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On common fixed points for Geraghty contraction mappings and an application to integral equations

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In this paper, few common fixed point theorems for Geraghty-type contractions are established in G_r-complete fuzzy b-metric spaces. Moreover, an example is constructed to authenticate the main result. Ultimately, an application of the results is obtained by exhibiting the existence of a solution for an integral equation.

Key words and phrases: fixed point, Geraghty contraction, fuzzy b-metric space, integral equation.

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1 Introduction

L.A. Zadeh [26] laid the basic frame work of fuzzy mathematics by introducing fuzzy sets in 1965. Afterwards, many researchers used this idea in general topology to define a new notion of a fuzzy metric space (FMS). The first example in this perspective is the work of I. Kramosil and J. Michálek [17], in which authors described the concept of a FMS and established some related fixed point results as well. In 1988, M. Grabiec [11] constructed a result, which is analogous to the renown BCP and Edelstien theorem in the context of FMSs by using the same idea. In 1994, A. George and P. Veeramani [9] modified the theme of a FMS by adopting the idea of continuous t-norms in the definition of I. Kramosil and J. Michálek [17] and proved some fixed point results. Subsequently, many new fixed point theorems are proved by many researchers in a FMS by using numerous mappings, such as compatible maps, weakly compatible maps, R-weakly compatible maps, and by using implicit relations. For details, one can consult [23–25].

In 1989, I.A. Bakhtin [6] initialized the idea of a b-metric space (BMS). In 1993, S. Czerwik [7] conceived this idea to formally define a BMS. Afterwards, a BMS becomes a stimulating topic for mathematicians and various results on this platform are established. Further related results are presented in the works [3–5, 14–16, 18, 22, 27, 28].

N. Hussain et al. [13] introduced the concept of a fuzzy b-metric space (FBMS) in 2015 and

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developed a relationship between the fuzzy b-metric and the parametric b-metric. In 2016, S. Nădăban [21] studied the notion of a FBMS. Recently, F. Mehmood et al. [19] presented the idea of extended fuzzy b-metric spaces and proved the Banach fixed point theorem. In 1973, M.A. Geraghty [10] introduced a family of functions to generalize the Banach's theorem. In this regard, O. Acar [1] presented further fixed point results for rational type ϕ-Geraghty contractions (see also [2]). In 2019, H. Faraji et al. [8] constructed some fixed point theorems for Geraghty type contraction mappings in BMSs accompanied with an application.

In this article, we merge the ideas presented in [8] and [13] to establish some common fixed point results for Geraghty type contractions in G-complete FBMSs. Our first two results are the extensions of the main results of H. Faraji et al. [8] and the third result is the generalization of the work of V. Gupta et al. [12] in the setting of G-complete FBMSs.

2 Preliminaries

We first recollect a few basic definitions which will help to understand the rest of the sections.

I. Kramosil and J. Michálek [17] presented the concept of a FMS in 1975 by using the concept of a fuzzy set and t-norm as follows.

Definition 1. *Take $\Omega \neq \varnothing$. Let $\mathcal{F}_m$ be a fuzzy set on $\Omega \times \Omega \times (0, \infty)$ and $*$ be a continuous t-norm. A triplet $(\Omega, \mathcal{F}_m, *)$ is said to be a FMS if for all $\varrho, \zeta, \vartheta \in \Omega$ and $\psi, \phi > 0$ the following conditions hold:*

(fm1) $\mathcal{F}_m(\varrho, \zeta, t) > 0$;

(fm2) $\mathcal{F}_m(\varrho, \zeta, \psi) = 1$ *if and only if* $\varrho = \zeta$;

(fm3) $\mathcal{F}_m(\varrho, \zeta, \psi) = \mathcal{F}_m(\zeta, \varrho, t)$;

(fm4) $\mathcal{F}_m(\varrho, \vartheta, \psi + \phi) \geq \mathcal{F}_m(\varrho, \zeta, \psi) * \mathcal{F}_m(\zeta, \vartheta, \phi)$;

(fm5) $\mathcal{F}_m(\varrho, \zeta, \cdot) \colon (0, \infty) \to [0, 1]$ *is left continuous.*

Many scholars further investigated the fixed point (FP for short) theory in a FMS. In 2016, S. Nădăban [21] gave the following definition of a FBMS.

Definition 2. *Take $\Omega \neq \varnothing$. Let $b \geq 1 \in \mathbb{R}$ be a real number and $*$ be a continuous t-norm. A fuzzy set $\mathcal{F}_{bm} \colon \Omega \times \Omega \times (0, \infty) \to [0, 1]$ is called a fuzzy b-metric on Ω if for $\psi, \phi > 0$ and for all $\varrho, \zeta, \vartheta \in \Omega$ it satisfies the following conditions:*

(fbm1) $\mathcal{F}_{bm}(\varrho, \zeta, t) > 0$;

(fbm2) $\mathcal{F}_{bm}(\varrho, \zeta, \psi) = 1$ *if and only if* $\varrho = \zeta$;

(fbm3) $\mathcal{F}_{bm}(\varrho, \zeta, \psi) = \mathcal{F}_{bm}(\zeta, \varrho, \psi)$;

(fbm4) $\mathcal{F}_{bm}(\varrho, \vartheta, b(\psi + \phi)) \geq \mathcal{F}_{bm}(\varrho, \zeta, \psi) * \mathcal{F}_{bm}(\zeta, \vartheta, \phi)$;

(fbm5) $\mathcal{F}_{bm}(\varrho, \zeta, \cdot) \colon (0, \infty) \to [0, 1]$ *is continuous, and* $\lim_{\psi \to \infty} \mathcal{F}_{bm}(\varrho, \zeta, \psi) = 1$.

The triplet $(\Omega, \mathcal{F}_{bm}, *)$ is called a FBMS.

Remark 1. *The class of FBMSs is the generalization of a FMS. Indeed, when $b = 1$, it coincides with a FMS setting.*

Example 1. *Let $\Omega = \mathbb{R}$ and define $d_b(\varrho, \zeta) = (\varrho - \zeta)^2$, then (Ω, d_b) is a BMS with coefficient $b = 2$. Let $\mathcal{F}_{bm} \colon \Omega \times \Omega \times (0, \infty) \to [0, 1]$ be defined by*

$$\mathcal{F}_{bm}(\varrho, \zeta, \psi) = e^{-\frac{(\varrho-\zeta)^2}{\psi}},$$

then $(\Omega, \mathcal{F}_{bm}, *)$ is a FBMS, but it is not a FMS (the continuous t-norm is taken as $a * b = ab$).

The notions of convergence, Cauchyness and completeness are similar to that of M. Grabiec [11]. In this article, these terms will be called G_r-convergent, G_r-Cauchy and G_r-complete, respectively.

Definition 3 ([21])**.** *A sequence $\{\varrho_n\}$ in a FBMS $(\Omega, \mathcal{F}_{bm}, *)$ is called G_r-convergent to $\varrho \in \Omega$ if $\lim\limits_{n\to\infty} \mathcal{F}_{bm}(\varrho_n, \varrho, \psi) = 1$ for all $\psi > 0$.*

Example 2. *Suppose $\Omega = [0, 1]$. Define $\mathcal{F}_{bm} \colon \Omega \times \Omega \times (0, \infty) \to [0, 1]$ by*

$$\mathcal{F}_{bm}(\varrho, \zeta, \psi) = \frac{\psi}{\psi + (\varrho - \zeta)^2},$$

then $(\Omega, \mathcal{F}_{bm}, *)$ is a FBMS. Consider a sequence $\{\varrho_n\}$ in Ω such that $\varrho_n = \frac{1}{n}$ for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \mathcal{F}_{bm}(\varrho_n, 0, \psi) = \lim_{n\to\infty} \frac{\psi}{\psi + \left(\frac{1}{n} - 0\right)^2} = \lim_{n\to\infty} \frac{\psi}{\psi + \frac{1}{n^2}} = 1.$$

This implies that ϱ_n is G_r-convergent to 0.

Definition 4 ([21])**.** *A sequence $\{\varrho_n\}$ in a FBMS $(\Omega, \mathcal{F}_{bm}, *)$ is said to be a G_r-Cauchy sequence if $\lim\limits_{n\to\infty} \mathcal{F}_{bm}(\varrho_n, \varrho_{n+p}, \psi) = 1$ for all $\psi > 0$ and $p \in \mathbb{N}$.*

If each G_r-Cauchy sequence in a FBMS is G_r-convergent, then it is called a G_r-complete FBMS.

Example 3. *Consider the FBMS $(\Omega, \mathcal{F}_{bm}, *)$ defined as in Example 2 and the sequence given as $\varrho_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Now, $\lim\limits_{n\to\infty} \mathcal{F}_{bm}(\varrho_n, \varrho_{n+p}, \psi) = \lim\limits_{n\to\infty} \frac{\psi}{\psi + \left(\frac{1}{n} - \frac{1}{n+p}\right)^2} = \frac{\psi}{\psi + 0} = 1$. Hence, $\{\varrho_n\}$ is a G_r-Cauchy sequence.*

In 1973, Geraghty's theorem opened new doors for researchers. Subsequently, various authors established new results for the existence of a FP by using Geraghty type contractive mappings. Recently, H. Faraji et al. [8] used BMS as a platform to obtain few FP theorems involving Geraghty type contractions. This research deals with the existence of common FP results for Geraghty type contractions in FBMSs.

3 Main results

Following [8], let F_b be the family of all functions $\beta \colon [0, \infty) \to [0, \frac{1}{b})$, where $b \geq 1$, which satisfy the following condition

$$\limsup_{n\to\infty} \beta(\psi_n) = \frac{1}{b} \qquad \Longrightarrow \qquad \lim_{n\to\infty} \psi_n = 1.$$

Lemma 1. *Let $(\Omega, \mathcal{F}_{bm}, *)$ be a G_r-complete FBMS. If for all $\varrho, \eta \in \Omega$, $\psi > 0$ and $\beta \in F_b$ we have $\mathcal{F}_{bm}(\varrho, \eta, \beta(\mathcal{F}_{bm}(\varrho, \eta, \psi))\psi) \geq \mathcal{F}_{bm}(\varrho, \eta, \psi)$, then $\varrho = \eta$.*

Proof. From *(fbm5)* it follows that $\lim_{\psi \to \infty} \mathcal{F}_{bm}(\varrho, \eta, \psi) = 1$, so $\mathcal{F}_{bm}(\varrho, \eta, \beta(\mathcal{F}_{bm}(\varrho, \eta, \psi))\psi) = 1$, by *(fbm2)*. It is clear that $\varrho = \eta$. □

The following is a common FP result for Geraghty type contractions in FBMSs.

Theorem 1. *Let $(\Omega, \mathcal{F}_{bm}, *)$ be a G_r-complete FBMS with $b \geq 1$. Let $\mathcal{T}, \mathcal{S}\colon \Omega \to \Omega$ be mappings satisfying the condition*

$$\mathcal{F}_{bm}(\mathcal{T}\varrho, S\eta, \beta(\mathcal{F}_{bm}(\varrho, \eta, \psi))\psi) \geq \lambda(\varrho, \eta, \psi), \tag{1}$$

where for all $\varrho, \eta \in \Omega$, $\beta \in F_b$ we have $\lambda(\varrho, \eta, \psi) = \min\{\mathcal{F}_{bm}(\varrho, \mathcal{T}\varrho, \psi), \mathcal{F}_{bm}(\eta, S\eta, \psi)\}$. If $\mathcal{T}$ or $\mathcal{S}$ is continuous, then $\mathcal{T}$ and $\mathcal{S}$ have a unique common FP.

Proof. Let $\varrho_0 \in \Omega$. Let us consider the iterative sequence defined by $\varrho_{2n+1} = \mathcal{T}\varrho_{2n}$ and $\varrho_{2n+2} = \mathcal{S}\varrho_{2n+1}$ for $n = 0, 1, 2, \ldots$. We have

$$\mathcal{F}_{bm}(\varrho_{2n+1}, \varrho_{2n+2}, \psi) = \mathcal{F}_{bm}(\mathcal{T}\varrho_{2n}, \mathcal{S}\varrho_{2n+1}, \psi) \geq \lambda\left(\varrho_{2n}, \varrho_{2n+1}, \frac{\psi}{\beta(\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi))}\right). \tag{2}$$

Now,

$$\begin{aligned}\lambda(\varrho_{2n}, \varrho_{2n+1}, \psi) &= \min\{\mathcal{F}_{bm}(\varrho_{2n}, \mathcal{T}\varrho_{2n}, \psi), \mathcal{F}_{bm}(\varrho_{2n+1}, \mathcal{S}\varrho_{2n+1}, \psi)\} \\ &= \min\{\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi), \mathcal{F}_{bm}(\varrho_{2n+1}, \varrho_{2n+2}, \psi)\}.\end{aligned}$$

If $\min\{\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi), \mathcal{F}_{bm}(\varrho_{2n+1}, \varrho_{2n+2}, \psi)\} = \mathcal{F}_{bm}(\varrho_{2n+1}, \varrho_{2n+2}, \psi)$, then we obtain the equality $\lambda(\varrho_{2n}, \varrho_{2n+1}, \psi) = \mathcal{F}_{bm}(\varrho_{2n+1}, \varrho_{2n+2}, \psi)$. The equation (2) implies

$$\mathcal{F}_{bm}(\varrho_{2n+1}, \varrho_{2n+2}, \psi) \geq \mathcal{F}_{bm}\left(\varrho_{2n+1}, \varrho_{2n+2}, \frac{\psi}{\beta(\mathcal{F}_{bm}(\varrho_{2n+1}, \varrho_{2n+2}, \psi))}\right),$$

then there is nothing to prove due to Lemma 1.

If $\min\{\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi), \mathcal{F}_{bm}(\varrho_{2n+1}, \varrho_{2n+2}, \psi)\} = \mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi)$, then we obtain $\lambda(\varrho_{2n}, \varrho_{2n+1}, \psi) = \mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi)$. Again, (2) implies

$$\begin{aligned}\mathcal{F}_{bm}(\varrho_{2n+1}, \varrho_{2n+2}, \psi) &\geq \mathcal{F}_{bm}\left(\varrho_{2n}, \varrho_{2n+1}, \frac{\psi}{\beta(\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi))}\right) \\ &\geq \mathcal{F}_{bm}\left(\varrho_{2n-1}, \varrho_{2n}, \frac{\psi}{\beta(\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi))\beta(\mathcal{F}_{bm}(\varrho_{2n-1}, \varrho_{2n}, \psi))}\right) \\ &\cdots \\ &\geq \mathcal{F}_{bm}\left(\varrho_0, \varrho_1, \frac{\psi}{\beta(\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi))\beta(\mathcal{F}_{bm}(\varrho_{2n-1}, \varrho_{2n}, \psi)) \ldots \beta(\mathcal{F}_{bm}(\varrho_0, \varrho_1, \psi))}\right).\end{aligned} \tag{3}$$

For any arbitrary $q \in \mathbb{N}$, and $\psi = \frac{\psi}{q} + \frac{\psi}{q} + \ldots + \frac{\psi}{q}$, using Definition 2 repeatedly, we get

$$\begin{aligned}&\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+p}, \psi) \\ &\quad \geq \mathcal{F}_{bm}\left(\varrho_{2n}, \varrho_{2n+1}, \frac{\psi}{qb}\right) * \mathcal{F}_{bm}\left(\varrho_{2n+1}, \varrho_{2n+2}, \frac{\psi}{qb^2}\right) * \ldots * \mathcal{F}_{bm}\left(\varrho_{2n+p-1}, \varrho_{2n+p}, \frac{\psi}{qb^p}\right).\end{aligned}$$

Using (3), we obtain

$$
\begin{aligned}
&\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+p}, \psi)\\
&\geq \mathcal{F}_{bm}\left(\varrho_0, \varrho_1, \frac{\psi}{qb\,\beta(\mathcal{F}_{bm}(\varrho_{2n-1}, \varrho_{2n}, \psi))\beta(\mathcal{F}_{bm}(\varrho_{2n-2}, \varrho_{2n-1}, \psi)) \ldots \beta(\mathcal{F}_{bm}(\varrho_0, \varrho_1, \psi))}\right)\\
&* \mathcal{F}_{bm}\left(\varrho_0, \varrho_1, \frac{\psi}{qb^2\beta(\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi))\beta(\mathcal{F}_{bm}(\varrho_{2n-1}, \varrho_{2n}, \psi)) \ldots \beta(\mathcal{F}_{bm}(\varrho_0, \varrho_1, \psi))}\right)\\
&\ldots\\
&* \mathcal{F}_{bm}\left(\varrho_0, \varrho_1, \frac{\psi}{qb^p\beta(\mathcal{F}_{bm}(\varrho_{2n+p-1}, \varrho_{2n+p}, \psi))\beta(\mathcal{F}_{bm}(\varrho_{2n+p-2}, \varrho_{2n+p-1}, \psi)) \ldots \beta(\mathcal{F}_{bm}(\varrho_0, \varrho_1, \psi))}\right),
\end{aligned}
$$

$$
\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+p}, \psi) \geq \mathcal{F}_{bm}\left(\varrho_0, \varrho_1, \frac{b^{2n-1}\psi}{q}\right) * \mathcal{F}_{bm}\left(\varrho_0, \varrho_1, \frac{b^{2n-1}\psi}{q}\right) * \ldots * \mathcal{F}_{bm}\left(\varrho_0, \varrho_1, \frac{b^{2n-1}\psi}{q}\right).
$$

Passing to the limit as $n \to \infty$, we obtain $\lim_{n\to\infty} \mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+p}, \psi) = 1$. Thus, $\{\varrho_n\}$ is a G_r-Cauchy sequence. As $(\Omega, \mathcal{F}_{bm}, *)$ is a G_r-complete FBMS, so there is $\sigma \in \Omega$ such that $\lim_{n\to\infty} \varrho_n = \sigma$. We show that σ is a FP of $\mathcal{T}$. Since $\mathcal{T}$ is continuous, one writes

$$
\begin{aligned}
\mathcal{F}_{bm}(\mathcal{T}\sigma, \sigma, \psi) &\geq \mathcal{F}_{bm}\left(\mathcal{T}\sigma, \mathcal{T}\varrho_n, \frac{\psi}{2b}\right) * \mathcal{F}_{bm}\left(\mathcal{T}\varrho_n, \sigma, \frac{\psi}{2b}\right)\\
&\geq \mathcal{F}_{bm}\left(\mathcal{T}\sigma, \mathcal{T}\sigma, \frac{\psi}{2b}\right) * \mathcal{F}_{bm}\left(\varrho_{n+1}, \varrho_n, \frac{\psi}{2b}\right) \longrightarrow 1 * 1 = 1 \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

This implies that $\mathcal{T}\sigma = \sigma$, so σ is a FP of $\mathcal{T}$. From (1), we have

$$
\mathcal{F}_{bm}(\sigma, \mathcal{S}\sigma, \beta(\mathcal{F}_{bm}(\sigma, \sigma, \psi))\psi) = \mathcal{F}_{bm}(\mathcal{T}\sigma, \mathcal{S}\sigma, \beta(\mathcal{F}_{bm}(\sigma, \sigma, \psi))\psi) \geq \lambda(\sigma, \sigma, \psi), \tag{4}
$$

where

$$
\lambda(\sigma, \sigma, \psi) = \min\{\mathcal{F}_{bm}(\sigma, \mathcal{T}\sigma, \psi), \mathcal{F}_{bm}(\sigma, b\sigma, \psi)\} = \min\{1, \mathcal{F}_{bm}(\sigma, \mathcal{S}\sigma, \psi)\} = \mathcal{F}_{bm}(\sigma, \mathcal{S}\sigma, \psi).
$$

The equation (4) implies $\mathcal{F}_{bm}(\sigma, \mathcal{S}\sigma, \beta(\mathcal{F}_{bm}(\sigma, \sigma, \psi))\psi) \geq \mathcal{F}_{bm}(\sigma, \mathcal{S}\sigma, \psi)$. By Lemma 1, $\mathcal{S}\sigma = \sigma$, and so σ is a FP of $\mathcal{S}$.

If we assume that $\mathcal{S}$ is continuous, then one can show analogously that $\mathcal{T}$ and $\mathcal{S}$ have a common FP.

Uniqueness. Assume that δ is any other common FP of $\mathcal{T}$ and $\mathcal{S}$, that is $\delta = \mathcal{T}\delta = \mathcal{S}\delta$, then from (1), we get $\mathcal{F}_{bm}(\sigma, \delta, \beta(\mathcal{F}_{bm}(\sigma, \delta, \psi))\psi) = \mathcal{F}_{bm}(\mathcal{T}\sigma, \mathcal{S}\delta, \beta(\mathcal{F}_{bm}(\sigma, \delta, \psi))\psi) \geq \lambda(\sigma, \delta, \psi)$, where $\lambda(\sigma, \delta, \psi) = \min\{\mathcal{F}_{bm}(\sigma, T\sigma, \psi), \mathcal{F}_{bm}(\delta, \mathcal{S}\delta, \psi)\} = \min\{1, 1\} = 1$. Thus, $\sigma = \delta$. Hence, $\mathcal{T}$ and $\mathcal{S}$ have a unique common FP. □

The following are immediate consequences of Theorem 1.

Corollary 1. *Let $(\Omega, \mathcal{F}_m, *)$ be a G_r-complete FMS. Let $\mathcal{T}, \mathcal{S} \colon \Omega \to \Omega$ be mappings satisfying the condition*

$$
\mathcal{F}_m(\mathcal{T}\varrho, \mathcal{S}\zeta, \beta(\mathcal{F}_m(\varrho, \zeta, \psi))\psi) \geq \lambda(\varrho, \zeta, \psi),
$$

where

$$
\lambda(\varrho, \zeta, \psi) = \min\{\mathcal{F}_m(\varrho, \mathcal{T}\varrho, \psi), \mathcal{F}_m(\zeta, \mathcal{S}\zeta, \psi)\}
$$

for all $\varrho, \zeta \in \Omega$, $\beta \in F_1$. If $\mathcal{T}$ or $\mathcal{S}$ is continuous, then $\mathcal{T}$ and $\mathcal{S}$ have a unique common FP.

By taking $\mathcal{T} = \mathcal{S}$ in Theorem 1, the following result can be obtained.

Corollary 2. *Let $(\Omega, \mathcal{F}_{bm}, *)$ be a G_r-complete FBMS with $b \geq 1$. Let $\mathcal{T}\colon \Omega \to \Omega$ be a mapping satisfying the condition*

$$\mathcal{F}_{bm}(\mathcal{T}\varrho, \mathcal{T}\zeta, \beta(\mathcal{F}_{bm}(\varrho, \zeta, \psi))\psi) \geq \lambda(\varrho, \zeta, \psi),$$

where

$$\lambda(\varrho, \zeta, \psi) = \min\{\mathcal{F}_{bm}(\varrho, \mathcal{T}\varrho, \psi), \mathcal{F}_{bm}(\zeta, \mathcal{T}\zeta, \psi)\},$$

for all $\varrho, \zeta \in \Omega$, $\beta \in F_b$. Further, if T is continuous, then $\mathcal{T}$ has a unique FP.

Analogous to [8, Theorem 4], we have a common FP theorem.

Theorem 2. *Let $(\Omega, \mathcal{F}_{bm}, *)$ be a G_r-complete FBMS with $b \geq 1$. Let $\mathcal{T}, \mathcal{S}\colon \Omega \to \Omega$ be a mapping satisfying the condition*

$$\mathcal{F}_{bm}(\mathcal{T}\varrho, \mathcal{S}\zeta, \beta(\mathcal{F}_{bm}(\varrho, \zeta, \psi))\psi) \geq \lambda(\varrho, \zeta, \psi),$$

where

$$\lambda(\varrho, \zeta, \psi) = \min\{\mathcal{F}_{bm}(\varrho, \zeta, \psi), \mathcal{F}_{bm}(\varrho, T\varrho, \psi), \mathcal{F}_{bm}(\zeta, \mathcal{S}\zeta, \psi)\},$$

for all $\varrho, \zeta \in \Omega$, $\beta \in F_b$. If $\mathcal{T}$ or $\mathcal{S}$ is continuous, then $\mathcal{T}$ and $\mathcal{S}$ have a unique common FP.

Proof. Let $\varrho_0 \in \Omega$. Let us consider the iterative sequence defined by $\varrho_{2n+1} = \mathcal{T}\varrho_{2n}$ and $\varrho_{2n+2} = \mathcal{S}\varrho_{2n+1}$ for $n = 0, 1, 2, \ldots$. We have

$$\mathcal{F}_{bm}(\varrho_{2n+1}, \varrho_{2n+2}, \psi) = \mathcal{F}_{bm}(\mathcal{T}\varrho_{2n}, \mathcal{S}\varrho_{2n+1}, \psi) \geq \lambda\left(\varrho_{2n}, \varrho_{2n+1}, \frac{\psi}{\beta(\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi))}\right) \tag{5}$$

and

$$\begin{aligned}
\lambda(\varrho_{2n}, \varrho_{2n+1}, \psi) &= \min\left\{\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi), \mathcal{F}_{bm}(\varrho_{2n}, \mathcal{T}\varrho_{2n}, \psi), \mathcal{F}_{bm}(\varrho_{2n+1}, \mathcal{S}\varrho_{2n+1}, \psi)\right\} \\
&= \min\left\{\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi), \mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi), \mathcal{F}_{bm}(\varrho_{2n+1}, \varrho_{2n+2}, \psi)\right\} \\
&= \min\left\{\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi), \mathcal{F}_{bm}(\varrho_{2n+1}, \varrho_{2n+2}, \psi)\right\}.
\end{aligned}$$

If $\min\left\{\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi), \mathcal{F}_{bm}(\varrho_{2n+1}, \varrho_{2n+2}, \psi)\right\} = \mathcal{F}_{bm}(\varrho_{2n+1}, \varrho_{2n+2}, \psi)$, then we get the equality $\lambda(\varrho_{2n}, \varrho_{2n+1}, \psi) = \mathcal{F}_{bm}(\varrho_{2n+1}, \varrho_{2n+2}, \psi)$. The equation (5) yields

$$\mathcal{F}_{bm}(\varrho_{2n+1}, \varrho_{2n+2}, \psi) \geq \mathcal{F}_{bm}\left(\varrho_{2n+1}, \varrho_{2n+2}, \frac{\psi}{\beta(\mathcal{F}_{bm}(\varrho_{2n+1}, \varrho_{2n+2}, \psi))}\right),$$

there is nothing to prove by Lemma 1.

If $\min\left\{\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi), \mathcal{F}_{bm}(\varrho_{2n+1}, \varrho_{2n+2}, \psi)\right\} = \mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi)$, then we obtain the equality $\lambda(\varrho_{2n}, \varrho_{2n+1}, \psi) = \mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi)$. The equation (5) implies

$$\begin{aligned}
&\mathcal{F}_{bm}(\varrho_{2n+1}, \varrho_{2n+2}, \psi) \\
&\quad \geq \mathcal{F}_{bm}\left(\varrho_{2n}, \varrho_{2n+1}, \frac{\psi}{\beta(\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi))}\right) \\
&\quad \geq \mathcal{F}_{bm}\left(\varrho_{2n-1}, \varrho_{2n}, \frac{\psi}{\beta(\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi))\beta(\mathcal{F}_{bm}(\varrho_{2n-1}, \varrho_{2n}, \psi))}\right) \\
&\quad \ldots \\
&\quad \geq \mathcal{F}_{bm}\left(\varrho_0, \varrho_1, \frac{\psi}{\beta(\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi))\beta(\mathcal{F}_{bm}(\varrho_{2n-1}, \varrho_{2n}, \psi)) \ldots \beta(\mathcal{F}_{bm}(\varrho_0, \varrho_1, \psi))}\right).
\end{aligned} \tag{6}$$

For any $q \in \mathbb{N}$, $\psi = \frac{\psi}{q} + \frac{\psi}{q} + \ldots + \frac{\psi}{q}$, applying Definition 2 repeatedly, we get

$$\begin{aligned}&\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+p}, \psi)\\ &\quad \geq \mathcal{F}_{bm}\left(\varrho_{2n}, \varrho_{2n+1}, \frac{\psi}{qb}\right) * \mathcal{F}_{bm}\left(\varrho_{2n+1}, \varrho_{2n+2}, \frac{\psi}{qb^2}\right) * \ldots * \mathcal{F}_{bm}\left(\varrho_{2n+p-1}, \varrho_{2n+p}, \frac{\psi}{qb^p}\right).\end{aligned}$$

Using (6), we obtain

$$\begin{aligned}&\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+p}, \psi)\\ &\geq \mathcal{F}_{bm}\left(\varrho_0, \varrho_1, \frac{\psi}{qb\,\beta(\mathcal{F}_{bm}(\varrho_{2n-1}, \varrho_{2n}, \psi))\beta(\mathcal{F}_{bm}(\varrho_{2n-2}, \varrho_{2n-1}, \psi)) \ldots \beta(\mathcal{F}_{bm}(\varrho_0, \varrho_1, \psi))}\right)\\ &* \mathcal{F}_{bm}\left(\varrho_0, \varrho_1, \frac{\psi}{qb^2\beta(\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi))\beta(\mathcal{F}_{bm}(\varrho_{2n-1}, \varrho_{2n}, \psi)) \ldots \beta(\mathcal{F}_{bm}(\varrho_0, \varrho_1, \psi))}\right)\\ &\ldots\\ &* \mathcal{F}_{bm}\left(\varrho_0, \varrho_1, \frac{\psi}{qb^p\beta(\mathcal{F}_{bm}(\varrho_{2n+p-1}, \varrho_{2n+p}, \psi))\beta(\mathcal{F}_{bm}(\varrho_{2n+p-2}, \varrho_{2n+p-1}, \psi)) \ldots \beta(\mathcal{F}_{bm}(\varrho_0, \varrho_1, \psi))}\right),\\ &\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+p}, \psi) \geq \mathcal{F}_{bm}\left(\varrho_0, \varrho_1, \frac{b^{2n-1}\psi}{q}\right) * \mathcal{F}_{bm}\left(\varrho_0, \varrho_1, \frac{b^{2n-1}\psi}{q}\right) * \ldots * \mathcal{F}_{bm}\left(\varrho_0, \varrho_1, \frac{b^{2n-1}\psi}{q}\right).\end{aligned}$$

Passing to the limit as $n \to \infty$, we obtain $\lim_{n\to\infty} \mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+p}, \psi) = 1$. Thus, $\{\varrho_n\}$ is a G_r-Cauchy sequence in Ω. By G_r-completeness of FBMS there is $\sigma \in \Omega$ such that $\lim\limits_{n\to\infty} \varrho_n = \sigma$.

Next, we need to prove that σ is a FP of $\mathcal{T}$. If $\mathcal{T}$ is continuous, then

$$\begin{aligned}\mathcal{F}_{bm}(\mathcal{T}\sigma, \sigma, \psi) &\geq \mathcal{F}_{bm}\left(\mathcal{T}\sigma, \mathcal{T}\varrho_n, \frac{\psi}{2b}\right) * \mathcal{F}_{bm}\left(\mathcal{T}\varrho_n, \sigma, \frac{\psi}{2b}\right)\\ &\geq \mathcal{F}_{bm}\left(\mathcal{T}\sigma, \mathcal{T}\sigma, \frac{\psi}{2b}\right) * \mathcal{F}_{bm}\left(\varrho_{n+1}, \varrho_n, \frac{\psi}{2b}\right) \longrightarrow 1 * 1 = 1 \quad \text{as} \quad n \to \infty.\end{aligned}$$

This implies that $\mathcal{T}\sigma = \sigma$, thus σ is a FP of $\mathcal{T}$. From (2), we get

$$\mathcal{F}_{bm}(\sigma, \mathcal{S}\sigma, \beta(\mathcal{F}_{bm}(\sigma, \sigma, \psi))\psi) = \mathcal{F}_{bm}(\mathcal{T}\sigma, \mathcal{S}\sigma, \beta(\mathcal{F}_{bm}(\sigma, \sigma, \psi))\psi) \geq \lambda(\sigma, \sigma, \psi), \tag{7}$$

where

$$\begin{aligned}\lambda(\sigma, \sigma, \psi) &= \min\left\{\mathcal{F}_{bm}(\sigma, \sigma, \psi), \mathcal{F}_{bm}(\sigma, \mathcal{T}\sigma, \psi), \mathcal{F}_{bm}(\sigma, \mathcal{S}\sigma, \psi)\right\}\\ &= \min\left\{1, 1, \mathcal{F}_{bm}(\sigma, \mathcal{S}\sigma, \psi)\right\} = \mathcal{F}_{bm}(\sigma, \mathcal{S}\sigma, \psi).\end{aligned}$$

The equation (7) implies $\mathcal{F}_{bm}(\sigma, \mathcal{S}\sigma, \beta(\mathcal{F}_{bm}(\sigma, \mathcal{S}\sigma, \psi))\psi) \geq \mathcal{F}_{bm}(\sigma, \mathcal{S}\sigma, \psi)$. So by Lemma 1, $\mathcal{S}\sigma = \sigma$. Thus, σ is a FP of $\mathcal{S}$.

Also, if $\mathcal{S}$ is continuous, then one can show in a similar way that $\mathcal{T}$ and $\mathcal{S}$ have a unique common FP.

Uniqueness. Let $\delta = \mathcal{T}\delta = \mathcal{S}\delta$ be another common FP of $\mathcal{T}$ and $\mathcal{S}$, then from (2), we get

$$\mathcal{F}_{bm}(\sigma, \delta, \beta(\mathcal{F}_{bm}(\sigma, \delta, \psi))\psi) = \mathcal{F}_{bm}(T\sigma, \mathcal{S}\delta, \beta(\mathcal{F}_{bm}(\sigma, \delta, \psi))\psi) \geq \lambda(\sigma, \delta, \psi), \tag{8}$$

where

$$\begin{aligned}\lambda(\sigma, \delta, \psi) &= \min\left\{\mathcal{F}_{bm}(\sigma, \delta, \psi), \mathcal{F}_{bm}(\sigma, \mathcal{T}\sigma, \psi), \mathcal{F}_{bm}(\delta, \mathcal{S}\delta, \psi)\right\}\\ &= \min\left\{\mathcal{F}_{bm}(\sigma, \delta, \psi), 1, 1\right\} = \mathcal{F}_{bm}(\sigma, \delta, \psi).\end{aligned}$$

The equation (8) implies that

$$\mathcal{F}_{bm}(\sigma, \delta, \beta(\mathcal{F}_{bm}(\sigma, \delta, \psi))\psi) \geq \mathcal{F}_{bm}(\sigma, \delta, \psi).$$

So by Lemma 1, $\sigma = \delta$. Hence $\mathcal{T}$ and $\mathcal{S}$ have unique common FP. $\square$

The following are immediate consequences of Theorem 2.

Corollary 3. *Suppose that* $(\Omega, \mathcal{F}_m, *)$ *is a* G_r*-complete FMS. Let* $\mathcal{T}, \mathcal{S} \colon \Omega \to \Omega$ *be mappings satisfying the condition*

$$\mathcal{F}_m(\mathcal{T}\varrho, \mathcal{S}\zeta, \beta(\mathcal{F}_m(\varrho, \zeta, \psi))\psi) \geq \lambda(\varrho, \zeta, \psi),$$

where

$$\lambda(\varrho, \zeta, \psi) = \min\left\{\mathcal{F}_m(\varrho, \zeta, \psi), \mathcal{F}_m(\varrho, T\varrho, \psi), \mathcal{F}_m(\zeta, \mathcal{S}\zeta, \psi)\right\}$$

for all $\varrho, \zeta \in \Omega$*,* $\beta \in F_1$*. Further, if* $\mathcal{T}$ *or* $\mathcal{S}$ *is continuous, then* $\mathcal{T}$ *and* $\mathcal{S}$ *have a unique common* FP*.*

If we take $\mathcal{T} = \mathcal{S}$ in Theorem 2, we get the following result.

Corollary 4. *Let* $(\Omega, \mathcal{F}_{bm}, *)$ *be a* G_r*-complete FBMS with* $b \geq 1$*. Let* $\mathcal{T} \colon \Omega \to \Omega$ *be a mapping satisfying the condition*

$$\mathcal{F}_{bm}(\mathcal{T}\varrho, \mathcal{T}\zeta, \beta(\mathcal{F}_{bm}(\varrho, \zeta, \psi))\psi) \geq \lambda(\varrho, \zeta, \psi),$$

where

$$\lambda(\varrho, \zeta, \psi) = \min\{\mathcal{F}_{bm}(\varrho, \zeta, \psi), \mathcal{F}_{bm}(\varrho, \mathcal{T}\varrho, \psi), \mathcal{F}_{bm}(\zeta, \mathcal{T}\zeta, \psi)\}$$

for all $\varrho, \zeta \in \Omega$*,* $\beta \in F_b$*. If* $\mathcal{T}$ *is continuous, then* $\mathcal{T}$ *has a unique* FP*.*

We now establish a common FP result, analogue to [12, Theorem 1] in G_r-complete FBMSs by using Geraghty type contractions.

Theorem 3. *Suppose that* $(\Omega, \mathcal{F}_{bm}, *)$ *is a* G_r*-complete FBMS with* $b \geq 1$*. Let* $\mathcal{T}, \mathcal{S} \colon \Omega \to \Omega$ *be mappings satisfying the condition*

$$\mathcal{F}_{bm}(\mathcal{T}\varrho, \mathcal{S}\zeta, \beta(\mathcal{F}_{bm}(\varrho, \zeta, \psi))\psi) \geq \lambda(\varrho, \zeta, \psi),$$

where

$$\lambda(\varrho, \zeta, \psi) = \min\left\{\frac{\mathcal{F}_{bm}(\zeta, \mathcal{S}\zeta, \psi)\left[1 + \mathcal{F}_{bm}(\varrho, \mathcal{T}\varrho, \psi)\right]}{1 + \mathcal{F}_{bm}(\varrho, \zeta, \psi)}, \mathcal{F}_{bm}(\varrho, \zeta, \psi)\right\}$$

for all $\varrho, \zeta \in \Omega$*,* $\psi > 0$ *and* $\beta \in F_b$*. If* $\mathcal{T}$ *or* $\mathcal{S}$ *is continuous, then* $\mathcal{T}$ *and* $\mathcal{S}$ *have a common* FP*.*

Proof. Let $\varrho_0 \in \Omega$. Let us consider the iterative sequence defined by $\varrho_{2n+1} = \mathcal{T}\varrho_{2n}$ and $\varrho_{2n+2} = \mathcal{S}\varrho_{2n+1}$ for $n = 0, 1, 2, \ldots$. We have

$$\mathcal{F}_{bm}(\varrho_{2n+1}, \varrho_{2n+2}, \psi) = \mathcal{F}_{bm}(\mathcal{T}\varrho_{2n}, \mathcal{S}\varrho_{2n+1}, \psi) \geq \lambda\left(\varrho_{2n}, \varrho_{2n+1}, \frac{\psi}{\beta(\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi))}\right). \quad (9)$$

Consider

$$\begin{aligned}
\lambda(\varrho_{2n}, \varrho_{2n+1}, \psi) &= \min\left\{\frac{\mathcal{F}_{bm}(\varrho_{2n+1}, \mathcal{S}\varrho_{2n+1}, \psi)\left[1 + \mathcal{F}_{bm}(\varrho_{2n}, \mathcal{T}\varrho_{2n}, \psi)\right]}{1 + \mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi)}, \mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi)\right\} \\
&= \min\left\{\frac{\mathcal{F}_{bm}(\varrho_{2n+1}, \varrho_{2n+2}, \psi)\left[1 + \mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi)\right]}{1 + \mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi)}, \mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi)\right\} \\
&= \min\left\{\mathcal{F}_{bm}(\varrho_{2n}, \varrho_{2n+1}, \psi), \mathcal{F}_{bm}(\varrho_{2n+1}, \varrho_{2n+2}, \psi)\right\}.
\end{aligned}$$

If $\min\{\mathcal{F}_{bm}(\varrho_{2n},\varrho_{2n+1},\psi),\mathcal{F}_{bm}(\varrho_{2n+1},\varrho_{2n+2},\psi)\} = \mathcal{F}_{bm}(\varrho_{2n+1},\varrho_{2n+2},\psi)$, then we obtain the equality $\lambda(\varrho_{2n},\varrho_{2n+1},\psi) = \mathcal{F}_{bm}(\varrho_{2n+1},\varrho_{2n+2},\psi)$. The equation (9) implies

$$\mathcal{F}_{bm}(\varrho_{2n+1},\varrho_{2n+2},\psi) \geq \mathcal{F}_{bm}\left(\varrho_{2n+1},\varrho_{2n+2},\frac{\psi}{\beta(\mathcal{F}_{bm}(\varrho_{2n+1},\varrho_{2n+2},\psi))}\right),$$

then there is nothing to prove by Lemma 1.

If $\min\{\mathcal{F}_{bm}(\varrho_{2n},\varrho_{2n+1},\psi),\mathcal{F}_{bm}(\varrho_{2n+1},\varrho_{2n+2},\psi)\} = \mathcal{F}_{bm}(\varrho_{2n},\varrho_{2n+1},\psi)$, then we obtain the equality $\lambda(\varrho_{2n},\varrho_{2n+1},\psi) = \mathcal{F}_{bm}(\varrho_{2n},\varrho_{2n+1},\psi)$. Again, the equation (9) implies

$$\begin{aligned}
&\mathcal{F}_{bm}(\varrho_{2n+1},\varrho_{2n+2},\psi)\\
&\quad\geq \mathcal{F}_{bm}\left(\varrho_{2n},\varrho_{2n+1},\frac{\psi}{\beta(\mathcal{F}_{bm}(\varrho_{2n},\varrho_{2n+1},\psi))}\right)\\
&\quad\geq \mathcal{F}_{bm}\left(\varrho_{2n-1},\varrho_{2n},\frac{\psi}{\beta(\mathcal{F}_{bm}(\varrho_{2n},\varrho_{2n+1},\psi))\beta(\mathcal{F}_{bm}(\varrho_{2n-1},\varrho_{2n},\psi))}\right)\\
&\quad\ldots\\
&\quad\geq \mathcal{F}_{bm}\left(\varrho_{0},\varrho_{1},\frac{\psi}{\beta(\mathcal{F}_{bm}(\varrho_{2n},\varrho_{2n+1},\psi))\beta(\mathcal{F}_{bm}(\varrho_{2n-1},\varrho_{2n},\psi))\ldots\beta(\mathcal{F}_{bm}(\varrho_{0},\varrho_{1},\psi))}\right).
\end{aligned}\tag{10}$$

For any $q \in \mathbb{N}$ and $\psi = \frac{\psi}{q}+\frac{\psi}{q}+\ldots+\frac{\psi}{q}$, using Definition 2 repeatedly, we obtain

$$\begin{aligned}
\mathcal{F}_{bm}(\varrho_{2n},\varrho_{2n+p},\psi) \geq \mathcal{F}_{bm}\left(\varrho_{2n},\varrho_{2n+1},\frac{\psi}{qb}\right) * \mathcal{F}_{bm}\left(\varrho_{2n+1},\varrho_{2n+2},\frac{\psi}{qb^2}\right)\\
* \cdots * \mathcal{F}_{bm}\left(\varrho_{2n+p-1},\varrho_{2n+p},\frac{\psi}{qb^p}\right).
\end{aligned}$$

In view of (10), we get

$$\begin{aligned}
&\mathcal{F}_{bm}(\varrho_{2n},\varrho_{2n+p},\psi)\\
&\geq \mathcal{F}_{bm}\left(\varrho_0,\varrho_1,\frac{\psi}{qb\,\beta(\mathcal{F}_{bm}(\varrho_{2n-1},\varrho_{2n},\psi))\beta(\mathcal{F}_{bm}(\varrho_{2n-2},\varrho_{2n-1},\psi))\ldots\beta(\mathcal{F}_{bm}(\varrho_0,\varrho_1,\psi))}\right)\\
&* \mathcal{F}_{bm}\left(\varrho_0,\varrho_1,\frac{\psi}{qb^2\beta(\mathcal{F}_{bm}(\varrho_{2n},\varrho_{2n+1},\psi))\beta(\mathcal{F}_{bm}(\varrho_{2n-1},\varrho_{2n},\psi))\ldots\beta(\mathcal{F}_{bm}(\varrho_0,\varrho_1,\psi))}\right)\\
&\ldots\\
&* \mathcal{F}_{bm}\left(\varrho_0,\varrho_1,\frac{\psi}{qb^p\beta(\mathcal{F}_{bm}(\varrho_{2n+p-1},\varrho_{2n+p},\psi))\beta(\mathcal{F}_{bm}(\varrho_{2n+p-2},\varrho_{2n+p-1},\psi))\ldots\beta(\mathcal{F}_{bm}(\varrho_0,\varrho_1,\psi))}\right),
\end{aligned}$$

$$M_b(\varrho_{2n},\varrho_{2n+p},\psi) \geq \mathcal{F}_{bm}\left(\varrho_0,\varrho_1,\frac{b^{2n-1}\psi}{q}\right) * \mathcal{F}_{bm}\left(\varrho_0,\varrho_1,\frac{b^{2n-1}\psi}{q}\right) * \ldots * \mathcal{F}_{bm}\left(\varrho_0,\varrho_1,\frac{b^{2n-1}\psi}{q}\right).$$

Passing to the limit as $n \to \infty$, we obtain $\lim_{n\to\infty} \mathcal{F}_{bm}(\varrho_{2n},\varrho_{2n+p},\psi) = 1$. Thus, $\{\varrho_n\}$ is a G_r-Cauchy sequence. Since $(\Omega,\mathcal{F}_{bm},*)$ is a G_r-complete FBMS, there exists $\sigma \in \Omega$ such that $\lim_{n\to\infty} \varrho_n = \sigma$. To prove that σ is a FP of $\mathcal{T}$, we proceed as follows. If $\mathcal{T}$ is continuous, then

$$\begin{aligned}
\mathcal{F}_{bm}(\mathcal{T}\sigma,\sigma,\psi) &\geq \mathcal{F}_{bm}\left(\mathcal{T}\sigma,\mathcal{T}\varrho_n,\frac{\psi}{2b}\right) * \mathcal{F}_{bm}\left(\mathcal{T}\varrho_n,\sigma,\frac{\psi}{2b}\right)\\
&\geq \mathcal{F}_{bm}\left(\mathcal{T}\sigma,\mathcal{T}\sigma,\frac{\psi}{2b}\right) * \mathcal{F}_{bm}\left(\varrho_{n+1},\sigma,\frac{\psi}{2b}\right) \longrightarrow 1*1=1 \quad \text{as} \quad n\to\infty.
\end{aligned}$$

This implies that $\mathcal{T}\sigma = \sigma$, thus σ is a FP of $\mathcal{T}$. From (3), we get

$$\mathcal{F}_{bm}(\sigma, \mathcal{S}\sigma, \beta(\mathcal{F}_{bm}(\sigma,\sigma,\psi))\psi) = \mathcal{F}_{bm}(\mathcal{T}\sigma, \mathcal{S}\sigma, \beta(\mathcal{F}_{bm}(\sigma,\sigma,\psi))\psi) \geq \lambda\left(\sigma,\sigma,\psi\right). \tag{11}$$

Consider,

$$\begin{aligned}\lambda(\sigma,\sigma,\psi) &= \min\left\{\frac{\mathcal{F}_{bm}(\sigma,\mathcal{S}\sigma,\psi)\left[1+\mathcal{F}_{bm}(\sigma,\mathcal{T}\sigma,\psi)\right]}{1+\mathcal{F}_{bm}(\sigma,\sigma,\psi)}, \mathcal{F}_{bm}(\sigma,\sigma,\psi)\right\}\\ &= \min\left\{\frac{\mathcal{F}_{bm}(\sigma,\mathcal{S}\sigma,\psi)\left[1+1\right]}{1+1}, 1\right\}\\ &= \min\left\{\mathcal{F}_{bm}(\sigma,\mathcal{S}\sigma,\psi), 1\right\} = \mathcal{F}_{bm}(\sigma,\mathcal{S}\sigma,\psi).\end{aligned}$$

The equation (11) implies $\mathcal{F}_{bm}(\sigma, \mathcal{S}\sigma, \beta(\mathcal{F}_{bm}(\sigma,\sigma,\psi))\psi) \geq \mathcal{F}_{bm}(\sigma,\mathcal{S}\sigma,\psi)$. So by Lemma 1, $\mathcal{S}\sigma = \sigma$. Thus, σ is a FP of $\mathcal{S}$. Similarly, if $\mathcal{S}$ is continuous, then one can show analogously that $\mathcal{T}$ and $\mathcal{S}$ have a common FP. □

The following are immediate consequences of Theorem 3.

Corollary 5. *Let $(\Omega, \mathcal{F}_m, *)$ be a G_r-complete FMS. Let $\mathcal{T}, \mathcal{S}\colon \Omega \to \Omega$ be mappings satisfying*

$$\mathcal{F}_m(\mathcal{T}\varrho, \mathcal{S}\zeta, \beta(\mathcal{F}_m(\varrho,\zeta,\psi))\psi) \geq \lambda(\varrho,\zeta,\psi),$$

where

$$\lambda(\varrho,\zeta,\psi) = \min\left\{\frac{\mathcal{F}_m(\zeta,\mathcal{S}\zeta,\psi)\left[1+\mathcal{F}_m(\varrho,\mathcal{T}\varrho,\psi)\right]}{1+\mathcal{F}_m(\varrho,\zeta,\psi)}, \mathcal{F}_m(\varrho,\zeta,\psi)\right\}$$

for all $\varrho, \zeta \in \Omega$, $\beta \in F_1$. If $\mathcal{T}$ or $\mathcal{S}$ is continuous, then $\mathcal{T}$ and $\mathcal{S}$ have a common FP.

Taking $\mathcal{T} = \mathcal{S}$ in Theorem 3, we obtain the following result.

Corollary 6. *Let $(\Omega, \mathcal{F}_{bm}, *)$ be a G-complete FBMS with $b \geq 1$. Let $\mathcal{T}\colon \Omega \to \Omega$ be a mapping satisfying the condition*

$$\mathcal{F}_{bm}(\mathcal{T}\varrho, \mathcal{T}\zeta, \beta(\mathcal{F}_{bm}(\varrho,\zeta,\psi))\psi) \geq \lambda(\varrho,\zeta,\psi),$$

where

$$\lambda(\varrho,\zeta,\psi) = \min\left\{\frac{\mathcal{F}_{bm}(\zeta,\mathcal{T}\zeta,\psi)\left[1+\mathcal{F}_{bm}(\varrho,T\varrho,\psi)\right]}{1+\mathcal{F}_{bm}(\varrho,\zeta,\psi)}, \mathcal{F}_{bm}(\varrho,\zeta,\psi)\right\}$$

for all $\varrho, \zeta \in \Omega$, $\beta \in F_b$. If $\mathcal{T}$ is continuous, then $\mathcal{T}$ has a FP.

The example given below illustrates Theorem 1.

Example 4. *Let $\Omega = [0,1]$. Define $\mathcal{F}_{bm}\colon \Omega \times \Omega \times (0,\infty) \to [0,1]$ by $\mathcal{F}_{bm}(\varrho,\zeta,\psi) = \frac{\psi}{\psi+(\varrho-\zeta)^2}$. It is easy to verify that $(\Omega, \mathcal{F}_{bm}, *)$ is a G-complete FBMS. Define mappings $\mathcal{T}, \mathcal{S}\colon \Omega \to \Omega$ by $\mathcal{T}\varrho = \frac{\varrho}{9}$ and $\mathcal{S}\zeta = \frac{\zeta}{8}$, respectively. Taking $\beta(\psi) = \dfrac{1}{3}$ for all $\psi > 0$, we have*

$$\begin{aligned}\mathcal{F}_{bm}(\mathcal{T}\varrho, \mathcal{S}\zeta, \beta(\mathcal{F}_{bm}(\varrho,\zeta,\psi))\psi) &= \mathcal{F}_{bm}\left(\frac{\varrho}{9}, \frac{\zeta}{8}, \frac{\psi}{3}\right) = \frac{\psi}{\psi + \dfrac{(8\varrho-9\zeta)^2}{1728}},\\ \mathcal{F}_{bm}(\varrho, \mathcal{T}\varrho, \psi) &= \mathcal{F}_{bm}\left(\varrho, \frac{\varrho}{9}, \psi\right) = \frac{\psi}{\psi + \dfrac{64\varrho^2}{81}},\\ \mathcal{F}_{bm}(\zeta, \mathcal{S}\zeta, \psi) &= \mathcal{F}_{bm}\left(\zeta, \frac{\zeta}{3}, \psi\right) = \frac{\psi}{\psi + \dfrac{49\zeta^2}{64}}.\end{aligned}$$

Case 1. For $\varrho > \zeta$, we have $8\varrho < \frac{64}{\sqrt{3}}\varrho$, which implies

$$8\varrho - 9\zeta < \frac{64}{\sqrt{3}}\varrho \;\Rightarrow\; (8\varrho-9\zeta)^2 < \frac{4096}{3}\varrho^2 \;\Rightarrow\; \frac{(8\varrho-9\zeta)^2}{64} < \frac{64}{3}\varrho^2 \;\Rightarrow\; \frac{(8\varrho-9\zeta)^2}{1728} < \frac{64}{81}\varrho^2.$$

From this, it follows that

$$\psi + \frac{(8\varrho-9\zeta)^2}{1728} < \psi + \frac{64}{81}\varrho^2 \;\Rightarrow\; \frac{1}{\psi + \frac{(8\varrho-9\zeta)^2}{1728}} > \frac{1}{\psi + \frac{64}{81}\varrho^2} \;\Rightarrow\; \frac{\psi}{\psi + \frac{(8\varrho-9\zeta)^2}{1728}} > \frac{\psi}{\psi + \frac{64}{81}\varrho^2},$$

and consequently $\mathcal{F}_{bm}(\mathcal{T}\varrho, \mathcal{S}\zeta, \beta(\mathcal{F}_{bm}(\varrho,\zeta,\psi))\psi) > \mathcal{F}_{bm}(\varrho, \mathcal{T}\varrho, \psi)$. Thus,

$$\mathcal{F}_{bm}(\mathcal{T}\varrho, \mathcal{S}\zeta, \beta(\mathcal{F}_{bm}(\varrho,\zeta,t))\psi) > \min\left\{\mathcal{F}_{bm}(\varrho,\mathcal{T}\varrho,\psi), \mathcal{F}_{bm}(\zeta,\mathcal{S}\zeta,\psi)\right\}.$$

Case 2. For $\varrho < \zeta$, we have $8\varrho < 21\sqrt{3}\zeta$, which implies

$$8\varrho - 9\zeta < 21\sqrt{3}\zeta \;\Rightarrow\; (8\varrho-9\zeta)^2 < 1223\zeta^2 \;\Rightarrow\; \frac{(8\varrho-9\zeta)^2}{27} < 49\zeta^2 \;\Rightarrow\; \frac{(8\varrho-9\zeta)^2}{1728} < \frac{49}{64}\zeta^2.$$

This leads to

$$\psi + \frac{(8\varrho-9\zeta)^2}{1728} < \psi + \frac{49}{64}\zeta^2 \;\Rightarrow\; \frac{1}{\psi + \frac{(8\varrho-9\zeta)^2}{1728}} > \frac{1}{\psi + \frac{49}{64}\zeta^2} \;\Rightarrow\; \frac{\psi}{\psi + \frac{(8\varrho-9\zeta)^2}{1728}} > \frac{\psi}{\psi + \frac{49}{64}\zeta^2},$$

which gives $\mathcal{F}_{bm}(\mathcal{T}\varrho, \mathcal{S}\zeta, \beta(\mathcal{F}_{bm}(\varrho,\zeta,\psi))\psi) > \mathcal{F}_{bm}(\zeta, \mathcal{S}\zeta, \psi)$. Thus,

$$\mathcal{F}_{bm}(\mathcal{T}\varrho, \mathcal{S}\zeta, \beta(\mathcal{F}_{bm}(\varrho,\zeta,\psi))\psi) > \min\left\{\mathcal{F}_{bm}(\varrho,\mathcal{T}\varrho,\psi), \mathcal{F}_{bm}(\zeta,\mathcal{S}\zeta,\psi)\right\}.$$

Therefore, all the conditions of Theorem 1 are fulfilled. Also note that $\mathcal{T}0 = 0$ and $\mathcal{S}0 = 0$, so 0 is the unique common FP of $\mathcal{S}$ and $\mathcal{T}$.

4 Applications

The theory of FPs is considered a strong method for studying the solution's existence of different forms of integral and differential equations, for instance see [4,13,20]. In this section, the existence of the solution of a specific non-linear integral equation has been proved with the help of the particular assumptions on the space as an application of FP results shown in previous section.

Consider all the real valued continuous functions on $\Omega = C[0, I]$. Define a function $\mathcal{F}_{bm}\colon \Omega \times \Omega \times (0,\infty) \to [0,1]$ by

$$\mathcal{F}_{bm}(\varrho,\zeta,\psi) = e^{-\frac{\sup_{s\in[0,I]}|\varrho(s)-\zeta(s)|^2}{\psi}}.$$

Then it is obvious that $(\Omega, \mathcal{F}_{bm}, *)$ is a G_r-complete $FBMS$.

Consider the integral equation

$$\varrho(\psi) = \varphi(\psi) + \int_0^I v(\psi,s)\mathcal{F}(\psi,s,\varrho(s))\,ds,$$

where the functions $\varphi\colon [0,I] \to \mathbb{R}$, $v\colon [0,I]\times[0,I] \to \mathbb{R}$ and $\mathcal{F}\colon [0,I]\times[0,I]\times\mathbb{R} \to \mathbb{R}$ are continuous.

Theorem 4. *Suppose that for all $\psi, s \in [0, I]$, $\varrho, \zeta \in \Omega$ and $\beta \in F_b$ the following conditions hold:*

(i) $|\mathcal{F}(\psi, s, \varrho(s)) - \mathcal{F}(\psi, s, \zeta(s))| < \sqrt{\beta(\mathcal{F}_{bm}(\varrho, \zeta, \psi))}\, |\varrho(s) - \zeta(s)|$;

(ii) $\sup\limits_{s\in[0,I]} \int_0^I (v(\psi, s))^2\, ds \leq \frac{1}{I}$.

Then the integral equation (4) *has a solution defined on* Ω.

Proof. Define the integral operators $\mathcal{T}, \mathcal{S} : \Omega \to \Omega$ by

$$\mathcal{T}\varrho(\psi) = \varphi(\psi) + \int_0^I v(\psi, s)\mathcal{F}(\psi, s, \varrho(s))\, ds, \quad \varrho \in \Omega, \text{ and } \psi, s \in [0, I]$$

$$\mathcal{S}\zeta(\psi) = \varphi(\psi) + \int_0^I v(\psi, s)\mathcal{F}(\psi, s, \zeta(s))\, ds, \quad \varrho \in \Omega, \text{ and } \psi, s \in [0, I].$$

For all $\varrho, \zeta \in \Omega$, we have

$$\begin{aligned}
\mathcal{F}_{bm}(\mathcal{T}\varrho, \mathcal{S}\zeta, \beta(\mathcal{F}_{bm}(\varrho, \zeta, \psi))\psi) &= e^{-\frac{\sup_{s\in[0,I]}\left|\mathcal{T}\varrho(\psi) - \mathcal{S}\zeta(\psi)\right|^2}{\beta(\mathcal{F}_{bm}(\varrho,\zeta,\psi))\psi}} \\
&= e^{-\frac{\sup_{s\in[0,I]}\left|\int_0^I v(\psi,s)\mathcal{F}(\psi,s,\varrho(s))ds - \int_0^I v(\psi,s)\mathcal{F}(\psi,s,\zeta(s))ds\right|^2}{\beta(\mathcal{F}_{bm}(\varrho,\zeta,\psi))\psi}} \\
&= e^{-\frac{\sup_{s\in[0,I]}\left|\int_0^I v(\psi,s)\{\mathcal{F}(\psi,s,\varrho(s)) - \mathcal{F}(\psi,s,\zeta(s)\}ds\right|^2}{\beta(\mathcal{F}_{bm}(\varrho,\zeta,\psi))\psi}} \\
&\geq e^{-\frac{\sup_{s\in[0,I]}\left|\int_0^I (v(\psi,s))^2 ds\right| \int_0^I \left|\mathcal{F}(\psi,s,\varrho(s)) - \mathcal{F}(\psi,s,\zeta(s)\right|^2 ds}{\beta(\mathcal{F}_{bm}(\varrho,\zeta,\psi))\psi}} \\
&\geq e^{-\frac{\sup_{s\in[0,I]}\frac{1}{I}\int_0^I \left\{\sqrt{\beta(\mathcal{F}_{bm}(\varrho,\zeta,\psi))}\, |\varrho(s) - \zeta(s)|\right\}^2 ds}{\beta(\mathcal{F}_{bm}(\varrho,\zeta,\psi))\psi}} \\
&\geq e^{-\frac{\sup_{s\in[0,I]}\beta(\mathcal{F}_{bm}(\varrho,\zeta,\psi))|\varrho(s) - \zeta(s)|^2}{\beta(\mathcal{F}_{bm}(\varrho,\zeta,\psi))\psi}} \\
&= e^{-\frac{\sup_{s\in[0,I]}|\varrho(s) - \zeta(s)|^2}{\psi}},
\end{aligned}$$

which implies $\mathcal{F}_{bm}(\mathcal{T}\varrho, \mathcal{S}\zeta, \beta(\mathcal{F}_{bm}(\varrho, \zeta, \psi))\psi) \geq \mathcal{F}_{bm}(\varrho, \zeta, \psi)$.

Hence, $\mathcal{T}$ and $\mathcal{S}$ have a common FP, and so the integral equation (4) has a solution. □

Conclusion

In this paper, we have established common FP results, analogue to [8, Theorem 4] for Geraghty type contractions in G_r-complete FBMS. An example was given to explain our main result. Moreover, we have established an analogue of the FP theorems of V. Gupta et al. [12] in the setting of G_r-complete FBMS. Our results are of interest for the readers/researchers in some specialized area of computer science.

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Батул С., Ашраф М.С., Саґхір Д.-е.-С., Мехмуд Ф., Айді Х., Кумар С. *Про спільні нерухомі точки для стискаючих відображень типу Гераті та застосування до інтегральних рівнянь* // Карпатські матем. публ. — 2026. — Т.18, №1. — С. 264–277.

У цій статті встановлено кілька теорем про спільну нерухому точку для стискаючих відображень типу Гераті в G_r-повних нечітких b-метричних просторах. Крім того, побудовано приклад для підтвердження основного результату. Зрештою, отримано застосування результатів шляхом демонстрації існування розв'язку для інтегрального рівняння.

Ключові слова і фрази: нерухома точка, стиснення Гераті, нечіткий b-метричний простір, інтегральне рівняння.