ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2025, **17** (1), 292–301
doi:10.15330/cmp.17.1.292-301

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2025, Т.17, №1, С.292–301

Groups of nilpotency class 3 of order p^4 as additive groups of local nearrings

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A nearring is a generalization of the notion of a ring in the sense that the addition needs not to be commutative and only one distributive law is assumed. A nearing with identity is called local if the set of all non-invertible elements forms a subgroup of its additive group. However, it is not true that any finite group is an additive group of a nearring with identity. Therefore the determination of the non-abelian finite p-groups, which are an additive groups of local nearrings, is an open problem.

We consider groups of nilpotency class 3 of order p^4, which are the additive groups of local nearrings. It is shown that for $p > 3$ there exists a local nearring on one of four such groups. Using the well-known system of computer algebra GAP, we construct examples of local nearrings with the specified properties.

Key words and phrases: local nearring, p-group, nilpotency class 3.

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Introduction

The classification of all nearrings up to certain orders is an open problem. It requires extensive computations, and the most suitable platform for their implementation is the computational algebra system GAP. Until 1990s the interest of pure mathematicians in nearring theory was stirred by, but in most cases also confined to the information that was produced by theoretical results for some research problems. However, 27 years ago the developers of the GAP [7] package SONATA [1] have shown the implementation of nearring theoretical algorithms. These are gradually becoming accepted both as standard tools for a nearring theoretician, like certain methods of proof, and as worthwhile objects of study, like connections between notions expressed in theorems. The package SONATA provides methods for the construction and analysis of finite nearrings, as well as the library of all nearrings up to order 15 and all nearrings with identity up to order 31. The current version of the LocalNR package [14] (not yet redistributed with GAP) contains all local nearrings of order not greater than 361, except those of orders 32, 64, 128, 243 and 256. We have already calculated some classes of local nearrings of orders 32, 64, 128, 243 and 625 (see, for example, [15]).

УДК 512.6
2020 *Mathematics Subject Classification:* 16Y30, 20D15.

The authors are grateful IIE-SRF for the support of their fellowships at the University of Warsaw.

In [8], it is proved that, up to isomorphism, there exist at least p local nearrings on elementary abelian additive groups of order p^3, which are not nearfields. Lower bounds for the number of local nearrings on groups of order p^3 were obtained in [9]. It is established that on each non-metacyclic non-abelian or metacyclic abelian groups of order p^3 there exist at least $p+1$ non-isomorphic local nearrings. It was proved that for $p > 2$ every finite non-metacyclic 2-generated p-group of nilpotency class 2 with cyclic commutator subgroup is the additive group of a local nearring and in particular of a nearring with identity [10].

Groups of nilpotency class 2 of order p^4, which are the additive group of local nearrings, were investigated in [11]. In this paper, we consider groups of nilpotency class 3 of order p^4, which are the additive group of local nearrings. There are examples of local nearrings on such groups constructed via GAP and the LocalNR package, i.e. of order 625 (see [12, 15]). It is shown that for $p > 3$ there exists a local nearring on one of four such groups.

1 Preliminaries

We consider groups of the nilpotency class 3 of order p^4.

Let $[n, i]$ be the i-th group of order n in the SmallGroups library in the computer system algebra GAP. We denote by $\mathbb{Z}_n$ the cyclic group of order n.

It is an easy exercise, for example in GAP, to get the following assertions.

Remark 1. *There are* 3 *non-isomorphic groups of nilpotency class* 3 *of order* $2^4 = 16$, *namely* D_{16} $[16, 7]$, QD_{16} $[16, 8]$, Q_{16} $[16, 9]$.

Remark 2. *There are* 4 *non-isomorphic groups of nilpotency class* 3 *of order* $3^4 = 81$, *namely* $(C_3 \times C_3 \times C_3) \rtimes C_3$ $[81, 7]$, $(C_9 \times C_3) \rtimes C_3$ $[81, 8]$, $(C_9 \rtimes C_3) \rtimes C_3$ $[81, 9]$, $(C_9 \rtimes C_3) \rtimes C_3$ $[81, 10]$.

Using the packages SONATA and LocalNR it is possible to check the following assertion (see also [4]).

Theorem 1. *There is no local nearring on the groups of nilpotency class* 3 *of orders* 16 *and* 81.

The following theorem contains the classification of groups of nilpotency class 3 of order p^4 with $p > 3$ (see, for example, [2, 5]).

Theorem 2. *There are* 4 *non-isomorphic groups of nilpotency class* 3 *of order* p^4 *with* $p > 3$, *namely*

$$H_1 = \langle a, b \colon a^p = b^p = c^p = [a, c]^p = [b, c] = e, [a, [a, c]] = [b, [a, c]] = e \rangle = (C_p \times C_p \times C_p) \rtimes C_p,$$
where $c = [a, b]$,

$$H_2 = \langle a, b \colon a^{p^2} = b^p = [a, b]^p = [b, [a, b]] = e, [a, [a, b]] = a^p \rangle = (C_{p^2} \rtimes C_p) \rtimes C_p,$$

$$H_3 = \langle a, b \colon a^{p^2} = b^p = [a, b]^p = [a, [a, b]] = e, [b, [a, b]] = a^p \rangle = (C_{p^2} \rtimes C_p) \rtimes C_p,$$

$$H_4 = \langle a, b \colon a^{p^2} = b^p = [a, b]^p = [a, [a, b]] = e, [b, [a, b]] = a^{2p} \rangle = (C_{p^2} \times C_p) \rtimes C_p.$$

Below we give the basic definitions.

Definition 1. *A non-empty set R with two binary operations "+ " and "· " is a nearring if*

1) $(R, +)$ *is a group with neutral element* 0*,*

2) $(R, \cdot)$ *is a semigroup,*

3) $x \cdot (y + z) = x \cdot y + x \cdot z$ *for all* $x, y, z \in R$*.*

Such a nearring is called a left nearring. If the third axiom is replaced by the following axiom $(x + y) \cdot z = x \cdot z + y \cdot z$ *for all* $x, y, z \in R$*, then we get a right nearring.*

The group $(R, +)$ of a nearring R is denoted by R^+ and is called the *additive group* of R. It is easy to see that for each subgroup M of R^+ and for each element $x \in R$ the set $xM = \{x \cdot y : y \in M\}$ is a subgroup of R^+ and, in particular, $x \cdot 0 = 0$. If in addition $0 \cdot x = 0$ for all $x \in R$, then the nearring R is called *zero-symmetric*. Furthermore, R is a *nearring with an identity* i if the semigroup $(R, \cdot)$ is a monoid with identity element i. In the latter case the group of all invertible elements of the monoid $(R, \cdot)$ is denoted by R^* and is called the *multiplicative group* of R. A subgroup M of R^+ is called R^**-invariant*, if $rM \leq M$ for each $r \in R^*$, and (R, R)*-subgroup*, if $xMy \subseteq M$ for arbitrary $x, y \in R$.

The following assertion is well-known (see, for instance, [6, Theorem 3]).

Lemma 1. *The exponent of the additive group of a finite nearring R with identity i is equal to the additive order of i, which coincides with the additive order of every invertible element of R.*

Definition 2. *A nearring R with identity is called local, if the set L of all non-invertible elements of R forms a subgroup of the additive group* R^+*.*

Throughout this paper, L denotes the subgroup of all non-invertible elements of R.

The following lemma characterizes the main properties of finite local nearrings (see [3, Lemma 3.2]).

Lemma 2. *Let R be a local nearring with identity i. Then the following statements hold*:

1) *L is an* (R, R)*-subgroup of* R^+*,*

2) *each proper* R^**-invariant subgroup of* R^+ *is contained in* L*,*

3) *the set* $i + L$ *forms a subgroup of the multiplicative group* R^**.*

Finite local nearrings with a cyclic subgroup of non-invertible elements are described in [13, Theorem 1].

Theorem 3. *Let R be a local nearring of order* p^n *with* $n > 1$*, whose subgroup L is cyclic and non-trivial. Then the additive group* R^+ *is either cyclic or is an elementary abelian group of order* p^2*. In the first case, R is a commutative local ring, which is isomorphic to residual ring* $\mathbb{Z}/p^n\mathbb{Z}$ *with* $n \geq 2$*. In the other case, there exist such p non-isomorphic nearrings R with* $|L| = p$*, from which* $p - 1$ *are zero-symmetric nearrings and their multiplicative groups* R^* *are isomorphic to a semidirect product of two cyclic subgroups of orders p and* $p - 1$*.*

As a direct consequence of Theorem 3 we have the following result.

Corollary 1. *Let R be a local nearring of order p^4 with non-abelian additive group and is not a nearfield. Then the subgroup of non-invertible elements L is a non-cyclic group of order p^3 or p^2.*

We define the binomial coefficient $\binom{n}{k}$ of integers n and k by

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{if } k > n \text{ or } k < 0. \end{cases} \tag{1}$$

1.1 The group H_1

Let H_1 be additively written group from Theorem 2. Then $H_1 = \langle a \rangle + \langle b \rangle + \langle c \rangle + \langle d \rangle$ for some elements a, b, c and d of R satisfying the relations $ap = bp = cp = dp = 0$, $b + c = c + b$, $a + d = d + a$, $b + d = d + b$, where $c = -a - b + a + b$ and $d = -a - c + a + c$.

Lemma 3. *For any integers k and m in the group H_1 the equalities $-ak - cm + ak + cm = dkm$ and $cm + ak = ak + cm - dkm$ hold.*

Proof. Since $-a - c + a + c = d$, we get the equality $-c + a + c = a + d$. Then we obtain $-cm + ak + cm = (a + dm)k = ak + dkm$. Therefore, $-ak - cm + ak + cm = dkm$ and so $cm + ak = ak + cm - dkm$. □

Corollary 2. *For any integers k and m in the group H_1 the equalities $-cm + ak = ak - cm + dkm$ and $-cm - ak = -ak - cm - dkm$ hold.*

Lemma 4. *For any integers k and l in the group H_1 the equalities $-ak - bl + ak + bl = ckl + dl\binom{k}{2}$ and $bl + ak = ak + bl - ckl + dl\binom{k}{2}$ hold.*

Proof. Let $k = 1$. Since $-a - b + a + b = c$, we get the equality $-b + a + b = a + c$. Then $-bl + a + bl = a + cl = a + cl - dl\binom{1}{2}$, where $dl\binom{1}{2} = 0$ by definition (1). Therefore, we obtain $-a - bl + a + bl = cl - dl\binom{1}{2}$ and so $bl + a = a + bl - cl + dl\binom{1}{2}$.

The proof is carried out by induction on k. For $k = 1$ the equality is valid. Assume that for a number k the equality holds, i.e. $-bl + ak + bl = ak + clk - dl\binom{k}{2}$.

Let us prove that the equality holds for $k + 1$, i.e.

$$\begin{aligned} -bl + a(k+1) + bl &= -bl + ak + a + bl = -bl + ak + bl + a + cl \\ &= ak + clk - dl\tbinom{k}{2} + a + cl = ak + a + clk - dlk - dl\tbinom{k}{2} \\ &= a(k+1) + cl(k+1) - dl\left(\tbinom{k}{2} + k\right) = a(k+1) + cl(k+1) - dl\tbinom{k+1}{2}. \end{aligned}$$

□

Corollary 3. *For any integers k and l in the group H_1 the equalities $-bl+ak = ak-bl+ckl-dl\binom{k}{2}$ and $-bl - ak = -ak - bl - ckl - dl\binom{-k}{2}$ hold.*

Lemma 5. *For any natural numbers k, l, n, m and r in the group H_1 the equality*

$$(ak + bl + cm + dn)r = akr + blr + c\left(mr - kl\tbinom{r}{2}\right) + d\left(nr + l\tbinom{k}{2}\tbinom{r}{2} - km\tbinom{r}{2} + k^2 l\tbinom{r}{r-3}\right)$$

holds.

Proof. The proof will be carried out by induction on r. For $r = 1$ the equality is valid. Let for r the equality holds, i.e.

$$(ak+bl+cm+dn)r = akr+blr+c\big(mr-kl\tbinom{r}{2}\big)+d\big(nr+l\tbinom{k}{2}\tbinom{r}{2}-km\tbinom{r}{2}+k^2l\tbinom{r}{r-3}\big).$$

Using Lemmas 3 and 4, let us prove the equality for $r+1$, i.e.

$$\begin{aligned}(ak+bl&+cm+dn)(r+1)\\ &= akr+blr+c\big(mr-kl\tbinom{r}{2}\big)+d\big(nr+l\tbinom{k}{2}\tbinom{r}{2}-km\tbinom{r}{2}+k^2l\tbinom{r}{r-3}\big)+ak+bl+cm+dn\\ &= akr+blr+ak+c\big(m(r+1)-kl\tbinom{r}{2}\big)-d\big(kmr-k^2l\tbinom{r}{2}\big)+bl\\ &\quad + d\big(n(r+1)+l\tbinom{k}{2}\tbinom{r}{2}-km\tbinom{r}{2}+k^2l\tbinom{r}{r-3}\big)\\ &= akr+ak+blr+bl-cklr+dlr\tbinom{k}{2}+c\big(m(r+1)-kl\tbinom{r}{2}\big)-d\big(kmr-k^2l\tbinom{r}{2}\big)\\ &\quad + d\big(n(r+1)+l\tbinom{k}{2}\tbinom{r}{2}-km\tbinom{r}{2}+k^2l\tbinom{r}{r-3}\big)\\ &= a(k(r+1))+b(l(r+1))+c\big(m(r+1)-kl\tbinom{r+1}{2}\big)\\ &\quad + d\big(n(r+1)+l\tbinom{k}{2}\tbinom{r+1}{2}-km\tbinom{r+1}{2}+k^2l\tbinom{r+1}{r-2}\big).\end{aligned}$$

Therefore, the equality holds for any r. □

1.2 Nearrings with identity whose additive groups are isomorphic to H_1

Let R be a nearring with identity, whose additive group of R^+ is isomorphic to H_1. Then $R^+ = \langle a\rangle + \langle b\rangle + \langle c\rangle + \langle d\rangle$ for some elements a, b, c and d of R satisfying the relations $ap = bp = cp = dp = 0$, $b+c = c+b$, $a+d = d+a$, $b+d = d+b$, where $c = -a-b+a+b$ and $d = -a-c+a+c$. In particular, each element $x \in R$ is uniquely written in the form $x = ax_1 + bx_2 + cx_3 + dx_4$ with coefficients $0 \le x_1 < p$, $0 \le x_2 < p$, $0 \le x_3 < p$ and $0 \le x_4 < p$.

Since order of the element a is equal to the exponent of group G, i.e. p, we can assume that a is an identity of R, i.e. $ax = xa = x$ for each $x \in R$. Furthermore, for each $x \in R$ there exist coefficients $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $\phi(x)$ such that $xb = a\alpha(x) + b\beta(x) + c\gamma(x) + d\phi(x)$. It is clear that they are uniquely defined modulo p, so that some mappings $\alpha\colon R \to \mathbb{Z}_p$, $\beta\colon R \to \mathbb{Z}_p$, $\gamma\colon R \to \mathbb{Z}_p$, $\phi\colon R \to \mathbb{Z}_p$ are determined.

Lemma 6. *Let R be a nearring with identity, whose additive group of R^+ is isomorphic to H_1. If a coincides with identity element of R, $x = ax_1+bx_2+cx_3+dx_4$, $y = ay_1+by_2+cy_3+dy_4 \in R$, $xb = a\alpha(x)+b\beta(x)+c\gamma(x)+d\phi(x)$, then*

$$\begin{aligned} xy &= a\big(x_1y_1+\alpha(x)y_2\big)+b\big(x_2y_1+\beta(x)y_2\big)\\ &\quad + c\big(x_3y_1 - x_1x_2\tbinom{y_1}{2} - x_2\alpha(x)y_1y_2+\gamma(x)y_2\big) - \alpha(x)\beta(x)\tbinom{y_2}{2}+x_1\beta(x)y_3 - x_2\alpha(x)y_3\big)\\ &\quad + d\big(x_4y_1+x_2\tbinom{x_1}{2}\tbinom{y_1}{2} - x_1x_3\tbinom{y_1}{2}+{x_1}^2x_2\tbinom{y_1}{y_1-3}+x_2y_1\tbinom{\alpha(x)y_2}{2} - \alpha(x)x_3y_1y_2\\ &\quad + x_1x_2\alpha(x)\tbinom{y_1}{2}y_2+\phi(x)y_2+\beta(x)\tbinom{\alpha(x)}{2}\tbinom{y_2}{2} - \alpha(x)\gamma(x)\tbinom{y_2}{2}+\alpha(x)^2\beta(x)\tbinom{y_2}{y_2-3}\\ &\quad + x_1\gamma(x)y_3 - \beta(x)\tbinom{x_1}{2}y_3+x_2\tbinom{\alpha(x)}{2}y_3 - x_3\alpha(x)y_3+x_1^2\beta(x)y_4 - x_1x_2\alpha(x)y_4\big). \end{aligned} \tag{2}$$

Moreover, for the mappings $\alpha\colon R \to \mathbb{Z}_p$, $\beta\colon R \to \mathbb{Z}_p$, $\gamma\colon R \to \mathbb{Z}_p$, $\phi\colon R \to \mathbb{Z}_p$ the following statements hold:

(0) $\alpha(0) \equiv 0 \pmod p$, $\beta(0) \equiv 0 \pmod p$, $\gamma(0) \equiv 0 \pmod p$ *and* $\phi(0) \equiv 0 \pmod p$ *if and only if the nearring R is zero-symmetric,*

(1) $xc = c\big(x_1\beta(x) - x_2\alpha(x)\big) + d\big(x_1\gamma(x) - \beta(x)\binom{x_1}{2} + x_2\binom{\alpha(x)}{2} - x_3\alpha(x)\big)$,

(2) $xd = d\big(x_1{}^2\beta(x) - x_1x_2\alpha(x)\big)$,

(3) $\alpha(xy) \equiv x_1\alpha(y) + \alpha(x)\beta(y) \pmod p$,

(4) $\beta(xy) \equiv x_2\alpha(y) + \beta(x)\beta(y) \pmod p$,

(5) $$\gamma(xy) \equiv x_3\alpha(y) - x_1x_2\binom{\alpha(y)}{2} - x_2\alpha(x)\alpha(y)\beta(y) + \gamma(x)\beta(y) - \alpha(x)\beta(x)\binom{\beta(y)}{2} + x_1\beta(x)\gamma(y) - x_2\alpha(x)\gamma(y) \pmod p,$$

(6) $$\begin{aligned}\phi(xy) \equiv{}& x_4\alpha(y) + x_2\tbinom{x_1}{2}\tbinom{\alpha(y)}{2} - x_1x_3\tbinom{\alpha(y)}{2} + x_1{}^2x_2\tbinom{\alpha(y)}{\alpha(y)-3} + x_2\alpha(y)\tbinom{\alpha(x)\beta(y)}{2}\\ &- \alpha(x)x_3\alpha(y)\beta(y) + x_1x_2\alpha(x)\tbinom{\alpha(y)}{2}y_2 + \phi(x)\beta(y) + \beta(x)\tbinom{\alpha(x)}{2}\tbinom{\beta(y)}{2}\\ &- \alpha(x)\gamma(x)\tbinom{\beta(y)}{2} + \alpha(x)^2\beta(x)\tbinom{\beta(y)}{\beta(y)-3} + x_1\gamma(x)\gamma(y) - \beta(x)\tbinom{x_1}{2}\gamma(y)\\ &+ x_2\tbinom{\alpha(x)}{2}\gamma(y) - x_3\alpha(x)\gamma(y) + x_1^2\beta(x)\phi(y) - x_1x_2\alpha(x)\phi(y) \pmod p.\end{aligned}$$

Proof. Since $0 \cdot a = a \cdot 0 = 0$, it follows that R is a zero-symmetric nearring if and only if $0 = 0 \cdot b = a\alpha(0) + b\beta(0) + c\gamma(0) + d\phi(0)$. Equivalently we have

$$\alpha(0) \equiv 0 \pmod p, \quad \beta(0) \equiv 0 \pmod p, \quad \gamma(0) \equiv 0 \pmod p, \quad \phi(0) \equiv 0 \pmod p.$$

Moreover, since $c = -a - b + a + b$, $d = -a - c + a + c$ and the left distributive, law we have $0 \cdot c = -0 \cdot a - 0 \cdot b + 0 \cdot a + 0 \cdot b = 0$ and $0 \cdot d = -0 \cdot a - 0 \cdot c + 0 \cdot a + 0 \cdot c = 0$, whence

$$0 \cdot x = 0 \cdot (ax_1 + bx_2 + cx_3 + dx_4) = (0 \cdot a)x_1 + (0 \cdot b)x_2 + (0 \cdot c)x_3 + (0 \cdot d)x_4 = 0.$$

So that statement *(0)* holds.

Further, using Lemmas 3 and 4, we derive

$$\begin{aligned} xc ={}& x(-a - b + a + b) = -xa - xb + xa + xb = -dx_4 - cx_3 - bx_2 - ax_1 - d\phi(x) - c\gamma(x)\\ &- b\beta(x) - a\alpha(x) + ax_1 + bx_2 + cx_3 + dx_4 + a\alpha(x) + b\beta(x) + c\gamma(x) + d\varphi(x)\\ ={}& -cx_3 - bx_2 - c\gamma(x) - ax_1 + dx_1\gamma(x) - b\beta(x) - a\alpha(x) + ax_1 + bx_2\\ &+ a\alpha(x) + cx_3 - dx_3\alpha(x) + b\beta(x) + c\gamma(x)\\ ={}& -cx_3 - bx_2 - c\gamma(x) + dx_1\gamma(x) - b\beta(x) - ax_1 + cx_1\beta(x) + d\beta(x)\tbinom{-x_1}{2} - a\alpha(x)\\ &+ a\alpha(x) + bx_2 - cx_2\alpha(x) + dx_2\tbinom{\alpha(x)}{2} + cx_3 - dx_3\alpha(x) + b\beta(x) + c\gamma(x)\\ ={}& -cx_3 - bx_2 - c\gamma(x) + dx_1\gamma(x) - b\beta(x) + cx_1\beta(x) - ax_1 - dx_1{}^2\beta(x)\\ &+ d\beta(x)\tbinom{-x_1}{2} + ax_1 + bx_2 - cx_2\alpha(x) + dx_2\tbinom{\alpha(x)}{2} + cx_3 - dx_3\alpha(x) + b\beta(x) + c\gamma(x)\\ ={}& c(x_1\beta(x) - x_2\alpha(x)) + d(x_1\gamma(x) - \beta(x)\tbinom{x_1}{2} + x_2\tbinom{\alpha(x)}{2} - x_3\alpha(x))\end{aligned}$$

and

$$\begin{aligned} xd ={}& x(-a - c + a + c) = -xa - xc + xa + xc = -dx_4 - cx_3 - bx_2 - ax_1\\ &- d\big(x_1\gamma(x) - \beta(x)\tbinom{x_1}{2} + x_2\tbinom{\alpha(x)}{2} - x_3\alpha(x)\big) - c(x_1\beta(x) - x_2\alpha(x)) + ax_1 + bx_2\\ &+ cx_3 + dx_4 + c(x_1\beta(x) - x_2\alpha(x)) + d\big(x_1\gamma(x) - \beta(x)\tbinom{x_1}{2} + x_2\tbinom{\alpha(x)}{2} - x_3\alpha(x)\big)\\ ={}& -cx_3 - bx_2 - c(x_1\beta(x) - x_2\alpha(x)) - ax_1 + d(x_1{}^2\beta(x) - x_1x_2\alpha(x))\\ &+ ax_1 + bx_2 + cx_3 + c(x_1\beta(x) - x_2\alpha(x))\\ ={}& d(x_1{}^2\beta(x) - x_1x_2\alpha(x)).\end{aligned}$$

So, we obtained statements *(1)* and *(2)* of the lemma. By the left distributive law and proved statements *(1)* and *(2)*, we have

$$\begin{aligned} xy &= x(ay_1) + x(by_2) + x(cy_3) + x(dy_4) \\ &= (ax_1 + bx_2 + cx_3 + dx_4)y_1 + \big(a\alpha(x) + b\beta(x) + c\gamma(x) + d(\phi(x))\big)y_2 \\ &\quad + c\big(x_1\beta(x) - x_2\alpha(x)\big) + d\big(x_1\gamma(x) - \beta(x)\tbinom{x_1}{2}\big)y_3 + d\big(x_1^2\beta(x) - x_1x_2\alpha(x)\big)y_4. \end{aligned}$$

Furthermore, Lemma 5 implies that

$$\begin{aligned} &(ax_1 + bx_2 + cx_3 + dx_4)y_1 = ax_1y_1 + bx_2y_1 + c\big(x_3y_1 - x_1x_2\tbinom{y_1}{2}\big) \\ &\qquad + d\big(x_4y_1 + x_2\tbinom{x_1}{2}\tbinom{y_1}{2} - x_1x_3\tbinom{y_1}{2} + \tbinom{y_1}{y_1-3}x_1^2x_2\big), \\ &(a\alpha(x) + b\beta(x) + c\gamma(x) + d\phi(x))y_2 = a\alpha(x)y_2 + b\beta(x)y_2 + c\big(\gamma(x)y_2 + \alpha(x)\beta(x)\tbinom{y_2}{2}\big) \\ &\qquad + d\big(\phi(x)y_2 + \beta(x)\tbinom{\alpha(x)}{2}\tbinom{y_2}{2} - \alpha(x)\gamma(x)\tbinom{y_2}{2} + \alpha(x)^2\beta(x)\tbinom{y_2}{y_2-3}\big), \\ &\big(c(x_1b(x) - x_2\alpha(x)) + d\big(x_1\gamma(x) - \beta(x)\tbinom{x_1}{2} + x_2\tbinom{\alpha(x)}{2} - x_3\alpha(x)\big)\big)y_3 \\ &\qquad = c(x_1\beta(x)y_3 - x_2\alpha(x)y_3) + d\big(x_1\gamma(x)y_3 - \beta(x)\tbinom{x_1}{2}y_3 + x_2\tbinom{\alpha(x)}{2}y_3 - x_3\alpha(x)y_3\big), \\ &d\big(x_1^2\beta(x) - x_1x_2\alpha(x)\big)y_4 = d\big(x_1^2\beta(x)y_4 - x_1x_2\alpha(x)y_4\big). \end{aligned}$$

By Lemmas 3 and 4, we have

$$\begin{aligned} xy &= a(x_1y_1 + \alpha(x)y_2) + b(x_2y_1 + \beta(x)y_2) \\ &\quad + c\big(x_3y_1 - x_1x_2\tbinom{y_1}{2} - x_2\alpha(x)y_1y_2 + \gamma(x)y_2 - \alpha(x)\beta(x)\tbinom{y_2}{2} + x_1\beta(x)y_3 - x_2\alpha(x)y_3\big) \\ &\quad + d\big(x_4y_1 + x_2\tbinom{x_1}{2}\tbinom{y_1}{2} - x_1x_3\tbinom{y_1}{2} + {x_1}^2x_2\tbinom{y_1}{y_1-3} + x_2y_1\tbinom{\alpha(x)y_2}{2} - \alpha(x)x_3y_1y_2 \\ &\quad + x_1x_2\alpha(x)\tbinom{y_1}{2}y_2 + \phi(x)y_2 + \beta(x)\tbinom{\alpha(x)}{2}\tbinom{y_2}{2} - \alpha(x)\gamma(x)\tbinom{y_2}{2} + \alpha(x)^2\beta(x)\tbinom{y_2}{y_2-3} \\ &\quad + x_1\gamma(x)y_3 - \beta(x)\tbinom{x_1}{2}y_3 + x_2\tbinom{\alpha(x)}{2}y_3 - x_3\alpha(x)y_3 + x_1^2\beta(x)y_4 - x_1x_2\alpha(x)y_4\big). \end{aligned}$$

Finally, the associativity of multiplication in R implies that

$$x(yb) = (xy)b = a\alpha(xy) + b\beta(xy) + c\gamma(xy) + d\phi(xy).$$

Furthermore, substituting $yb = a\alpha(y) + b\beta(y) + c\gamma(y) + d\phi(y)$ instead of y in formula (2), we also have

$$\begin{aligned} x(yb) &= a(x_1\alpha(y) + \alpha(x)\beta(y)) + b(x_2\alpha(y) + \beta(x)\beta(y)) + c\big(x_3\alpha(y) - x_1x_2\tbinom{\alpha(y)}{2} \\ &\quad - x_2\alpha(x)\alpha(y)\beta(y) + \gamma(x)\beta(y) - \alpha(x)\beta(x)\tbinom{\beta(y)}{2} + x_1\beta(x)\gamma(y) - x_2\alpha(x)\gamma(y)\big) \\ &\quad + d\big(x_4\alpha(y) + x_2\tbinom{x_1}{2}\tbinom{\alpha(y)}{2} - x_1x_3\tbinom{\alpha(y)}{2} + {x_1}^2x_2\tbinom{\alpha(y)}{\alpha(y)-3} + x_2\alpha(y)\tbinom{\alpha(x)\beta(y)}{2} \\ &\quad - \alpha(x)x_3\alpha(y)\beta(y) + x_1x_2\alpha(x)\tbinom{\alpha(y)}{2}y_2 + \phi(x)\beta(y) + \beta(x)\tbinom{\alpha(x)}{2}\tbinom{\beta(y)}{2} \\ &\quad - \alpha(x)\gamma(x)\tbinom{\beta(y)}{2} + \alpha(x)^2\beta(x)\tbinom{\beta(y)}{\beta(y)-3} + x_1\gamma(x)\gamma(y) - \beta(x)\tbinom{x_1}{2}\gamma(y) \\ &\quad + x_2\tbinom{\alpha(x)}{2}\gamma(y) - x_3\alpha(x)\gamma(y) + x_1^2\beta(x)\phi(y) - x_1x_2\alpha(x)\phi(y)\big). \end{aligned}$$

Comparing the coefficients under a, b, c and d in two expressions obtained for $x(yb)$, we derive statements *(3)–(6)* of the lemma. □

1.3 Local nearrings whose additive groups are isomorphic to H_1

Let R be a local nearring, whose additive group of R^+ is isomorphic to H_1. Then $R^+ = \langle a\rangle + \langle b\rangle + \langle c\rangle + \langle d\rangle$ for some elements a, b, c and d of R satisfying the relations $ap = bp = cp = dp = 0$, $b + c = c + b$, $a + d = d + a$, $b + d = d + b$, where $c = -a - b + a + b$ and $d = -a - c + a + c$. In particular, each element $x \in R$ is uniquely written in the form $x = ax_1 + bx_2 + cx_3 + dx_4$ with coefficients $0 \le x_1 < p$, $0 \le x_2 < p$, $0 \le x_3 < p$ and $0 \le x_4 < p$.

Since order of the element a is equal to the exponent of group G, i.e. p, it follows that we can assume that a is an identity of R, i.e. $ax = xa = x$ for all $x \in R$. Furthermore, for each $x \in R$ there exist $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $\phi(x)$ such that $xb = a\alpha(x) + b\beta(x) + c\gamma(x) + d\phi(x)$. It is clear that they are uniquely defined modulo p, so that some mappings $\alpha\colon R \to \mathbb{Z}_p$, $\beta\colon R \to \mathbb{Z}_p$, $\gamma\colon R \to \mathbb{Z}_p$, $\phi\colon R \to \mathbb{Z}_p$ are determined.

By Corollary 1, L is the normal subgroup of order p^3 or p^2 in R.

Throughout this section, let R be a local nearring with $|R : L| = p$.

Theorem 4. *Let R be a local nearring, whose additive group R^+ is isomorphic to a group H_1 and $|R : L| = p$. Then $R^+ = \langle a\rangle + \langle b\rangle + \langle c\rangle + \langle d\rangle$, $ap = bp = cp = dp = 0$, $b + c = c + b$, $a + d = d + a$ $b + d = d + b$, where $c = -a - b + a + b$, $d = -a - c + a + c$. If a coincides with identity element of R, then the following statements hold:*

1) $L = \langle b\rangle + \langle c\rangle + \langle d\rangle$ *and* $R^* = \{ax_1 + bx_2 + cx_3 + dx_4 : x_1 \not\equiv 0 \pmod p\}$,

2) $xb = b\beta(x) + c\gamma(x) + d\phi(x)$,

3) $xc = cx_1\beta(x) + d\big(x_1\gamma(x) - \beta(x)\binom{x_1}{2}\big)$,

4) $xd = dx_1{}^2\beta(x)$.

Proof. Since L consists the derived subgroup of R^+, it follows that we can choose generators b, d and c, such that $c = -a - b + a + b$ and $d = -a - c + a + c$. If $|L| = p^3$, then $L = \langle b\rangle + \langle c\rangle + \langle d\rangle$ and L is the (R, R)-subgroup in R^+. By statement *1)* of Lemma 2, we have $xb \in L$, hence $a\alpha(x) \in L$ for each $x \in R$. Thus $\alpha(x) \equiv 0 \pmod p$, and so $xb = b\beta(x) + c\gamma(x) + d\phi(x)$, i.e. statement *2)* holds.

Since $R^* = R \setminus L$, it follows that $R^* = \{ax_1 + bx_2 + cx_3 + cx_4 : x_1 \not\equiv 0 \pmod p\}$ and $x = ax_1 + bx_2 + cx_3 + dx_4$ is invertible if and only if $x_1 \not\equiv 0 \pmod p$, as claimed in statement *1)*.

Further, substituting $\alpha(x) \equiv 0 \pmod p$ to *1)* and *2)* of Theorem 6, we derive the equalities $xc = cx_1\beta(x) + d\big(x_1\gamma(x) - \beta(x)\binom{x_1}{2}\big)$ and $xd = dx_1{}^2\beta(x)$. □

As a consequence of Lemma 6 and Theorem 4 we get the following assertion.

Corollary 4. *If a coincides with identity element of a nearring R, $x = ax_1 + bx_2 + cx_3 + dx_4$, $y = ay_1 + by_2 + cy_3 + dy_4 \in R$ and $|R : L| = p$, then*

$$\begin{aligned} xy = a(x_1y_1) &+ b\big(x_2y_1 + \beta(x)y_2\big) + c\big(x_3y_1 - x_1x_2\tbinom{y_1}{2} + x_1\beta(x)y_3 + \gamma(x)y_2\big) \\ &+ d\big(x_4y_1 + x_2\tbinom{x_1}{2}\tbinom{y_1}{2} - x_1x_3\tbinom{y_1}{2} + x_1{}^2x_2\tbinom{y_1}{y_1-3} \\ &+ \phi(x)y_2 + x_1\gamma(x)y_3 - \beta(x)\tbinom{x_1}{2}y_3 + x_1{}^2\beta(x)y_4\big). \end{aligned} \tag{3}$$

Moreover, for the mappings $\beta\colon R \to \mathbb{Z}_p$, $\gamma\colon R \to \mathbb{Z}_p$, $\phi\colon R \to \mathbb{Z}_p$ the following statements hold:

(0) $\beta(0) \equiv 0 \pmod{p}$, $\gamma(0) \equiv 0 \pmod{p}$, $\phi(0) \equiv 0 \pmod{p}$ *if and only if the nearring R is zero-symmetric,*

(1) $\beta(xy) \equiv \beta(x)\beta(y) \pmod{p}$,

(2) $\gamma(xy) \equiv x_1\beta(x)\gamma(y) \pmod{p}$,

(3) $\phi(xy) \equiv \phi(x)\beta(y) + x_1\gamma(x)\gamma(y) - \beta(x)\binom{x_1}{2}\gamma(y) + x_1^2\beta(x)\phi(y) \pmod{p}$.

Next, we give examples of local nearrings, whose additive group R^+ is isomorphic to H_1.

Lemma 7. *Let R be a local nearring, whose additive group of R^+ is isomorphic to H_1 and $|R : L| = p$. If $x = ax_1 + bx_2 + cx_3 + dx_4$, $y = ay_1 + by_2 + cy_3 + dy_4 \in R$, then the mappings $\beta\colon R \to \mathbb{Z}_p$, $\gamma\colon R \to \mathbb{Z}_p$ and $\phi\colon R \to \mathbb{Z}_p$ from multiplication* (3) *can be $\beta(x) \equiv {x_1}^2 \pmod{p}$, $\gamma(x) \equiv 0 \pmod{p}$ and $\phi(x) = \begin{cases} {x_2}^2, & \text{if } x_1 \equiv 0 \pmod{p}, \\ 0, & \text{if } x_1 \not\equiv 0 \pmod{p}. \end{cases}$*

Proof. It is easy to check that the functions β and γ satisfy conditions *1), 2)* of Corollary 4. Since $\gamma(x) \equiv 0 \pmod{p}$ it follows that $\phi(xy) = \phi(x)\beta(y) + x_1^2\beta(x)\phi(y)$. If $x_1y_1 \not\equiv 0 \pmod{p}$, then $\phi(xy) = 0 = \phi(x)\beta(y)$. For $x_1y_1 \equiv 0 \pmod{p}$ we need check the following three cases:

1) if $x_1 \equiv 0 \pmod{p}$ and $y_1 \not\equiv 0 \pmod{p}$, then $\phi(xy) = (x_2y_1)^2 = {x_2}^2{y_1}^2 = \phi(x)\beta(y)$,

2) if $x_1 \not\equiv 0 \pmod{p}$ and $y_1 \equiv 0 \pmod{p}$, then $\phi(xy) = (x_2^2y_1)^2 = {x_2}^4{y_1}^2 = \phi(x)\beta(y)$,

3) if $x_1 \equiv 0 \pmod{p}$ and $y_1 \equiv 0 \pmod{p}$, then $\phi(xy) = 0 = \phi(x)\beta(y)$.

We derive condition *3)* of Corollary 4. □

As a consequence of Lemma 7 we get the following result.

Theorem 5. *For each prime $p > 3$ there exists a local nearring R, whose additive group R^+ is isomorphic to H_1.*

Example 1. *Let $G \cong (C_5 \times C_5 \times C_5) \rtimes C_5$. If $x = ax_1 + bx_2 + cx_3 + dx_4$ and $y = ay_1 + by_2 + cy_3 + dy_4 \in G$ and $(G, +, \cdot)$ is a local nearring, then, by Lemma 7, " $\cdot$ " can be*

$$\begin{aligned} x \cdot y =& a(x_1y_1) + b(x_2y_1 + \beta(x)y_2) + c\big(x_3y_1 - x_1x_2\tbinom{y_1}{2} + x_1\beta(x)y_3\big) \\ &+ d\big(x_4y_1 + x_2\tbinom{x_1}{2}\tbinom{y_1}{2} - x_1x_3\tbinom{y_1}{2} + {x_1}^2x_2\tbinom{y_1}{y_1-3} + \phi(x)y_2 - \beta(x)\tbinom{x_1}{2}y_3 + {x_1}^2\beta(x)y_4\big), \end{aligned}$$

where $\beta(x) \equiv {x_1}^2 \pmod{5}$ and $\phi(x) = \begin{cases} {x_2}^2, & \text{if } x_1 \equiv 0 \pmod{5}, \\ 0, & \text{if } x_1 \not\equiv 0 \pmod{5}. \end{cases}$

The computer program, which verified that for $p = 5$ the nearring obtained in Lemma 7, is indeed a local nearring (see Example 1), is deposited on GitHub:

`https://github.com/raemarina/Examples/blob/main/LNR_625-7.txt`

From the package LocalNR and [15] we have the following number of all non-isomorphic zero-symmetric local nearrings on H_1 of order 625.

$IdGroup(R^+)$	$IdGroup(R^*)$	$StructureDescription(R^*)$	$n(R^*)$
[625, 7]	[500, 21]	$((C_5 \times C_5) \rtimes C_5) \rtimes C_4$	45
103	[500, 42]	$C_5 \times ((C_5 \times C_5) \rtimes C_4)$	34
	[500, 44]	$C_5 \times ((C_5 \times C_5) \rtimes C_4)$	6
	[500, 46]	$(C_5 \times C_5 \times C_5) \rtimes C_4$	18

References

[1] Aichinger E., Binder F., Ecker Ju., Mayr P., Noebauer C. SONATA – system of near-rings and their applications, GAP package, Version 2.9.1, 2018. `https://gap-packages.github.io/sonata/`

[2] Al-Hasanat B.N., Almazaydeh A. *On classification of groups of order p^4, where p is an odd prime*. Int. J. Math. Comp. Sci. 2022, **17** (4), 1569–1593.

[3] Amberg B., Hubert P., Sysak Ya. *Local near-rings with dihedral multiplicative group*. J. Algebra 2004, **273**, 700–717.

[4] Boykett T.H.H., Nöbauer C. A class of groups which cannot be the additive groups of near-rings with identity. In: Contributions to General Algebra, **10**. Klagenfurt, Heyn, 1998.

[5] Burnside W. Theory of groups of finite order. Cambridge, University press, 1897,

[6] Clay J.R., Malone Jr. *The near-rings with identities on certain finite groups*. Math. Scand 1966, **19**, 146–150.

[7] The GAP Group, GAP — Groups, Algorithms, and Programming, Version 4.12.0 (2022). `https://www.gap-system.org`

[8] Raievska I.Yu., Raievska M.Yu. *Local nearrings on elementary Abelian groups of order p^3*. Sci. Bull. Uzhhorod Univ. Ser. Math. Inform. 2021, **1** (38), 85–93. (in Ukrainian)

[9] Raievska I., Raievska M. *Lower bounds for the number of local nearrings on groups of order p^3*. Math. Commun. 2024, **29**, 177–191.

[10] Raievska I.Yu., Raievska M.Yu. *Local nearrings on finite non-abelian 2-generated p-groups*. Carpathian Math. Publ. 2020, **12** (1), 199–207. doi:10.15330/cmp.12.1.199-207

[11] Raievska I., Raievska M. *Groups of the nilpotency class 2 of order p^4 as additive groups of local nearrings*. Algebra Discrete Math. 2024, **38** (1), 93–114. doi:10.12958/adm2269

[12] Raievska I., Raievska M. *Groups of order p^4 as additive groups of local nearrings*. Ukrainian Math. J. 2024, **76** (6), 1005–1024. doi:10.1007/s11253-024-02369-w (translation of Ukrain. Mat. Zh. 2024, **76** (6), 890–906. doi:10.3842/umzh.v76i5.8053 (in Ukrainian))

[13] Raievska I.Yu., Raievska M.Yu. *Local nearrings with restrictions on the multiplicative groups and the subgroups of non-invertible elements*. Sci. J. Dragomanov Ped. Univ. Ser. 1. Phys.-math. Sci. 2013, **14**, 134–145 (in Ukrainian).

[14] Raievska I., Raievska M., Sysak Y. LocalNR, Package of local nearrings, Version 1.0.4 (2024) (GAP package). `https://gap-packages.github.io/LocalNR`

[15] Raievska I., Raievska M., Sysak Ya. (2023). DatabaseEndom625: (v0.2) [Data set]. Zenodo. doi:10.5281/zenodo.7613145

Received 07.09.2023

Раєвська І.Ю., Раєвська М.Ю. *Групи класу нільпотентності 3 порядку p^4 як адитивні групи локальних майже-кілець* // Карпатські матем. публ. — 2025. — Т.17, №1. — С. 292–301.

Майже-кільця — це узагальнення кілець в сенсі того, що додавання не повинно бути абелевим і тільки один дистрибутивний закон має місце. Майже-кільце з одиницею називають локальним, якщо множина всіх необоротних елементів утворює підгрупу в адитивній групі. Відомо, що не є вірним, що кожна скінченна група є адитивною групою майже-кільця з одиницею. Тому визначення неабелевих скінченних p-груп, які є адитивними групами локальних майже-кілець, є відкритою проблемою.

Ми розглянули групи класу нільпотентсності 3 порядку p^4 як адитивні групи локальних майже-кілець. Було показано, що для $p > 3$ існують локальні майже-кільця на одній з чотирьох таких груп. Базуючись на добре відомій системі комп'ютерної алгебри GAP, ми побудували приклади локальних майже-кілець з вказаними властивостями.

Ключові слова і фрази: локальне майже-кільце, p-група, клас нільпотентності 3.