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Cauchy problem for the double nonlinear parabolic equation not in divergent form with a time-dependent source or absorption

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This paper studies the properties of solutions for a double nonlinear time-dependent parabolic equation with variable density, not in divergence form, with a source or absorption. The problem is formulated as a partial differential equation with a nonlinear term that depends on the solution and the time. The main results are the existence of weak solutions in suitable function spaces; regularity and positivity of solutions; asymptotic behavior of solutions as time goes to infinity; comparison principles; and maximum principles for solutions. The proofs are based on comparison methods and asymptotic techniques. Some examples and applications are also given to illustrate the features of the problem.

Key words and phrases: degenerate parabolic equation, global solvability, weak solution, critical Fujita, asymptotic.

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1 Introduction

In the domain $Q = \left\{(t,x) : t > t_0 > 0,\, x \in \mathbb{R}^N\right\}$, we consider the following Cauchy problem to the double nonlinear parabolic equation with variable density, not in divergence form, with a time-dependent source or absorption,

$$|x|^{-n}\,\partial_t u = u^q div\left(|x|^{n_1} u^{m-1}\left|\nabla u^k\right|^{p-2}\nabla u\right) + \varepsilon |x|^{-n} t^l u^{\beta}, \quad (t,x) \in Q, \tag{1}$$

$$u\,(0,x) = u_0\,(x) \geq 0, \quad x \in \mathbb{R}^N, \tag{2}$$

where $k, m \geq 1$, $p \geq 2$, $0 < q < 1$, $\varepsilon = \pm 1$ and nonnegative n, l, β are given numerical parameters.

Problem (1)–(2) arises in different applications [14, 15, 20]. Equation (1) is of degenerate type. Therefore, in the domain Q, where $u = 0, \nabla u = 0$, it is of degenerate type. Therefore, in this case, we need to consider a weak solution that has physical meaning. The qualitative properties of problem (1)–(2) depending on $\varepsilon = 1$ and $\varepsilon = -1$ are different (see [5, 10, 11]).

Equation (1) for the particular value of numerical parameters when $\varepsilon = 1$ and $\varepsilon = -1$ intensively studied by many authors (see [3, 4, 6, 8] and references therein).

Weak solutions play an essential role in modelling real-world phenomena, since many differential equations are unable to admit sufficiently smooth solutions, and the weak formulation is the only method available for solving them. It often proves advantageous to prove the

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existence of weak solutions first and then show that those solutions are sufficiently smooth, even when an equation has differentiable solutions [19]. The elegant and beautiful work [15] revealed a new phenomenon of nonlinear PDEs and stimulated the study of similar phenomena for various parabolic, hyperbolic, and nonlinear Schrödinger-type equations [24]. Many significant discoveries have emerged since then, but it is not possible to provide a thorough summary in such a short paper. The interested reader is referred to the paper by Sh. Kamin and J.L. Vázquez [18] and references therein for a good account of related works. A brief review of some related results on parabolic equations is provided below.

Investigating qualitative properties of the problem, such as Fujita type global solvability, asymptotic solution, localisation of solution, finite speed propagation of distribution, blow-up solution, and so on, by many authors based on self-similar solutions (for example, see [9, 10] and references therein).

A.V. Martynenko and A.F. Tedeev [21] studied the Cauchy problem in the case $q = n = 0$, $k = 1, l = 0, \varepsilon = 1$. They showed that under some restrictions on the parameters, any nontrivial solution to the Cauchy problem blows up in a finite time and established a sharp universal estimate of the solution near the blow-up point. D. Andreucci and A.F. Tedeev [2] studied the Neumann problem in the case $n = l = 0, k = m = 1, \gamma = 1$. They established a sharp universal estimate of the solution near the blow-up point and showed the condition for the solutions to exist and to which class they belonged.

Also, D. Andreucci and A.F. Tedeev [1] proved prior supremum bounds for solutions in the case $n = l = 0$, $\gamma = 1$ as t approaches the time, when u becomes unbounded. Such bounds are universal in the sense that they do not depend on u.

Furthermore, we recall some well-known results. In particular, when $\gamma = 1$, $l = 0$, the authors of the work [6] studied the heat conduction equation with a nonlinear source term. They showed that for the Cauchy problem, the critical Fujita exponent is

$$\beta_c = m + k(p-2) + q + \frac{p(1-q)}{N-n}.$$

In the case $l = n = 0$, $k = 1$ and without a source term, the authors of [8] proved that, depending on the values of the numerical parameters and the initial value, there are global solutions to the Cauchy problem.

The purpose of this paper is to investigate the influence of a variable density and a time-dependent term on the evaluation of nonlinear processes. It is proved that the Fujita-type global solvability and asymptotic properties of the self-similar solution are based on an algorithm of nonlinear splitting. The problem has been solved with an initial approximation for the numerical solution of the problem (1)–(2), and it is suggested that the numerical scheme method and algorithm for solution keep the nonlinear properties solution of fast diffusion, slow diffusion, critical and singular cases.

Definition 1. *A weak solution to the problem* (1) – (2) *in* (t_0, T) *with* $0 < T \leq +\infty$ *is the function* $u(t, x)$ *with the property*

$$0 \leq u, \quad |x|^{n_1} u^{m-1} \left|\nabla u^k\right|^{p-2} \nabla u \in C(Q)$$

and satisfying the equation (1) *in a distributional sense.*

The existence of a weak solution is proved as in [10], where doubly nonlinear parabolic equations are considered.

2 Global solvability

Let us introduce the notation $v = u^{1-q}$ and put this into problem (1)–(2), we get

$$L(v) \equiv -|x|^{-n}v_t + div\left(|x|^{n_1}v^{m_2-1}\left|\nabla v^{k_2}\right|^{p-2}\nabla v\right) + \varepsilon(1-q)|x|^{-n}t^l v^{\beta_2}, \tag{3}$$

$$v\,|_{t=t_0} = v_0\,(x) = [u_0(x)]^{1/(1-q)}, \tag{4}$$

where $m_2 = \dfrac{m}{1-q}$, $k_2 = \dfrac{k}{1-q}$, $\beta_2 = \dfrac{\beta - q}{1-q}$.

We are looking for the solution $v(t,r)$ that has the following form

$$v(t,r) = \overline{v}(t)w\big(\tau(t), \varphi(|x|)\big),$$

where

$$\overline{v}(t) = \begin{cases} l_1 t^{\frac{1+l}{1-\beta_2}}, & \text{if } l \neq -1, \beta_2 \neq 1, \\ l_2\,(\ln t)^{\frac{1}{1-\beta_2}}, & \text{if } l = -1, \beta_2 \neq 1, \\ e^{-l_3 t^{l+1}}, & \text{if } l \neq -1, \beta_2 = 1, \\ t^{-l_4}, & \text{if } \beta_2 = -l = 1, \end{cases}$$

with $l_4 = \varepsilon(1-q)$, $l_3 = \dfrac{l_4}{1+l}$, $l_2 = [l_4\,(\beta_2 - 1)]^{\frac{1}{1-\beta_2}}$, $l_1 = [l_3\,(\beta_2 - 1)]^{\frac{1}{1-\beta_2}}$,

$$\tau(t) = \begin{cases} \dfrac{l_6\,(1-\beta_2)\,t^{\frac{l_5}{1-\beta_2}+1}}{l_5 + 1 - \beta_2}, & \text{if } l_5 \neq \beta_2 - 1, \\ l_6 \ln t, & \text{if } l_5 = \beta_2 - 1, \end{cases}$$

with $l_5 = (1+l)\,(m_2 + k_2\,(p-2) - 1)$, $l_6 = l_1^{m_2 + k_2(p-2)-1}$, and

$$\varphi(r) = \begin{cases} \dfrac{p r^{\frac{p-n-n_1}{p}}}{p - n - n_1}, & \text{if } p \neq n + n_1, \\ \ln r, & \text{if } p = n + n_1. \end{cases}$$

It is easy to verify that, for $p - n - n_1 > 0$, the unknown function w satisfies the following "radially symmetric" equation

$$\frac{\partial w}{\partial \tau} = \varphi^{1-s}\frac{\partial}{\partial \varphi}\left(\varphi^{s-1}w^{m_2-1}\left|\frac{\partial w^{k_2}}{\partial \varphi}\right|^{p-2}\frac{\partial w}{\partial \varphi}\right) + \frac{w^{\beta_2} + w}{l_7 \tau}, \tag{5}$$

where $s = \dfrac{p\,(N-n)}{p - n - n_1}$, $l_7 = -\dfrac{l_5 + 1 - \beta_2}{1+l}$, $n < N$, $p > n + n_1$.

Let us put in (5) the function

$$w\,(\tau, \varphi) = f\,(\xi), \quad \xi = \varphi\tau^{-1/p}. \tag{6}$$

Then, for defining function $f\,(\xi)$, we have degenerate type self-similarly differential equation

$$Af \equiv \xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1}f^{m_2-1}\left|\frac{df^{k_2}}{d\xi}\right|^{p-2}\frac{df}{d\xi}\right) + \frac{\xi}{p}\frac{df}{d\xi} + \frac{f + f^{\beta_2}}{l_7} = 0. \tag{7}$$

Notice that in the singular case $p = n + n_1$ the equation (1) after transformation (5) have the following form

$$\frac{\partial w}{\partial \tau} = \frac{\partial}{\partial \varphi}\left(w^{m_2-1}\left|\frac{\partial w^{k_2}}{\partial \varphi}\right|^{p-2}\frac{\partial w}{\partial \varphi}\right) + \tau(N-n)\left(w^{m_2-1}\left|\frac{\partial w^{k_2}}{\partial \varphi}\right|^{p-2}\frac{\partial w}{\partial \varphi}\right) + \frac{w^{\beta_2}+w}{l_7 \tau}. \tag{8}$$

We construct self-similar solutions in two ways, forward and backward, and then prove that all solutions satisfying equation (7) have the following asymptotic

$$\overline{f}(\xi) = \begin{cases} (a - b\xi^{\gamma_1})_+^{\gamma_2}, & \text{if } m_2 + k_2(p-2) \neq 1, \\ e^{-b\xi^{\gamma_1}}, & \text{if } m_2 + k_2(p-2) = 1, \end{cases} \tag{9}$$

where $b = \left[\gamma_1 \gamma_2 p k_2^{p-2}\right]^{-\frac{1}{p-1}}$, $\gamma_1 = \frac{p}{p-1}$, $\gamma_2 = \frac{p-1}{m_2 + k_2(p-2) - 1}$, $a = \text{const} \geq 0$ and $(d)_+ = \max\{d, 0\}$.

Let us introduce new notation

$$z(t,x) = \overline{v}(t)\,\overline{f}(\xi), \tag{10}$$

where $\overline{v}(t) = l_1 t^{\frac{1+l}{1-\beta_2}}, l > -1, \beta_2 \neq 1$.

Theorem 1. *Let* $m_2 + k_2(p-2) - 1 \geq 0$, $p > n + n_1$, $u(0,x) \leq z(0,x)$, $x \in \mathbb{R}^N$. *Then for solution of problem* (1)–(2) *the estimate* $u(t,x) \leq z(t,x)$ *holds in* Q.

Proof. We use the comparison of the solutions approach described in [10]. For the comparison of solution is taken the function $z(t,x)$. Substituting (10) into (3), we get

$$L\big(z(t,x)\big) = [\overline{v}(t)]^{m_2+k_2(p-2)}\left(\xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1}\overline{f}^{m_2-1}\left|\frac{d\overline{f}^{k_2}}{d\xi}\right|^{p-2}\frac{d\overline{f}}{d\xi}\right) + \frac{\xi}{p}\frac{d\overline{f}}{d\xi} + \frac{\overline{f}+\overline{f}^{\beta_2}}{l_7}\right). \tag{11}$$

Now, to prove Theorem 1, we should show that $L\big(z(t,x)\big) \leq 0$ in

$$D = \left\{(t,x) : t \geq t_0 > 0,\ |x| \leq \left(\left(\frac{a}{b}\right)^{p-1}\left(\frac{p-n-n_1}{p}\right)^{p}\tau\right)^{\frac{1}{p-n-n_1}}\right\}.$$

For this goal, we need to show that

$$\xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1}\overline{f}^{m_2-1}\left|\frac{d\overline{f}^{k_2}}{d\xi}\right|^{p-2}\frac{d\overline{f}}{d\xi}\right) + \frac{\xi}{p}\frac{d\overline{f}}{d\xi} + \frac{\overline{f}+\overline{f}^{\beta_2}}{l_7} \leq 0. \tag{12}$$

It is obvious that for the function $\overline{f}(\xi)$, the inequality (12) can be rewritten as follows

$$(a - b\xi^{\gamma_1})_+^{\gamma_2}\left[\frac{1}{l_7} - \frac{s}{p} + \frac{(a-b\xi^{\gamma_1})_+^{\gamma_2\beta_2-1}}{l_7}\right] \leq 0,$$

or

$$\frac{1}{l_7} - \frac{s}{p} + \frac{(a-b\xi^{\gamma_1})_+^{\gamma_2\beta_2-1}}{l_7} \leq 0 \quad \text{in} \quad \xi^{p/(p-1)} \leq \frac{a}{b}.$$

To execute the last inequality, it is necessary that $\frac{1}{l_7} - \frac{s}{p} < 0$, i.e.

$$\frac{1}{l_7} - \frac{s}{p} = -\frac{1+l}{(1+l)\left(m_2 + k_2\left(p-2\right)-1\right)+1-\beta_2} - \frac{N-n}{p-n-n_1} < 0,$$

or

$$\beta_2 \geq \beta_{2c} = 1 + (1+l)\left[m_2 + k_2\left(p-2\right) - 1 + \frac{p-n-n_1}{N-n}\right]$$

for sufficiently small a.

Then, according to the hypothesis of Theorem 1 and comparison principle, we obtain $v\left(t,x\right) = \left[u(t,x)\right]^{1/(1-q)} \leq z\left(t,x\right)$ in Q, if $v_0(x) \leq z\left(0,x\right)$, $x \in \mathbb{R}^N$.

The proof of Theorem 1 is completed. The value of β_{2c} is called the Fujita type critical exponent. □

2.1 Global solvability of Fujita type in singular case $(p = n + n_1)$

Let us introduce the following new functions

$$\begin{aligned} &w(\tau,\varphi) = f_1(\eta), \quad \eta = \frac{\ln r}{\tau^{1/p}}, \\ &z_1(t,x) = \overline{v}(t) f_1(\eta), \end{aligned} \tag{13}$$

where $f_1(\eta)$ is a Barenblatt profile type function and satisfies the equation (8) (see [12]).

Theorem 2. *Assume that* $m_2 + k_2(p-2) - 1 \geq 0$, $\frac{1}{p} \geq \frac{a^{\gamma_2\beta_2 - 1}+1}{l_7}$, $v_0(x) \leq z_1\left(0,x\right)$, $x \in \mathbb{R}^N \setminus \{\mathbf{0}\}$. *Then, for sufficiently small* $v_0\left(x\right)$, *the following inequality*

$$v\left(t,x\right) \leq z_1\left(t,x\right) \tag{14}$$

holds in $Q \setminus \{\mathbf{0}\}$, *where the function* $z_1\left(t,x\right)$ *is defined above.*

Proof. Theorem 2 is proved by the comparison solution method [10]. For comparing a solution, the function $z_1\left(t,x\right)$ is considered. Substituting (13) into (3), we obtain

$$\begin{aligned} L\big(z_1(t,x)\big) = \left[\overline{v}(t)\right]^{m_2+k_2(p-2)} \Bigg(&\frac{d}{d\eta}\left(f_1^{m_2-1}\left|\frac{df_1^{k_2}}{d\eta}\right|^{p-2}\frac{df_1}{d\eta}\right) \\ &+ \tau(N-n)\left(z_1^{m_2-1}\left|\frac{\partial z_1^{k_2}}{\partial\varphi}\right|^{p-2}\frac{\partial z_1}{\partial\varphi}\right) + \frac{\eta}{p}\frac{df_1}{d\eta} + \frac{f_1 + f_1^{\beta_2}}{l_7}\Bigg). \end{aligned} \tag{15}$$

Since the function $f_1(\eta)$ has compact support (see [23]), it is enough to show that $L\big(z_1(t,x)\big) \leq 0$ in

$$D_1 = \left\{ (t,x) : t \geq t_0 > 0,\ |x| \leq \exp\left((a/b)^{\frac{1}{\gamma_1}}\left[\tau(t)\right]^{1/p}\right)\right\}.$$

Since $\tau(N-n)\left(z_1^{m_2-1}\left|\frac{\partial z_1^{k_2}}{\partial\varphi}\right|^{p-2}\frac{\partial z_1}{\partial\varphi}\right) \leq 0$ in D_1, we get the following

$$\frac{d}{d\eta}\left(f_1^{m_2-1}\left|\frac{df_1^{k_2}}{d\eta}\right|^{p-2}\frac{df_1}{d\eta}\right) + \frac{\xi}{p}\frac{df_1}{d\eta} + \frac{f_1 + f_1^{\beta_2}}{l_7} \leq 0. \tag{16}$$

For the function $f_1(\eta)$, the inequality (16) can be rewritten as follows

$$(a-b\eta^{\gamma_1})_+^{\gamma_2}\left[\frac{1}{l_7}-\frac{1}{p}+\frac{(a-b\eta^{\gamma_1})_+^{\gamma_2\beta_2-1}}{l_7}\right]\leq 0,$$

or

$$\frac{1}{l_7}-\frac{1}{p}+\frac{(a-b\eta^{\gamma_1})_+^{\gamma_2\beta_2-1}}{l_7}\leq 0 \quad \text{in } D_1.$$

Since $(a-b\eta^{\gamma_1})_+^{\gamma_2(\beta_2-1)}\leq a^{\gamma_2(\beta_2-1)}$, we have $\dfrac{a^{\gamma_2\beta_2-1}+1}{l_7}\leq\dfrac{1}{p}$.

Then, according to the hypothesis of Theorem 2 and the comparison principle, we obtain $v(t,x)\leq z_1(t,x)$ in $Q\setminus\{\mathbf{0}\}$, if $v_0(x)\leq z_1(0,x)$, $x\in\mathbb{R}^N\setminus\{\mathbf{0}\}$. The proof of Theorem 2 is completed. □

3 Asymptotic of compactly supported weak solution

We will investigate nontrivial, non-negative solutions of the equation (7) that satisfy the following conditions:

$$f(0)=c>0,\quad f(a_1)=0,\quad a_1<\infty. \tag{17}$$

For this purpose, we consider the function

$$\overline{f}(\xi)=(a-b\xi^{\gamma_1})_+^{\gamma_2}, \tag{18}$$

where b, γ_1, γ_2 are defined above.

It is easy to show that the function (18) is a sub-solution of the equation

$$\xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1}f^{m_2-1}\left|\frac{df^{k_2}}{d\xi}\right|^{p-2}\frac{df}{d\xi}\right)+\frac{\xi}{p}\frac{df}{d\xi}+\frac{f+f^{\beta_2}}{l_7}=0.$$

We will demonstrate that the function (18) is an asymptotic of solutions of the problem (1)–(2).

Now we study the asymptotic behavior of the problem (7), (17) as $\xi\to(a/b)_-^{\frac{1}{\gamma_1}}$. We seek a solution of the equation (7) in the following form

$$f(\xi)=\overline{f}(\xi)\,w(\eta),\quad \eta=-\ln(a-b\xi^{\gamma_1}), \tag{19}$$

where $w(\eta)$ is an unknown function.

It is easy to see that, $\eta\to+\infty$ as $\xi\to(a/b)_-^{\frac{1}{\gamma_1}}$. After the transformation (19), equation (7) takes the following form

$$\left(w^{\mu}\left|Lw\right|^{p-2}Lw\right)'+a_1w^{\mu}\left|Lw\right|^{p-2}Lw+a_2Lw+a_3w+a_4w^{\beta_2}=0, \tag{20}$$

where $Lw=w'-\gamma_2 w$, $\mu=1-(p-1)\left(1-\frac{1}{\gamma_2}\right)$, $a_1=\frac{s}{\gamma_1}a_0-\gamma_2$, $a_0=\frac{e^{-\eta}}{a-e^{-\eta}}$, $a_2=\frac{\gamma_1\gamma_3}{p}$, $\gamma_3=b^{-\frac{p}{\gamma_1}}\gamma_1^{-p}k_2^{p-2}$, $a_3=\frac{\gamma_3}{l_7}a_0$, $a_4=a_3e^{\gamma_2(1-\beta_2)\eta}$.

First, we demonstrate that the solution of equation (7) has a finite limit w_0 as $\eta\to+\infty$.

Let us consider the function

$$z(\eta)=w^{\mu}\left|Lw\right|^{p-2}Lw.$$

The expression (20) is transformed to the following form

$$\begin{cases} w' = \gamma_2 w + w^{-\frac{\mu}{p-1}} |z|^{\gamma_1 - 2} z, \\ z' = -a_1 z - a_2 w^{-\frac{\mu}{p-1}} |z|^{\gamma_1 - 2} z - a_3 w - a_4 w^{\beta_2}. \end{cases} \tag{21}$$

Lemma 1. *Let us suppose that* $0 < K_1 \le K_2, \theta_1 < \theta_2 \le 0$. *Let* $(w_1, z_1), (w_2, z_2)$ *be the solutions of system* (21) *with the initial value conditions* $w_i(\eta_0) = K_i$, $z_i(\eta_0) = \theta_i$, $i = 1, 2$. *If* w_1 *and* w_2 *are positive in* $[\eta_0, +\infty)$, *then* $w_1(\eta) \le w_2(\eta), z_1(\eta) < z_2(\eta)$ *for any* $\eta \in [\eta_0, +\infty)$.

Proof. Since $0 < w_1(\eta_0) \le w_2(\eta_0)$, $z_1(\eta_0) < z_2(\eta_0) \le 0$, we get

$$\begin{aligned} w_2'(\eta_0) &= \gamma_2 w_2(\eta_0) + w_2^{-\frac{\mu}{p-1}}(\eta_0) |z_2(\eta_0)|^{\gamma_1 - 2} z_2(\eta_0) \\ &> \gamma_2 w_1(\eta_0) + w_1^{-\frac{\mu}{p-1}}(\eta_0) |z_1(\eta_0)|^{\gamma_1 - 2} z_1(\eta_0) = w_1'(\eta_0). \end{aligned}$$

Hence, we get $w_2'(\eta_0) > w_1'(\eta_0)$. Then there must exist a constant $\delta > 0$ such that $w_1(\eta) \le w_2(\eta), z_1(\eta) \le z_2(\eta)$ on $[\eta_0, \eta_0 + \delta]$ (see [17] and references therein).

Analogously, we will see that this result is also valid for the segment $[\eta_0 + \delta, \eta_0 + 2\delta]$ (see [9, 13]). Consequently, we obtain that inequalities $w_1(\eta) \le w_2(\eta)$, $z_1(\eta) \le z_2(\eta)$ hold true for $\eta \in [\eta_0 + \delta(j-1), \eta_0 + \delta j]$, $j \ge 1$. Moreover, by repeating this process many times, we can derive that inequalities $w_1(\eta) \le w_2(\eta)$, $z_1(\eta) \le z_2(\eta)$ are valid for all $\eta \in [\eta_0, +\infty)$. □

Lemma 2. *Assume that* $0 < K_1 \le K_2$, $0 \ge \theta_1 \ge \theta_2$. *Let* (w_1, z_1), (w_2, z_2) *be the solutions of system* (21) *with the initial value conditions* $w_i(\eta_0) = K_i$, $z_i(\eta_0) = \theta_i$, $i = 1, 2$. *If* w_1 *and* w_2 *are positive in* $[\eta_0, +\infty)$, *then* $w_1(\eta) \le w_2(\eta), z_1(\eta) \ge z_2(\eta)$ *for any* $\eta \in [\eta_0, +\infty)$.

Proof. Based on the hypotheses, we get

$$\begin{aligned} z_2'(\eta_0) + a_1 z_2(\eta_0) &= -a_2 w_2^{-\frac{\mu}{p-1}} |z_2|^{\gamma_1 - 2} z_2 - a_3 w_2 - a_4 w_2^{\beta_2} \Big|_{\eta = \eta_0}, \\ z_2'(\eta_0) &= -a_2 \left(w_2^{-\frac{\mu}{p-1}} |z_2|^{\gamma_1 - 2} z_2 + \gamma_2 w_2 \right) - (a_3 - a_2) w_2 - a_4 w_2^{\beta_2}(\eta_0) \\ &= -a_2 w_2'(\eta_0) - (a_3 - a_2) w_2 - a_4 w_2^{\beta_2}(\eta_0) \\ &< -a_2 w_1^{-\frac{\mu}{p-1}} |z_1|^{\gamma_1 - 2} z_1 - a_3 w_1 - a_4 w_1^{\beta_2} \Big|_{\eta = \eta_0} \\ &= -a_2 w_1'(\eta_0) - (a_3 - a_2) w_1 - a_4 w_1^{\beta_2}(\eta_0) \\ &= z_1'(\eta_0) + a_1 z_1(\eta_0) \\ &= z_1'(\eta_0). \end{aligned}$$

This means that $z_2'(\eta_0) < z_1'(\eta_0)$, $w_2'(\eta_0) > w_1'(\eta_0)$. Then, taking into account the proof of Lemma 1, we obtain $w_2(\eta) \ge w_1(\eta)$, $z_1(\eta) > z_2(\eta)$ for all $\eta \in [\eta_0, +\infty)$. □

Theorem 3. *Let* $m_2 + k_2(p-2) > 0$. *Then a finite solution of the problem* (7), (17) *has an asymptotic* $f(\xi) = C\overline{f}(\xi)\left(1 + o(1)\right)$ *as* $\xi \to \left(\frac{a}{b}\right)_{-}^{\frac{1}{\gamma_1}}$, *where* C *is solution of an algebraic equation*

$$\gamma_2^p C^{m_2 k_2 (p-2) - 1} - \frac{b^{-\frac{p}{\gamma_1}} \gamma_1^{1-p} \gamma_2 k_2^{p-2}}{p} + \overset{0}{a}_4 C^{\beta_2 - 1} = 0.$$

Proof. We first show that the solution of system (21) has a finite limit w_0 as $\eta \to +\infty$. It is well known that the function w_η is bounded. So, it is sufficient to show that it is monotone non-increasing in $[\eta_0, +\infty)$. Let us take $w \equiv 1$. We can see from (19) that it is a sub-solution. Then for other solutions $w_1(\eta)$ we have $w_1'(\eta) \le 0$. This shows that any solution is not increasing in $[\eta_0, \eta_1)$ for any $\eta_1 > \eta_0$, where difference $\eta_1 - \eta_0$ is sufficiently small. By considering Lemma 1 and Lemma 2, we can find two solutions w_1, w_2 such that $w_1(\eta_1) = w_2(\eta_0)$. We conclude that $w_1(\eta_1) \ge w_1(\eta_0)$. We can see that w_1 is monotone in $[\eta_0, +\infty)$, due to the arbitrariness of η_1. Consequently, it has a limit as $\eta \to +\infty$. So, we get

$$\lim_{\eta \to +\infty} \frac{be^{-\eta}}{a - e^{-\eta}} \to 0, \quad w' = 0.$$

Consequently, we obtain $\lim\limits_{\eta \to +\infty} a_0 = \lim\limits_{\eta \to +\infty} a_3 = 0$, $\lim\limits_{\eta \to +\infty} a_1 = -\gamma_2$ and

$$\overset{0}{a}_4 = \lim_{\eta \to +\infty} a_4 = \begin{cases} \dfrac{\gamma_3}{al_7}, & \gamma_2(1-\beta_2) = 1, \\ 0, & \gamma_2(1-\beta_2) < 1, \\ +\infty, & \gamma_2(1-\beta_2) > 1. \end{cases}$$

From (20) we obtain the algebraic equation

$$\gamma_2^p w_0^{m_2 k_2(p-2)-1} - \frac{b^{-\frac{p}{\gamma_1}} \gamma_1^{1-p} \gamma_2 k_2^{p-2}}{p} + \overset{0}{a}_4 w_0^{\beta_2 - 1} = 0$$

as $\xi \to (a/b)^{\frac{1}{\gamma_1}}_{-}$. Theorem 3 has been proved. □

Asymptotic of solution in the absorption case. In the work [8], authors established a large time asymptotic solution of the problem (1)–(2) for the case $q = l = 0$, $\varepsilon = -1$, $\beta = \beta_c$. They established the following asymptotic of the solution

$$u(t,x) \sim (t \ln t)^{-\frac{1}{\beta_c - 1}} \exp\left(-|x|^2/t\right), \quad \beta_c = 1 = 2/N \quad \text{for } t \sim \infty.$$

Generalisation of large time asymptotic and behaviour of the front of this result for a degenerate nonlinear parabolic equation considered in the works [6,8]. Authors of these works showed the following large-time asymptotic

$$u(t,x) \sim \left((T+t)\ln(T+t)\right)^{-\frac{1}{\beta_c - 1}} \exp\left(-|x|^2/t\right)$$

of a solution of the problem (1)–(2). It was proved in the critical exponent case

$$\beta_c = 1 + (1+l)\left[m_2 + k_2(p-2) - 1 + \frac{p - n_1 - n}{N - n}\right], \quad n < N, \;\; p - n_1 - n > 0,$$

and free-boundary behaviour for a particular value of the numerical parameters, namely for $l = 1, q = 0$.

4 Fast diffusion case

We will take into consideration nontrivial, non-negative solutions of equation (7), that satisfy the conditions

$$f(\infty) = f'(0) = 0. \tag{22}$$

In that case, various families of solutions oscillate near $\xi = +\infty$ (see, e.g., [14,18–22]). These types of solutions are called eigenfunctions in nonlinear media [16]. By using the nonlinear splitting method [17], we obtain the following upper solution

$$g(\xi) = A\left(a + \xi^{\gamma_1}\right)^{\gamma_2}$$

of the problem (1)–(2), where $A = \left[\dfrac{(\gamma_1\gamma_2)^{1-p} k_2^{2-p}}{p}\right]^{\frac{\gamma_2}{p-1}}$, $a = \text{const} > 0$.

Theorem 4. *Let $\gamma_2(\beta_2 - 1) \le 0$. Then a finite solution of the problem (7), (22) has an asymptotic $f(\xi) = Cg(\xi)\big(1 + o(1)\big)$, where C is solution of an algebraic equation*

$$\left(\frac{s}{\gamma_1} + \gamma_2\right)\gamma_2^{p-1} C^{\mu+p-2} + A^{\frac{1-p}{\gamma_2}}\gamma_1^{-p}\left(\frac{\gamma_1\gamma_2}{p} + \frac{1}{l_7}\right) + \overset{0}{b_4} C^{\beta_2-1} = 0.$$

Proof. We will seek the solution of equation (7) in the form

$$f(\xi) = g(\xi)\, w(\eta), \quad \eta = \ln\left(a + \xi^{\gamma_1}\right). \tag{23}$$

And it is easy to see that $\eta \to +\infty$ as $\xi \to +\infty$.

After the transformation (23), we get

$$\left(w^{\mu}\,|Lw|^{p-2}\,Lw\right)' + b_1 w^{\mu}\,|Lw|^{p-2}\,Lw + b_2 Lw + b_3 w + b_4 w^{\beta_2} = 0, \tag{24}$$

where

$$Lw = w' + \gamma_2 w, \quad \mu = \frac{p-1}{\gamma_2} + 2 - p, \quad b_1 = \frac{s}{\gamma_1} b_0 + \gamma_2, \quad b_0 = \frac{1}{1 - ae^{-\eta}},$$

$$b_2 = \frac{\gamma_1^{1-p} A^{\frac{1-p}{\gamma_2}}}{p}, \quad b_3 = \frac{p\, b_0\, b_2}{\gamma_1 l_7}, \quad b_4 = \frac{\gamma_1^{-p} A^{\beta_2 - m_2 - k_2(p-2)} b_0}{l_7} e^{\gamma_2(\beta_2-1)\eta}.$$

Considering that $w(\eta)$ has a finite limit (see the proof of Theorem 3), we get

$$\lim_{\eta\to+\infty} \frac{1}{1 - ae^{-\eta}} \to 1, \quad w' = 0.$$

Consequently, we obtain the following

$$\lim_{\eta\to+\infty} b_1 = \frac{s}{\gamma_1} + \gamma_2, \quad \lim_{\eta\to+\infty} b_2 = \frac{\gamma_1^{1-p} A^{\frac{1-p}{\gamma_2}}}{p}, \quad \lim_{\eta\to+\infty} b_3 = \frac{\gamma_1^{-p} A^{\frac{1-p}{\gamma_2}}}{l_7},$$

$$\overset{0}{b_4} = \lim_{\eta\to+\infty} b_4 = \begin{cases} \dfrac{\gamma_1^{-p} A^{\beta_2 - m_2 - k_2(p-2)}}{l_7}, & \gamma_2(\beta_2 - 1) = 0, \\ 0, & \gamma_2(\beta_2 - 1) < 0, \\ +\infty, & \gamma_2(\beta_2 - 1) > 0. \end{cases}$$

We come to the following algebraic equation

$$\left(\frac{s}{\gamma_1} + \gamma_2\right)\gamma_2^{p-1} C^{\mu+p-2} + A^{\frac{1-p}{\gamma_2}}\gamma_1^{-p}\left(\frac{\gamma_1\gamma_2}{p} + \frac{1}{l_7}\right) + \overset{0}{b_4} C^{\beta_2-1} = 0,$$

as $\eta \to +\infty$ and $f(\xi) = Cg(\xi)\big(1 + o(1)\big)$, where C is the root of last algebraic equation. The theorem has been proved. □

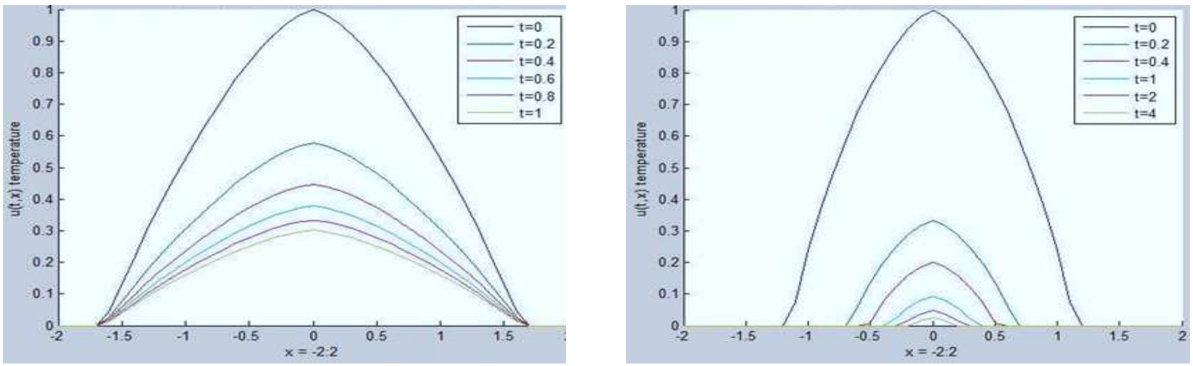
5 Numerical analysis of solutions

As is prominent for the numerical computation of a nonlinear problem, the choice of the initial approximation is essential, which preserves the properties of the final celerity of propagation, spatial localisation, and bounded and blow-up solutions, which guarantees convergence with a given precision to the solution of the problem with a minimum number of iterations. We note that due non-uniqueness of solutions, many different cases arise in the numerical study of the problems of type (1)–(2). Therefore, the question of selecting a good initial approximation and preserving properties of nonlinearity arises [7]. Depending on the parameters of the equation, this difficulty is overcome by an appropriate choice of initial approximations, which are accepted by the asymptotic formula established above.

We produced the numerical calculations based on the obtained qualitative properties. The numerical results show fast convergence of the iterative process to the solution of the Cauchy problem (1)–(2) due to the successful choice of the initial approximation. Below, we give two results of numerical experiments for different values of the numerical parameters.

In the slow diffusion case, as an initial approximation, the function $u_0(t,x) = l_1 t^{\frac{1+l}{1-\beta_2}} \overline{f}(\xi)$ is used.

The work for nonlinear modelling of processes, diffusion, filtration, heat dissipation etc. shows the effect of the finite speed of perturbation, by self-similar analysis of an asymptotically compactly-supported solution is established. The important problem of choosing an initial appropriate approximation is solved. This work shows that the computational scheme suggested by Samarskii-Sobol is effective for the numerical analysis of solutions. The given results verified this phenomenon. The obtained results consist of cases of the porous medium equation, p-Laplacian, and so on, which are not in divergence form. The number of iterations did not exceed 3. Figures illustrate an evolution of the diffusion process with a finite speed of perturbation in the one and two-dimensional cases.

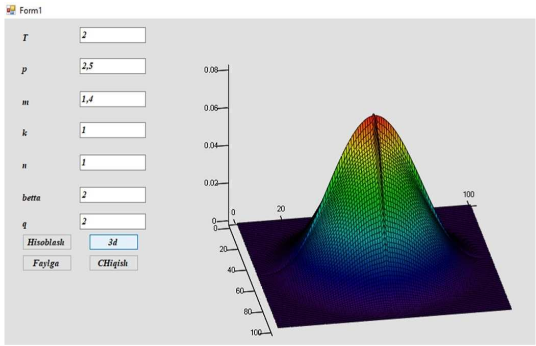


(a) $q = 0.8, m = k = 1, p = 3, n = 0, \gamma = 0$ (b) $q = 0.9, m = k = 1, p = 3, n = 0, \gamma = 0$

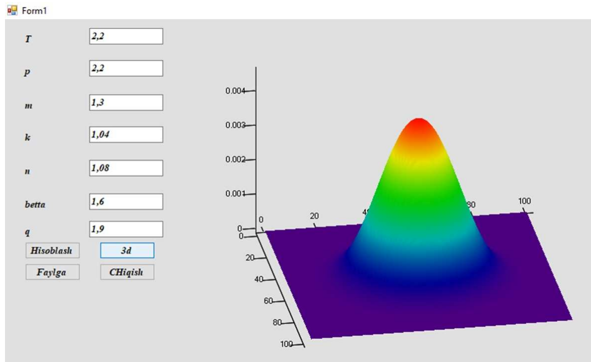
Figure 1. One-dimensional case

6 Conclusion

Global solvability of the problem Cauchy for a degenerate double nonlinear parabolic equation in non-divergent form with variable density and source or absorption is studied.



(a) $q = 0$, $m = 3$, $k = 3.2$, $p = 4$, $n = 0.2$, $\gamma = 1$, $l = 0$, $\beta = 1.3$



(b) $q = 0$, $m = 3.1$, $k = 3$, $p = 3.4$, $n = 0.3$, $\gamma = 1$, $l = 0$, $\beta = 3$

Figure 2. Two-dimensional case

The condition of global solvability of Fujita type, an estimate of the solution for the slow diffusion case, and a free boundary are established. It is shown that the properties of the solution depend on the value of numerical parameters characterising nonlinear media. The properties solution of the Cauchy problem in slow diffusion, fast diffusion, and critical cases were investigated. For every case, asymptotic self-similar solutions are established.

The problem of choosing an appropriate initial approximation for the numerical solution is considered a problem by the iteration method. The results of the numerical analysis are discussed.

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Аріпов М., Бобокандов М. *Задача Коші для подвійного нелінійного параболічного рівняння з поглинанням у неоднорідному середовищі* // Карпатські матем. публ. — 2025. — Т.17, №2. — С. 693–705.

У цій статті досліджуються властивості розв'язків подвійного нелінійного параболічного рівняння зі змінною густиною, що залежить від часу, не у формі дивергенції з джерелом або поглинанням. Задачу сформульовано у вигляді диференціального рівняння з частинними похідними з нелінійним членом, який залежить як від розв'язку, так і від часу. Основними результатами є існування слабких розв'язків у відповідних функціональних просторах; регулярність і додатність розв'язків; асимптотична поведінка розв'язків при прямуванні часу до нескінченності; принципи порівняння; принципи максимуму для розв'язків. Доведення ґрунтуються на методах порівняння та асимптотичних методах. Також наведено кілька прикладів і застосувань, які ілюструють особливості проблеми.

Ключові слова і фрази: вироджене параболічне рівняння, глобальна розв'язність, слабкий розв'язок, критична Фуджіта, асимптотика.