}(A_n^{-1} - F'(x^*))\| \le \omega_1(\|x_n - x^*\|, \|y_n - x^*\|) + \omega_2(\|x_n - x^*\|, \|y_n - x^*\|) < 1,$$

so

$$\begin{aligned}\|A_nF'(x^*)\| &\leq \frac{1}{1-\Big(\omega_1(\|x_n-x^*\|,\|y_n-x^*\|)+\omega_2(\|x_n-x^*\|,\|y_n-x^*\|)\Big)}\\ &\leq \frac{1}{1-(\omega_1(t,g_1(t)t)+\omega_2(t,g_1(t)t))},\\ z_n-x^* &= y_n-x^*-\left[2[y_n,x_n;F]-F'(x_n)\right]^{-1}F(y_n)\\ &= A_n\left[2[y_n,x_n;F]-F'(x_n)-[y_n,x^*;F]\right](y_n-x^*).\end{aligned}$$

Thus,

$$\begin{aligned}\|z_n-x^*\| &\leq \|A_nF'(x^*)\|\|F'(x^*)^{-1}[([y_n,x_n;F]-F'(x_n))+([y_n,x_n;F]-[y_n,x^*;F])]\|\|y_n-x^*\|\\ &\leq g_2(\|x_n-x^*\|)\|x_n-x^*\| \leq \|x_n-x^*\| < r,\end{aligned}$$

$$\begin{aligned}\|F'(x^*)^{-1}(C_n-F'(x^*))\| &\leq \omega_1(\|y_n-x^*\|,\|z_n-x^*\|)\\ &\quad+\omega_4(\|x_n-x^*\|,\|y_n-x^*\|,\|z_n-x^*\|)=p_n<1,\end{aligned}$$

so

$$\|C_n^{-1}F'(x^*)\| \leq \frac{1}{1-p_n}=q_n.$$

Moreover, we can write in turn that

$$\begin{aligned}&z_n-x^*-[z_n,x_n;F]^{-1}[z_n,y_n;F]C_n^{-1}[z_n,x^*;F](z_n-x^*)\\ &\quad=[z_n,x_n;F]^{-1}\big\{\big([z_n,x_n;F]-[z_n,y_n;F]\big)\\ &\qquad+[z_n,y_n;F]C_n^{-1}\big[2[z_n,y_n;F]-[z_n,x_n;F]-[z_n,x^*;F]\big]\big\}(z_n-x^*),\end{aligned}$$

so

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|z_n-x^*-B_nF(z_n)\|\\ &\leq \frac{\left[\omega_4(\|x_n-x^*\|,\|y_n-x^*\|,\|z_n-x^*\|)+q_ns_n\omega_5(\|z_n-x^*\|,\|y_n-x^*\|)\right]\|z_n-x^*\|}{1-\omega_1(\|x_n-x^*\|,\|z_n-x^*\|)}\\ &\leq g_3(\|x_n-x^*\|)\|x_n-x^*\| \leq \|x_n-x^*\|.\end{aligned}$$

Hence, we arrived to the following assertion.

Theorem 2. *Suppose that conditions* (h1), (h2), (h3), (h6) *and* (h7) *hold. Then the conclusions of Theorem 1 hold.*

Concerning the uniqueness of the solution we can also use function ω_1 (see (h7)) instead of ω_0 (see (h2)).

Proposition 2. *Assume that there exists a simple solution $x^* \in U(x^*,\rho)$ of equation $F(x)=0$ for some $\rho>0$. Let condition* (h7) *holds and there exists $\rho^* \geq \rho$ such that*

$$\omega_1(0,\rho^*)<1. \tag{10}$$

Then the point x^* is the only solution of equation $F(x)=0$ in the set $U_2 := U[x^*,\rho^*]\cap\Omega$.

Proof. Let $y^* \in U_2$ with $F(y^*) = 0$. Define the linear operator $T_1 = [x^*, y^*; F]$. Then, by using the first condition in (h7) and (10), we get

$$\|F'(x^*)^{-1}(T_1 - F'(x^*))\| \leq \omega_1(0, \|y^* - x^*\|) \leq \omega_1(0, \rho^*) < 1,$$

implying $x^* = y^*$. □

Comments similar to Remark 1 can follow.

Case 2. Choose $d_4(x_n, y_n) = y_n - A_n F(y_n)$, where $A_n = [y_n, x_n; F]^{-1} F'(x_n) [y_n, x_n; F]^{-1}$. Then, using that the iterates x_n, y_n exist, we have the estimate

$$\begin{aligned} z_n - x^* &= y_n - x^* - [y_n, x_n; F]^{-1} F'(x_n) [y_n, x_n; F]^{-1} [y_n, x^*; F](y_n - x^*) \\ &= \left(I - [y_n, x_n; F]^{-1} F'(x_n) [y_n, x_n; F]^{-1} [y_n, x^*; F]\right)(y_n - x^*). \end{aligned}$$

The expression inside the branches can be written as

$$[y_n, x_n; F]^{-1} \left[([y_n, x_n; F] - F'(x_n)) + F'(x_n)[y_n, x_n; F]^{-1}([y_n, x_n; F] - [y_n, x^*; F]) \right].$$

Composing by $F'(x^*)^{-1}$ and using the condition (h7), we get

$$\begin{aligned} &\|F'(x^*)^{-1} \left([y_n, x_n; F] - F'(x_n)\right)\| \\ &\qquad + \|F'(x^*)^{-1} F'(x_n)\| \|[y_n, x_n; F]^{-1} F'(x^*)\| \|F'(x^*)^{-1}([y_n, x_n; F] - [y_n, x^*; F])\| \\ &\qquad \leq \omega_2(\|x_n - x^*\|, \|y_n - x^*\|) + \frac{\omega_6(\|x_n - x^*\|)\omega_3(\|x_n - x^*\|, \|y_n - x^*\|)}{1 - \omega_1(\|x_n - x^*\|, \|y_n - x^*\|)} = e_n, \end{aligned}$$

so

$$\|z_n - x^*\| \leq \frac{e_n \|y_n - x^*\|}{1 - \omega_1(\|x_n - x^*\|, \|y_n - x^*\|)}.$$

It follows that function g_2 can be defined by

$$g_2(t) = \frac{1}{1 - \omega_1(t, g_1(t)t)} \left(\omega_2(t, g_1(t)t) + \frac{\omega_6(t)\omega_3(t, g_1(t)t)}{1 - \omega_1(t, g_1(t)t)} \right).$$

Case 3. Choose $d_4(x_n, y_n) = y_n - A_n F(y_n)$, where $A_n = 2[y_n, x_n; F]^{-1} - F'(x_n)^{-1}$. This time we get

$$z_n - x^* = y_n - x^* - A_n [y_n, x^*; F](y_n - x^*) = (I - A_n [y_n, x^*; F]) (y_n - x^*).$$

The expression inside the branches can be written as

$$F'(x_n)^{-1}(F'(x_n) - [y_n, x^*; F]) - 2[y_n, x_n; F]^{-1}(F'(x_n) - [y_n, x_n; F])F'(x_n)^{-1}[y_n, x^*; F].$$

Using the conditions (h7), we obtain

$$\begin{aligned} &\|F'(x_n)^{-1} F'(x^*)\| \|F'(x^*)^{-1}(F'(x_n) - [y_n, x^*; F])\| + 2\|[y_n, x_n; F]^{-1} F'(x^*)\| \\ &\quad \times \|F'(x^*)^{-1}(F'(x_n) - [y_n, x_n; F])\| \|F'(x_n)^{-1} F'(x^*)\| \|F'(x^*)^{-1}[y_n, x^*; F]\| \\ &\quad \leq \frac{\omega_7(\|y_n - x^*\|, \|x_n - x^*\|)}{1 - \omega_6(\|x_n - x^*\|)} + 2\frac{\omega_2(\|x_n - x^*\|, \|y_n - x^*\|)\omega_8(\|y_n - x^*\|)}{(1 - \omega_1(\|x_n - x^*\|, \|y_n - x^*\|))(1 - \omega_6(\|x_n - x^*\|))} \\ &\quad = \lambda_n. \end{aligned}$$

Therefore

$$\|z_n - x^*\| \leq \lambda_n \|y_n - x^*\|.$$

Hence, the function g_2 can be defined by

$$g_2(t) = \frac{\omega_7(g_1(t)t, t)}{1 - \omega_6(t)} + 2\frac{\omega_2(t, g_1(t)t)\omega_8(g_1(t)t)}{(1 - \omega_1(t, g_1(t)t))(1 - \omega_6(t))}.$$

3 Numerical Experiments

Example 1. *Let* $X = Y = \mathbb{R}$. *Define* $\Omega = (0,2)$ *and* $F : \Omega \to \mathbb{R}$ *by* $F(x) = x^3 - 1$. *Then* $F'(x) = 3x^2$ *and* $[x,y;F] = x^2 + xy + y^2$. *The solution of equation* $F(x) = 0$ *is* $x^* = 1$.

Let us compute the convergence radii of the method (2) *with* d_4 *and* d_8 *defined in* Case 1. *Functions* ω *and* ω_i, $i = \overline{0,5}$, *are linear by its arguments. Let us show it.*

Since $F'(x) - F'(y) = 3x^2 - 3y^2 = 3(x+y)(x-y)$, *then* $\omega_0(|x-x^*|) = A_0|x-x^*|$ *and* $\omega(|x-y|) = A|x-y|$, *where* $A_0 = \max\limits_{x\in\Omega} \frac{|x+x^*|}{(x^*)^2}$ *and* $A = \max\limits_{x,y\in U_0} \frac{|x+y|}{(x^*)^2}$.

Next, we can write

$$
\begin{aligned}
[x,y;F] - F'(x^*) &= x^2 + xy + y^2 - 3(x^*)^2 \\
&= x^2 - (x^*)^2 + y^2 - (x^*)^2 + xy - xx^* + xx^* - (x^*)^2 \\
&= (x-x^*)(x+2x^*) + (y-x^*)(x+y+x^*), \\
[x,y;F] - F'(x) &= x^2 + xy + y^2 - 3x^2 \\
&= (y-x)(2x+y) \\
&= (2x+y)\big((y-x^*) + (x^*-x)\big), \\
[y,x;F] - [y,x^*;F] &= x^2 + xy + y^2 - y^2 - yx^* - (x^*)^2 \\
&= x^2 - y^2 + xy - yx^* + y^2 - (x^*)^2 \\
&= (x-y)(x+y) + y(x-x^*) + (y-x^*)(y+x^*), \\
[z,x;F] - [z,y;F] &= z^2 + zx + x^2 - z^2 - zy - y^2 \\
&= x^2 - y^2 + zx - x^*x + x^*x - x^*y + x^*y - zy \\
&= (x-y)(x+y+x^*) + x(z-x^*) + (x^*-z)y, \\
[z,y;F] &= z^2 + zy + y^2 - zx^* + zx^* \\
&= z(z-x^*) + z(y-x^*+2x^*) + y^2 \\
&= z(z-x^*) + z(y-x^*) + 2zx^* + y^2.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\omega_1(|x-x^*|,|y-x^*|) &= A_1|x-x^*| + B_1|y-x^*|, \\
&\quad A_1 = \max_{x\in U_0} \frac{|x+2x^*|}{3(x^*)^2},\ B_1 = \max_{x,y\in U_0} \frac{|x+y+x^*|}{3(x^*)^2}, \\
\omega_2(|x-x^*|,|y-x^*|) &= A_2|x-x^*| + B_2|y-x^*|,\ A_2 = B_2 = \max_{x,y\in U_0} \frac{|2x+y|}{3(x^*)^2}, \\
\omega_3(|x-x^*|,|y-x^*|) &= A_3|x-x^*| + B_3|y-x^*|, \\
&\quad A_3 = \max_{x,y\in U_0} \frac{|x+2y|}{3(x^*)^2},\ B_3 = \max_{x,y\in U_0} \frac{|x+y|+|y+x^*|}{3(x^*)^2}, \\
\omega_4(|x-x^*|,|y-x^*|,|z-x^*|) &= A_4|x-x^*| + B_4|y-x^*| + C_4|z-x^*|, \\
&\quad A_4 = B_4 = \max_{x,y\in U_0} \frac{|x+y+x^*|}{3(x^*)^2},\ C_4 = \max_{x,y\in U_0} \frac{|x|+|y|}{3(x^*)^2}, \\
\omega_5(|z-x^*|,|y-x^*|) &= B_5|y-x^*| + C_5|z-x^*| + D_5, \\
&\quad B_5 = C_5 = \max_{z\in U_0} \frac{|z|}{3(x^*)^2},\ D_5 = \max_{y,z\in U_0} \frac{|2zx^*+y^2|}{3(x^*)^2}.
\end{aligned}
$$

We get $\rho_0 = \dfrac{1}{3}$ *and* $U_0 = \left(\dfrac{2}{3}, \dfrac{4}{3}\right)$. *Convergence radii for different* α *are shown in Table 1. We see that as* α *increases, the radius of the convergence region increases.*

	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
γ_1	0.1386	0.1560	0.1802	0.2179
γ_2	0.1174	0.1343	0.1586	0.1982
γ	0.1174	0.1343	0.1586	0.1982
r_1	0.0458	0.0984	0.1593	0.2308
r_2	0.0758	0.0957	0.1253	0.1748
r_3	0.0669	0.0826	0.1069	0.1516
r	0.0458	0.0826	0.1069	0.1516

Table 1. Radii of convergence

Let us choose $x_0 = 1.14$. *The method* (2) *for* $\alpha = 1$ *gives the solution* x^* *at 2 iterations for* $\varepsilon = 10^{-10}$ *and estimates* (5) – (7) *hold. The obtained results are shown in Table 2.*

n	$e_n = \|x_n - x^*\|$	$g_3(e_{n-1})e_{n-1}$	$\|y_n - x^*\|$	$g_1(e_n)e_n$	$\|z_n - x^*\|$	$g_2(e_n)e_n$
0	1.4000e-01	–	1.6489e-02	4.5057e-02	1.8580e-04	4.2224e-02
1	5.6775e-09	6.4332e-02	3.2234e-17	4.2979e-17	6.9270e-34	6.5070e-25
2	0	1.1205e-32				

Table 2. Error estimates

Example 2. *Consider the system of* k *equations*

$$\sum_{j=1}^{k} x_j + e^{x_i} - 1 = 0, \quad i = 1, \dots, k.$$

Here $X = Y = \mathbb{R}^k$, $\Omega \subseteq R^k$ *and* $x^* = (0, \dots, 0)^T$.

Let us choose $k = 10$, $x_0 = (15, \dots, 15)^T$ *and investigate the influence of the parameter* α *on the convergence speed of the considered methods. Results in Table 3 show that method* (2) *converges fastest if* $\alpha = 1$. *The influence of the operator* A_n *on the number of iterations is insignificant. However, it is easy to see that* Case 1 *has the smallest computational complexity.*

α	Case1	Case2	Case3
0.25	10	10	10
0.5	8	8	9
0.75	7	7	7
1	4	5	5

Table 3. Number of iterations

4 Conclusion

We studied a method with eighth-order convergence for solving nonlinear operator equations in Banach space. We provide the local convergence analysis under ω-type conditions.

This method and its special cases were studied earlier in the Euclidean space provided the ninth derivative of the operator. We use conditions only on the linear operators which exist in the method. Hence, the conditions are weaker and the applicability of this method is extended. The future topics include the study of the convergence of other iterative methods, where this process can be used, since it does not depend on the method.

We avoid that using conditions which are on the method and in Banach space.

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Аргирос І.К., Регмі С., Шахно С.М., Ярмола Г.П. *Розширений аналіз локальної збіжності оптимального методу восьмого порядку для розв'язування рівнянь у банаховому просторі* // Карпатські матем. публ. — 2026. — Т.18, №1. — C. 171–180.

Проведено аналіз локальної збіжності методу восьмого порядку для розв'язування нелінійного рівняння в банаховому просторі за ω-умов. Дослідження цього методу було проведено раніше. Але, щоб показати збіжність у скінченновимірному евклідовому просторі $\mathbb{R}^k$, вимагали існування похідної дев'ятого порядку. Однак похідні високого порядку не використовуються в методі. Крім того, оцінки похибок не було отримано. Таким чином, попередні дослідження не можуть забезпечити збіжність, якщо ці похідні не існують, хоча метод може збігатися. У цій статті розглядаються ці питання. Зокрема, нові умови збіжності вимагають існування лише першої похідної, яка використовується в методі. Крім того, отримано оцінки похибок та визначено область єдиності розв'язку рівняння. Числовий приклад доповнює теорію. Новизна розробленого підходу дослідження ітераційних процесів дозволяє використовувати його для інших методів, оскільки він не залежить від методу.

Ключові слова і фрази: локальна збіжність, восьмий порядок збіжності, область збіжності, банахів простір.