ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2025, **17** (1), 246–254
doi:10.15330/cmp.17.1.246-254

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2025, Т.17, №1, С.246–254

r-Subhypermodules over Krasner hypermodules

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In this study, we introduce the notion of r-subhypermodule of an $\mathcal{R}$-hypermodule, where $\mathcal{R}$ is a commutative Krasner hyperring. A proper subhypermodule N of M is said to be an r-subhypermodule if $a \cdot m \in N$ with $ann_M(a) = 0_M$ implies that $m \in N$ for each $a \in \mathcal{R}$, $m \in M$. We investigate the relations between the concept of prime subhypermodules and r-subhypermodules. We also give some results about r-subhypermodules.

Key words and phrases: r-hyperideal, r-subhypermodule, prime subhypermodule.

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Dedicated to Professor Thomas Vougiouklis on the occasion of his 75th birthday

1 Introduction

Hyperstructures were presented at the 8th Congress of Scandinavian Mathematicians in 1934 by the French mathematician F. Marty [9]. Hundreds of articles and books have been produced on the subject since then (see [1, 3–5, 15]). The composition of two elements is an element in a classical algebraic structure, whereas the composition of two elements is a set in an algebraic hyperstructure. An overview of the foundations of this theory has been recently published in [10].

In the sense of F. Marty, for $G \neq \varnothing$, a mapping $\circ : G \times G \longrightarrow P^{\star}(G)$ is a hyperoperation and $(G, \circ)$ is a hypergroupoid. For a hypergroupoid $(G, \circ)$, if $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in G$, which means $\bigcup\limits_{u \in x \circ y} u \circ z = \bigcup\limits_{v \in y \circ z} x \circ v$, then G is a semihypergroup. When $(G, \circ)$ is a semihypergroup with $x \circ G = G \circ x = G$ for all $x \in G$, then $(G, \circ)$ is called a hypergroup.

A nonempty set G along with the hyperoperation "$+$" is called [11] a canonical hypergroup if the following axioms hold:

i) $x + (y + z) = (x + y) + z$ for $x, y, z \in G$;

ii) $x + y = y + x$ for $x, y \in G$;

iii) there exists $0 \in G$ such that $x + 0 = \{x\}$ for any $x \in G$;

iv) for any $x \in G$, there exists a unique element $x' \in G$ such that $0 \in x + x'$ (x' is called the opposite of x and it is denoted by $-x$);

v) $z \in x + y$ implies that $y \in -x + z$ and $x \in z - y$, that is $(G, +)$ is inversible.

УДК 512.55
2020 *Mathematics Subject Classification:* 13A15, 13C05, 13C13.

In 1956, M. Krasner [7] introduced the notion of hyperfield. Later on, he introduced the notions of hyperring and hypermodule over a hyperring, known nowadays as Krasner hyperrings and Krasner hypermodules [8]. If $(G, +)$ is a canonical hypergroup, $(G, \circ)$ is a semigroup having 0 as a bilaterally absorbing element for all elements, and $\circ$ is distributive with respect to "+", then $(G, +, \circ)$ is known as Krasner hyperring. Besides them, there are also other kinds of hyperrings, as multiplicative hyperrings defined by R. Rota [14], or generalized hyperrings defined by T. Vougiouklis [16]. In this study, we deal with Krasner hyperrings.

A non-empty set M is called [2] a left hypermodule over a hyperring $\mathcal{R}$ ($\mathcal{R}$-hypermodule) if $(M, +)$ is a canonical hypergroup and there exists the map $\cdot : \mathcal{R} \times M \to M$ defined by $(r, m) \to r \cdot m$ such that for all $r_1, r_2 \in \mathcal{R}$ and $m_1, m_2 \in M$ we have

i) $r_1 \cdot (m_1 + m_2) = r_1 \cdot m_1 + r_1 \cdot m_2$;

ii) $(r_1 + r_2) \cdot m_1 = r_1 \cdot m_1 + r_2 \cdot m_1$;

iii) $(r_1 \cdot r_2) \cdot m_1 = r_1(r_2 \cdot m_1)$.

Furthermore, in this definition, if $\mathcal{R}$ is a Krasner hyperring and $r \cdot 0_R = 0_M$, then M is said to be a left Krasner hypermodule over $\mathcal{R}$. Similarly, a right Krasner hypermodule is defined. If M is both right and left Krasner hypermodule, M is called Krasner $\mathcal{R}$-hypermodule.

In 2015, R. Mohamadian [12] introduced the concept of r-ideals and pr-ideals, while r-submodules have been studied in [6]. S. Omidi et. al. [13] defined r-hyperideals in commutative hyperrings. Recently, in [17], r-hyperideals and pr-hyperideals in commutative Krasner hyperrings have been studied. Recall from [13] that a proper hyperideal N in a commutative Krasner hyperring G is called an r-hyperideal (respectively, pr-hyperideal), if $a \cdot b \in N$ with $ann(a) = 0$ implies that $b \in N$ (respectively, $b^n \in N$ for some $n \in \mathbb{N}$) for each $a, b \in G$.

In this paper, we generalize this concept to hypermodules. We concentrate on the similarities and differences between r-subhypermodules and prime subhypermodules. We investigate basic notions of r-subhypermodules.

2 r-Subhypermodules

Throughout this section, all hyperrings are commutative Krasner hyperrings with $1 \neq 0$ and all related hypermodules are unitary and nonzero Krasner hypermodules, unless otherwise stated.

Definition 1. *Let M be an $\mathcal{R}$-hypermodule and N be a subhypermodule of M such that $N \neq M$. Then N is called an r-subhypermodule if $r \cdot m \in N$ with $ann_M(r) = 0_M$ implies that $m \in N$ for each $r \in \mathcal{R}, m \in M$.*

Example 1. *Consider $\mathbb{Z}$-hypermodule $M = \{(x, y) : x, y \in \mathbb{Z}\}$ with trivial hyperoperations. $N_1 = \{(x, x) : x \in \mathbb{Z}\}$ is subhypermodule of M.*

Let $a \cdot (x, y) \in N_1$ with $ann(a) = 0_M$ for $a \in \mathbb{Z}, (x, y) \in M$. Then $(a \cdot x, a \cdot y) \in N$, hence $a \cdot x = a \cdot y$ and therefore $x = y$. Thus, N_1 is an r-subhypermodule of M.

Similarly, $N_2 = \{(x, 0) : x \in \mathbb{Z}\}$ and $N_3 = \{(0, x) : x \in \mathbb{Z}\}$ are also r-subhypermodules of M.

Corollary 1.

i) *The zero subhypermodule is an r-subhypermodule.*

ii) *The intersection of an arbitrary nonempty set of r-subhypermodules is an r-subhypermodule.*

iii) *The sum of two r-subhypermodules may not be an r-subhypermodule (see Example 2).*

iv) *We know that if N is a prime subhypermodule of M, then $(N :_{\mathcal{R}} M)$ is a prime hyperideal. But this is not always true for r-subhypermodules (see Example 3).*

Example 2. *Consider $\mathbb{Z}$-hypermodule $\mathbb{Z}_{12}$. We know that $I = <\overline{3}>$ and $J = <\overline{4}>$ are two proper subhypermodules of $\mathbb{Z}_{12}$. We easily see that I and J are also r-subhypermodules of $\mathbb{Z}_{12}$. But $I + J = \mathbb{Z}_{12}$. Since it is not proper, it is not an r-subhypermodule.*

Example 3. *$K = <\overline{2}>$ is an r-subhypermodule of $\mathbb{Z}$-hypermodule $\mathbb{Z}_{12}$. Then $(K :_{\mathbb{Z}} \mathbb{Z}_{12}) = 2\mathbb{Z}$. We see $2 \cdot 3 \in 2\mathbb{Z}$ and $ann_{\mathbb{Z}}(2) = 0$, but $3 \notin 2\mathbb{Z}$. So, $2\mathbb{Z}$ is not an r-hyperideal of $\mathbb{Z}$.*

We will denote the set $\{a \in \mathcal{R} : ann_M(a) \neq 0_M\}$ by $Z(M)$. Then N is an r-subhypermodule of M if and only if $Z(M/N) \subseteq Z(M)$. Moreover, the r-subhypermodules of $\mathcal{R}$-hypermodule are definitely the r-hyperideals of $\mathcal{R}$.

Lemma 1. *Let N be an r-subhypermodule of M. Then $(N :_{\mathcal{R}} M) \subseteq Z(M)$.*

Proof. We know that $(N :_{\mathcal{R}} M) = Ann(M/N) \subseteq Z(M/N)$. Since N is an r-subhypermodule of M, then $Z(M/N) \subseteq Z(M)$. Thus, $(N :_{\mathcal{R}} M) \subseteq Z(M)$, as requested. □

A subhypermodule P of a hypermodule H is called a prime subhypermodule if for any two subhypermodules A and B of H such that their product AB is contained in P at least one of A or B is contained in P.

In the context of abstract algebra, particularly in the study of hypermodules, prime subhypermodules and r-subhypermodules are distinct concepts. Here is an example that illustrates the difference between these concepts.

Example 4. *Consider the hypermodule $H = R^2$, where R is a ring. Let $M = \{(a, 0) : a \in R\}$ be a subhypermodule of H. M is an r-subhypermodule, because for any $(a, 0)$ in M and r in R, the product $(a, 0)r = (ar, 0)$ is also in M, satisfying the definition of an r-subhypermodule. However, M may not necessarily be a prime subhypermodule, as it does not have to satisfy the prime subhypermodule condition. The key difference between prime subhypermodules and r-subhypermodules is their definitions and the properties they must satisfy. Prime subhypermodules are defined based on containment in products, while r-subhypermodules are defined based on closure under right multiplication by ring elements. The example provided illustrates the distinction between these two concepts.*

However, if N is prime and $(N :_{\mathcal{R}} M) \subseteq Z(M)$, then N is an r-subhypermodule of M, since $Z(M/N) = (N :_{\mathcal{R}} M)$.

A nonempty set S of $\mathcal{R}$ is multiplicatively closed when $a \cdot b \in S$ for all $a, b \in S$. Suppose that S is a multiplicatively closed set of $\mathcal{R}$ and M is an $\mathcal{R}$-hypermodule. Then the fraction hypermodule at S is denoted by $S^{-1}M$. Therefore, the mapping $\varphi : M \to S^{-1}M$, given by $a = a/1$ for all $a \in M$, is natural $\mathcal{R}$-homomorphism.

Theorem 1. *Suppose that N is a proper subhypermodule of M. Then the following are equivalent:*

i) *N is an r-subhypermodule of M;*

ii) *$(r \cdot M) \cap N = r \cdot N$ for any $r \in \mathcal{R} \backslash Z(M)$;*

iii) *$N = (N :_M r)$ for any $r \in \mathcal{R} \backslash Z(M)$;*

iv) *$N = \varphi^{-1}(L)$, where $S = \mathcal{R} \backslash Z(M)$ and L is an $S^{-1}\mathcal{R}$-subhypermodule of $S^{-1}M$.*

Proof. *i)* $\Rightarrow$ *ii)* Let N be an r-subhypermodule of M. Obviously, $r \cdot N \subseteq (r \cdot M) \cap N$ for every $r \in \mathcal{R}$. We need to show that $(r \cdot M) \cap N \subseteq r \cdot N$. Let $x \in (r \cdot M) \cap N$. Then $x \in (r \cdot M)$ and $x \in N$. We get $x = r \cdot m \in N$ for some $m \in M$. We conclude that $m \in N$, since N is an r-subhypermodule and $r \notin Z(M)$.

ii) $\Rightarrow$ *iii)* We know that $N \subseteq (N :_M r)$ for every $r \in \mathcal{R}$. Let $m \in (N :_M r)$. Hence $m \cdot r \in N$. From *ii)*, we get $m \cdot r \in (r \cdot M) \cap N = r \cdot N$. This implies that $m \cdot r = n \cdot r$ for $n \in N$. Since $ann_M(r) = 0_M$, we get $m = n \in N$. Hence, $m \in N$. Thus, $(N :_M r) \subseteq N$.

iii) $\Rightarrow$ *iv)* It is known, that $N \subseteq \varphi^{-1}(S^{-1}N)$. Let $m \in \varphi^{-1}(S^{-1}N)$. So we have $\varphi(m) = m/1 \in S^{-1}N$ and $r \cdot m \in N$ for some $r \in S$. By *iii)*, we get $m \in (N :_M r) = N$.

iv) $\Rightarrow$ *i)* Assume that $N = \varphi^{-1}(L)$, where $S = \mathcal{R} \backslash Z(M)$ and L is an $S^{-1}\mathcal{R}$-subhypermodule of $S^{-1}M$. Let $r \cdot m \in N$ and $ann_M(r) = 0_M$. Then $\varphi(r \cdot m) = (r \cdot m)/1 = r/1 \cdot m/1 \in L$. Since $r \in S$ and L is an $S^{-1}\mathcal{R}$-subhypermodule, we get $1/r \cdot (r \cdot m)/1 = m/1 = \varphi(m) \in L$. Thus, $m \in \varphi^{-1}(L) = N$. □

Recall that subhypermodule N is said to be pure subhypermodule of M if $a \cdot N = N \cap a \cdot M$ for every $a \in \mathcal{R}$.

Corollary 2. *If N is r-subhypermodule of M, then N is pure subhypermodule.*

Proof. It is trivial to the previous theorem. □

The reverse of Corollary 2 is not necessarily true. In other words, just because a subhypermodule N is pure in M does not mean it is an r-subhypermodule of M. Here is a counterexample to illustrate this.

Example 5. *Consider the hypermodule $M = \mathbb{Z}_6$ with addition modulo 6, and let $N = \{0, 3\}$. Then N is a pure subhypermodule of M. However, N is not an r-subhypermodule of M. Consider the element 2 in M. If N were an r-subhypermodule, there should exist an element r in M such that $2 \cdot r = 3$, since $3 \in N$. However, there is no such r in M because the equation $2r \equiv 3 (mod\, 6)$ has no solution in integers. Therefore, the reverse of the given corollary is not true, and the fact that a pure subhypermodule does not imply that it is necessarily an r-subhypermodule.*

Proposition 1. *Every proper subhypermodule of $\mathcal{R}$-hypermodule M is an r-subhypermodule if and only if for every subhypermodule N of M we have $r \cdot N = N$ for every $r \in \mathcal{R} \backslash Z(M)$.*

Proof. Let every proper subhypermodule of M be an r-subhypermodule, N be a subhypermodule and $r \in \mathcal{R} \backslash Z(M)$. If $N = M$ and $r \cdot M \neq M$, then $r \cdot M$ is an r-subhypermodule. Since $r \cdot m \in r \cdot M$ for every $m \in M$ and $ann_M(r) = 0_M$, then we have $m \in r \cdot M$ and so $r \cdot M = M$, which is a contradiction. Let N be a proper subhypermodule of M. Then $r \cdot N \subseteq N \neq M$ and $r \cdot N$ is an r-subhypermodule. Similarly, we get $r \cdot N = N$.

Conversely, if for proper subhypemodule N of M, $r \cdot N = N$ for every $r \in \mathcal{R} \backslash Z(M)$, then we have $r \cdot M \cap N = r \cdot N$. Thus, by Theorem 1, N is an r-subhypermodule. □

Theorem 2. *Let M_1, M_2 be $\mathcal{R}$-hypermodules and $\varphi : M_1 \longrightarrow M_2$ is a good homomorphism. Then the following statements hold.*

i) If φ is a monomorphism and N_2 is an r-subhypermodule of M_2 with $\varphi^{-1}(N_2) \neq M_1$, then $\varphi^{-1}(N_2)$ is an r-subhypermodule of M_1.

ii) If φ is an epimorphism, N_1 is an r-subhypermodule of M_1 and $Ker(\varphi) \subseteq N_1$, then $\varphi(N_1)$ is an r-subhypermodule of M_2.

Proof. $i)$ Let φ be a monomorphism and N_2 is an r-subhypermodule of M_2 such that $\varphi^{-1}(N_2) \neq M_1$. Suppose that $x \cdot y \in \varphi^{-1}(N_2)$ and $ann_{M_1}(x) = 0_{M_1}$ for $x \in \mathcal{R}$, $y \in M_1$. Then $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) \in N_2$. Since $ann_{M_1}(x) = 0_{M_1}$, then $ann_{M_2}(\varphi(x)) = 0_{M_2}$. Since N_2 is an r -subhypermodule, then $\varphi(y) \in N_2$ and so $y \in \varphi^{-1}(N_2)$. Thus, $\varphi^{-1}(N_2)$ is an r-subhypermodule.

$ii)$ Let us assume that φ is an epimorphism and N_1 is an r-subhypermodule of M_1 and $Ker(\varphi) \subseteq N_1$. Let $x \cdot y \in \varphi(N_1)$ and $ann_{M_2}(x) = 0_{M_2}$ for $x \in \mathcal{R}$, $y \in M_2$. Since φ is onto, there exist $a, m \in \mathcal{R}$ such that $x = \varphi(a)$ and $y = \varphi(m)$. Then $x \cdot y = \varphi(a) \cdot \varphi(m) = \varphi(a \cdot m) \in \varphi(N_1)$. Moreover, since $Ker(\varphi) \subseteq N_1$, we get $a \cdot m \in N_1$. Since $ann_{M_2}(x) = 0_{M_2}$, we obtain $ann_{M_2}(\varphi(a)) = 0_{M_2}$ and so $ann_{M_1}(a) = 0_{M_1}$. We see that $m \in N_1$, because N_1 is an r-subhypermodule. Thus, $\varphi(m) = y \in \varphi(N_1)$, as requested. □

Corollary 3. *If I is a subhypermodule of M, then the following statements hold.*

i) If N is an r-subhypermodule of M with $I \nsubseteq N$, then $N \cap I$ is an r-subhypermodule of I.

ii) If N is an r-subhypermodule of M with $I \subseteq N$, then N/I is an r-subhypermodule of M/I.

Proof. $i)$ Let us assume that $x \cdot y \in N \cap I$ with $ann_I(x) = 0_I$ for $x \in \mathcal{R}$, $y \in I$. Then we have $x \cdot y \in N$ and $x \cdot y \in I$. Therefore, $ann_M(x) = 0_M$. Since N is an r-subhypermodule, we get $y \in N$. By our assumption, $y \in N \cap I$. Thus, $N \cap I$ is an r-subhypermodule of I.

$ii)$ Let us assume that N is an r-subhypermodule and $I \subseteq N$. Let $(x \oplus I) \cdot (y \oplus I) \subseteq N/I$ and $ann_{M/I}(x \oplus I) = 0_{M/I} = 0_M + I$ for any $x \in \mathcal{R}$, $y \in M$. Then $x \cdot y \oplus I \subseteq N/I$ and $ann_M(x) = 0_M$. We have $x \cdot y \in N$. Since N is an r-subhypermodule, we get that $y \in N$ and so $y \oplus I \subseteq N/I$. □

Proposition 2. *Let N and K/N be r-subhypermodules of M/N such that $N \subseteq K$. Then K is an r-subhypermodule of M.*

Proof. Assume that N and K/N are r-subhypermodules of M/N such that $N \subseteq K$. Let $x \cdot y \in K$ with $ann_M(x) = 0_M$ for $x \in \mathcal{R}$, $y \in M$. We have $x \cdot y \oplus N \subseteq K/N$ and so $(x \oplus N) \cdot (y \oplus N) \subseteq K/N$. Since $ann_{M/N}(x \oplus N) = 0_{M/N} = 0_M + N$ and K/N is an r-subhypermodule, we get $y \oplus N \in K/N$. Hence, $y \in K$. □

Proposition 3. *Let N be an r-subhypermodule of M and $\varnothing \neq K \subseteq \mathcal{R}$ such that $K \nsubseteq (N :_{\mathcal{R}} M)$. Then $(N :_M K)$ is an r-subhypermodule of M. Particularly, if $K \nsubseteq Ann_{\mathcal{R}}(M)$, then $(0_M :_M K)$ is always an r-subhypermodule.*

Proof. Let $x \cdot y \in (N :_M K)$ with $ann_M(x) = 0_M$ for $x \in \mathcal{R}, y \in M$. Then $x \cdot y \cdot k \in N$ for $k \in K$. Thus, $y \cdot k \in N$ and so $y \in (N :_M K)$, since N is an r-subhypermodule.

Moreover, when $K \nsubseteq Ann_{\mathcal{R}}(M)$, if we take 0_M instead of N, then $(0_M :_M K)$ is an r-subhypermodule. □

Corollary 4. *If $a \notin Ann_{\mathcal{R}}(M)$, then $ann_M(a)$ is an r-subhypermodule of M.*

Proposition 4. *Let M be an $\mathcal{R}$-hypermodule. If the zero subhypermodule is the only r-subhypermodule, then:*

i) the zero subhypermodule is a prime subhypermodule of M;

ii) $Ann_{\mathcal{R}}(M)$ is a prime hyperideal of $\mathcal{R}$.

Proof. i) Let $r \cdot m = 0_M$ and $r \notin Ann_{\mathcal{R}}(M)$. We know that $ann_M(r)$ is an r-subhypermodule. We get that $ann_M(r) = 0_M$, since we accept the zero subhypermodule is the only r-subhypermodule. Thus, $m = 0_M$ and so the zero subhypermodule is a prime subhypermodule.

ii) It is obvious from i). □

Recall that a proper subhypermodule P of M is prime iff when I is a hyperideal of $\mathcal{R}$ and J is a subhypermodule of M such that $I \cdot J \subseteq P$, then $I \subseteq (P :_{\mathcal{R}} M)$ or $J \subseteq P$. In the following, we give a similar result for r-subhypermodules.

Theorem 3. *Let us assume that N is proper subhypermodule of M. The following statements hold.*

i) N is an r-subhypermodule of M if and only if K is a hyperideal of $\mathcal{R}$ and L is a subhypermodule of M with $K \cap (\mathcal{R}\backslash Z(M)) \neq \varnothing$ and $K \cdot L \subseteq N$, then $L \subseteq N$.

ii) Suppose that $(N :_{\mathcal{R}} M) \subseteq Z(M)$ and N is not an r-subhypermodule of M. Then there exist a hyperideal K of $\mathcal{R}$ and a subhypermodule L of M with $K \cap (\mathcal{R}\backslash Z(M)) \neq \varnothing$, $N \subsetneqq L$, $(N :_{\mathcal{R}} M) \subsetneqq K$ and $K \cdot L \subseteq N$.

Proof. i) Let us suppose that N is an r-subhypermodule, K is a hyperideal of $\mathcal{R}$ and L is a subhypermodule of M with $K \cap (\mathcal{R}\backslash Z(M)) \neq \varnothing$ and $K \cdot L \subseteq N$. We find an element $r \in K$ such that $ann_M(r) = 0_M$. Since $r \cdot l \in N$ for every $l \in L$ and N is an r-subhypermodule, we get $l \in N$ and so $L \subseteq N$.

Conversely, let $r \cdot m \in N$ and $ann_M(r) = 0_M$ for $r \in \mathcal{R}, m \in M$. We take $K = r \cdot \mathcal{R}$ and $L = \mathcal{R} \cdot m$. Note that $K \cap (\mathcal{R}\backslash Z(M)) \neq \varnothing$ and $I \cdot L \subseteq N$. Then by assumption, we have $\mathcal{R} \cdot m \subseteq N$ and so $m \in N$. Hence, N is an r-subhypermodule.

ii) Since N is not an r-subhypermodule, there exist $r \in \mathcal{R}, m \in M$ such that $r \cdot m \in N$ with $ann_M(r) = 0_M$ and $m \notin N$. We take $K = (N :_{\mathcal{R}} m)$. Note that $r \in K$ and $r \notin (N :_{\mathcal{R}} M)$, because $ann_M(r) = 0_M$. Thus, we have $(N :_{\mathcal{R}} M) \subsetneqq K$. Let us take $L = (N :_M K)$. Since $m \notin N$ and $m \in L$, we obtain $N \subsetneqq L$. Hence, we get $N \subsetneqq L$, $(N :_{\mathcal{R}} M) \subsetneqq I$ and $K \cdot L = K \cdot (N :_M K) \subseteq N$. □

Theorem 4. *Let I, J and L be subhypermodules of M and K be a hyperideal of $\mathcal{R}$ such that $K \cap (\mathcal{R} \backslash Z(M)) \neq \varnothing$. Then the following statements hold.*

i) If I, J are r-subhypermodules of M with $K \cdot I = K \cdot J$, then $I = J$.

ii) If $K \cdot L$ is an r-subhypermodule of M, then $K \cdot L = L$. Especially, L is an r-subhypermodule.

Proof. $i)$ By the previous theorem, we have $I \subseteq J$, because $K \cdot I \subseteq J$ and J is r-subhypermodule. Similarly, $J \subseteq I$. Thus, $I = J$.

$ii)$ Similarly to $i)$, the proof is obvious. □

Theorem 5. *Let $P_1, P_2, P_3, \dots, P_n$ be prime subhypermodules of M such that $(P_i :_{\mathcal{R}} M)$s are not comparable. If $\bigcap\limits_{i=1}^{n} P_i$ is an r-subhypermodule, then P_i is an r-subhypermodule for each $i \in \{1, 2, \dots, n\}$.*

Proof. Let $r \cdot m \in P_k$ and $ann_M(r) = 0_M$ for $r \in \mathcal{R}, m \in M$. We have

$$x \in \Big(\bigcap_{i=1, i \neq k}^{n} (P_i :_{\mathcal{R}} M) \Big) \backslash (P_k :_{\mathcal{R}} M)$$

for some $x \in \mathcal{R}$. Then we get $x \cdot r \cdot m \in \bigcap\limits_{i=1}^{n} P_i$ and so $x \cdot m \in \bigcap\limits_{i=1}^{n} P_i \subseteq P_k$ since $\bigcap\limits_{i=1}^{n} P_i$ is an r-subhypermodule. Since P_k is a prime subhypermodule and $x \notin (P_k :_{\mathcal{R}} M)$, we get $m \in P_k$. □

Proposition 5. *Let N be a maximal r-subhypermodule of M. Then N is a prime subhypermodule.*

Proof. Assume that $r \cdot m \in N$, $m \notin N$ and $r \notin (N :_{\mathcal{R}} M)$. By Proposition 3, $(N :_M r)$ is an r-subhypermodule. N is a maximal r-subhypermodule, so we have $(N :_M r) = N$, which is a contradiction. Therefore, $r \in (N :_{\mathcal{R}} M)$. □

Now we give a similar proposition with prime avoidance lemma for r-subhypermodules.

Proposition 6. *Let $N, N_1, N_2, \dots, N_n$ be subhypermodules of M such that $N \subseteq \bigcup\limits_{i=1}^{n} N_i$ is an r-subhypermodule. Suppose that N_j is an r-subhypermodule and $(N_i :_{\mathcal{R}} M) \cap (\mathcal{R} \backslash Z(M)) \neq \varnothing$ for every $i \neq j$. If $N \nsubseteq \bigcup\limits_{i \neq j} N_i$, then $N \subseteq N_j$.*

Proof. Let $j = 1$. Then there exists $a \in N$ such that $a \notin \bigcup\limits_{i=2}^{n} N_i$ so $a \in N_1$.

Let $b \in N \cap N_2 \cap N_3 \cap \dots \cap N_n$. Then we have $a + b \subseteq N \backslash \bigcup\limits_{i=2}^{n} N_i$ and so $a + b \subseteq N_1$. Thus $b \in N_1$ and $N \cap N_2 \cap N_3 \cap \dots \cap N_n \subseteq N_1$. There exists $a_i \in (N_i :_{\mathcal{R}} M)$ such that $ann_M(a_i) = 0_M$ for $i = 2, 3, \dots, n$, since $(N_i :_{\mathcal{R}} M) \cap (\mathcal{R} \backslash Z(M)) \neq \varnothing$. Obviously, $ann_M(a_2 \cdot a_3 \cdot \dots \cdot a_n) = 0_M$. We have $a_2 \cdot a_3 \cdot \dots \cdot a_n \in \bigcap\limits_{i=2}^{n} (N_i :_{\mathcal{R}} M) \cap (\mathcal{R} \backslash Z(M))$. Since

$$\bigcap_{i=2}^{n} (N_i :_{\mathcal{R}} M) \cap N \subseteq N \cap N_2 \cap N_3 \cap \dots \cap N_n \subseteq N_1 \quad \text{and} \quad \bigcap_{i=2}^{n} (N_i :_{\mathcal{R}} M) \cap (\mathcal{R} \backslash Z(M)) \neq \varnothing,$$

from Theorem 3 we obtain $N \subseteq N_1$. □

Let M_1 be an $\mathcal{R}_1$-hypermodule and M_2 be an $\mathcal{R}_2$-hypermodule, where $\mathcal{R}_1$ and $\mathcal{R}_2$ are commutative Krasner hyperrings with identity. Let $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2$ and $M = M_1 \times M_2$. Then M is an $\mathcal{R}$-hypermodule with coordinate-wise addition and the scalar multiplication. Therefore, every subhypermodule can be written as $N = N_1 \times N_2$, where N_1 is a subhypermodule of M_1 and N_2 is a subhypermodule of M_2. Next theorem characterizes the r-subhypermodule of Cartesian product of hypermodules.

Lemma 2. *Let $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2$ and $M = M_1 \times M_2$, where M_1 is an $\mathcal{R}_1$-hypermodule and M_2 is an $\mathcal{R}_2$-hypermodule. If $N = N_1 \times N_2$ is a subhypermodule of M, then the following statements are equivalent:*

i) N is an r-subhypermodule of M;

ii) $N_1 = M_1$ and N_2 is an r-subhypermodule of M_2 or N_1 is an r-subhypermodule of M_1 and $N_2 = M_2$ or N_1, N_2 are r-subhypermodules of M_1 and M_2, respectively.

Proof. $i) \Rightarrow ii)$ Suppose that $N = N_1 \times N_2$ is an r-subhypermodule of $M = M_1 \times M_2$. Let $(a,b) \cdot (c,d) \in N_1 \times N_2$ with $ann_{M_1 \times M_2}((a,b)) = (0_{M_1}, 0_{M_2})$ for $(a,b) \in \mathcal{R}_1 \times \mathcal{R}_2$ and $(c,d) \in M_1 \times M_2$. This means $a \cdot c \in N_1$ and $b \cdot d \in N_2$. Since we assume $N_1 \times N_2$ is an r-subhypermodule, we get $(c,d) \in N_1 \times N_2$ and so $c \in N_1$ and $d \in N_2$. Since $ann_{M_1 \times M_2}((a,b)) = (0_{M_1}, 0_{M_2})$, we obtain $ann_{M_1}(a) = 0_{M_1}$ or $ann_{M_2}(b) = 0_{M_2}$. Thus, N_1, N_2 are r-subhypermodules of M_1 and M_2, respectively. The rest follows easily.

$ii) \Rightarrow i)$ Suppose that N_1, N_2 are r-subhypermodules of M_1 and M_2, respectively. Let $(a,b) \cdot (c,d) \in N_1 \times N_2$ with $ann_{M_1 \times M_2}((a,b)) = (0_{M_1}, 0_{M_2})$ for $(a,b) \in \mathcal{R}_1 \times \mathcal{R}_2$ and $(c,d) \in M_1 \times M_2$. Then $(a,b) \cdot (c,d) = (a \cdot c, b \cdot d) \in N_1 \times N_2$, which means $a \cdot c \in N_1$ and $b \cdot d \in N_2$. Since $ann_{M_1 \times M_2}((a,b)) = (0_{M_1}, 0_{M_2})$, we get $ann_{M_1}(a) = 0_{M_1}$ or $ann_{M_2}(b) = 0_{M_2}$. Moreover, since we assume that N_1, N_2 are r-subhypermodules of M_1 and M_2, we obtain $c \in N_1$ and $d \in N_2$. We conclude that $(c,d) \in N_1 \times N_2 = N$, as requested. □

Theorem 6. *Let $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_n$ and $M = M_1 \times M_2 \times \cdots \times M_n$, where M_i is an $\mathcal{R}_i$-hypermodule for $n \geq 1$ and $1 \leq i \leq n$. Let $N = N_1 \times N_2 \times \cdots \times N_n$ be a subhypermodule of M. The following statements are equivalent:*

i) N is an r-subhypermodule of M;

ii) $N_i = M_i$ for $i \in \{k_1, k_2, \ldots, k_j : j < n\}$ and N_i is an r-subhypermodule of M_i for $i \in \{1, 2, \ldots, n\} \backslash \{k_1, k_2, \ldots, k_j\}$.

Proof. If $n = 1$, it is obvious. If $n = 2$, by the previous lemma, $(i) \Leftrightarrow (ii)$. Suppose that the claim is true for $n = k \geq 3$. This means that $I = N_1 \times N_2 \times \cdots \times N_k$ is an r-subhypermodule of M if and only if $N_k = M_k$ and N_k is an r-subhypermodule of M_k. We need to show that it is also true for $n = k + 1$. By the previous lemma, $I \times N_{k+1}$ is an r-subhypermodule of M if and only if $N_{k+1} = M_{k+1}$ and N_{k+1} is an r-subhypermodule. □

3 Conclusion

In this study, we introduced the definition of r-subhypermodule and some basic algebraic properties are obtained. Moreover, we observed that the prime subhypermodules and r-subhypermodules have some similar characteristics. The r-subhypermodules of the Cartesian product of hypermodules has been investigated.

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Received 22.08.2023
Revised 22.11.2023

Болат М., Кая Е., Онар С., Ерсой Б.А., Хіла К. *r-Підгіпермодулі над гіпермодулями Краснера* // Карпатські матем. публ. — 2025. — Т.17, №1. — С. 246–254.

У цій роботі ми вводимо поняття r-підгіпермодуля $\mathcal{R}$-гіпермодуля, де $\mathcal{R}$ — комутативне гіперкільце Краснера. Власний підгіпермодуль N модуля M називають r-підгіпермодулем, якщо з умови $a \cdot m \in N$ та $ann_M(a) = 0_M$ випливає, що $m \in N$ для кожного $a \in \mathcal{R}$, $m \in M$. Ми досліджуємо зв'язок між поняттями простого підгіпермодуля та r-підгіпермодуля. Також наведено деякі результати щодо r-підгіпермодулів.

Ключові слова і фрази: r-гіперідеал, r-підгіпермодуль, простий підгіпермодуль.