



Nonlocal boundary value problem for the inhomogeneous equation with the differentiation operator $z \frac{\partial}{\partial z}$ and homogeneous boundary conditions

Il'kiv V.S., Strap N.I., Volianska I.I.

The paper investigates a nonlocal problem with homogeneous boundary conditions for a non-homogeneous partial differential equation involving the generalized differentiation operator $z \frac{\partial}{\partial z}$, acting on a function of the complex variable z with scalar coefficients. The uniqueness and existence theorems for the solution of the problem in the space $H_q^n(\mathcal{D})$ are proved. The Hadamard well-posedness of the problem is established. This distinguishes the problem from Hadamard ill-posed nonlocal problems with a vector complex variable z .

Key words and phrases: boundary value problem, inhomogeneous partial differential equation, nonlocal boundary condition, function of complex variable.

Lviv Polytechnic National University, 12 Bandera str., 79013, Lviv, Ukraine

E-mail: il'kivvv@i.ua (Il'kiv V.S.), n.strap@i.ua (Strap N.I.), i.volyanska@i.ua (Volianska I.I.)

Introduction

One of the central problems in the general theory of partial differential equations is the formulation of conditions ensuring the well-posedness of boundary value problems. This challenge arises from the impossibility of establishing correctness using only local boundary conditions. Moreover, such problems often serve as mathematical models for numerous real-world processes.

In general, nonlocal problems for partial differential equations are ill-posed, and their solvability within bounded domains is closely related to the small denominator problem. Using the metric approach, some studies [2–6, 11, 12] have established conditions for the existence and uniqueness of solutions to nonlocal problems in various functional spaces. The well-posedness of nonlocal boundary value problems for partial differential equations has also been examined by numerous authors (see, e.g., [8, 10]) under the assumption of additional restrictions imposed on the equations, boundary conditions and domains.

Nonlocal boundary value problems for partial differential equations with both constant and variable coefficients have also been studied in domains that are unbounded with respect to spatial variables [1, 9, 13].

The article investigates a semi-homogeneous nonlocal boundary value problem for a non-homogeneous partial differential equation involving the generalized differentiation operator $B = z \frac{\partial}{\partial z}$, considered in the case of a single complex spatial variable z . The criteria for the unique solvability of the problem in the space $H_q^n(\mathcal{D})$ have been established. In the case of

one spatial variable the corresponding denominators are not very small and are estimated from below by some constants. Previously, in the work [7], the authors investigated a semi-homogeneous problem in the refined Sobolev scale, namely, a nonlocal boundary value problem for a homogeneous partial differential equation with nonhomogeneous initial conditions, in which the absence of the small denominator problem was also established.

1 Introduction of spaces and basic notations

Denote by $\mathcal{S}$ a spatial domain in $\mathbb{C} \setminus \{0\}$. Define the cylinder $\mathcal{D} = [0, T] \times \mathcal{S}$, where $T > 0$ is its height.

Let $\mathbf{W}$ be the linear space of finite sums (basic functions) of the form $P(z) = \sum_k P_k z^k$, where $z \in \mathcal{S}$, P_k are complex coefficients, $k \in \mathbb{Z}$. Each basic function $P(z)$ can be represented as $P(z) = P_1(z) + P_2(1/z)$, where $P_1(z) = \sum_k P_k z^k$, $P_2(w) = \sum_k P_{-k} w^k$ are polynomials and $P_2(0) = 0$.

Let $\mathbf{W}'$ denote the dual space of $\mathbf{W}$. This is the space of generalized functions, i.e. linear continuous functionals $Q : \mathbf{W} \rightarrow \mathbb{C}$, represented by formal Laurent series $Q(z) = \sum_{k \in \mathbb{Z}} Q_k z^k$, which act on a basic function $P \in \mathbf{W}$ as follows $\langle Q, P \rangle = \sum_k Q_k \bar{P}_k$, where $\bar{P}_k$ denotes the complex conjugate of P_k .

Let us introduce two scales of spaces $\{\mathbf{H}_q(\mathcal{S})\}_{q \in \mathbb{R}}$ and $\{\mathbf{H}_q^n(\mathcal{D})\}_{q \in \mathbb{R}}$. Here $\mathbf{H}_q(\mathcal{S})$ is the Hilbert space of functions $\psi = \sum_{k \in \mathbb{Z}} \psi_k z^k$ with scalar product $(\psi_1, \psi_2)_{\mathbf{H}_q(\mathcal{S})} = \sum_{k \in \mathbb{Z}} \tilde{k}^{2q} \psi_{1k} \bar{\psi}_{2k}$ and $\tilde{k} = \sqrt{1 + k^2}$. The space $\mathbf{H}_q^n(\mathcal{D})$ with an integer $n \geq 0$ is the Banach space consisting of functions $u = u(t, z)$, whose partial derivatives with respect to t are given by $\frac{\partial^r u(t, z)}{\partial t^r} = \sum_{k \in \mathbb{Z}} u_k^{(r)}(t) z^k$ for $r = 0, 1, \dots, n$, and, for each $t \in [0, T]$, these derivatives belong to the corresponding spaces $\mathbf{H}_{q-r}(\mathcal{S})$ and depend continuously on t with respect to the topology of these spaces.

The norms in these spaces are defined as follows

$$\|\psi\|_{\mathbf{H}_q(\mathcal{S})}^2 = (\psi, \psi)_{\mathbf{H}_q(\mathcal{S})} = \sum_{k \in \mathbb{Z}} \tilde{k}^{2q} |\psi_k|^2, \quad \|u\|_{\mathbf{H}_q^n(\mathcal{D})}^2 = \sum_{r=0}^{n-1} \max_{[0, T]} \left\| \frac{\partial^r u(t, \cdot)}{\partial t^r} \right\|_{\mathbf{H}_{q-r}(\mathcal{S})}^2.$$

2 The problem formulation and construction of solution. Correctness conditions of the problem

In the scales of spaces $\{\mathbf{H}_q(\mathcal{S})\}_{q \in \mathbb{R}}$ and $\{\mathbf{H}_q^n(\mathcal{D})\}_{q \in \mathbb{R}}$ we consider the nonhomogeneous problem with two-point nonlocal conditions for the differential operator equation with constant coefficients

$$Lu = L\left(\frac{\partial}{\partial t}, B\right) = \sum_{s_0 + s_1 \leq n} a_{s_0, s_1} B^{s_1} \frac{\partial^{s_0} u}{\partial t^{s_0}} = f(t, z), \quad (1)$$

$$M_m u = \mu \frac{\partial^m u}{\partial t^m} \Big|_{t=0} - \frac{\partial^m u}{\partial t^m} \Big|_{t=T} = 0, \quad m = 0, 1, \dots, n-1, \quad (2)$$

where $a_{s_0, s_1} \in \mathbb{C}$, $\mu \in \mathbb{C} \setminus \{0\}$, $a_{n, 0} = 1$, $u = u(t, z)$ is the required function, $f = f(t, z)$ is the given function.

For any $q \in \mathbb{R}$ the operators B , L , M_m define mappings between the following functional spaces

$$(B, L, M_m) : \mathbf{H}_q^n(\mathcal{D}) \rightarrow (\mathbf{H}_{q-1}^n(\mathcal{D}), \mathbf{H}_{q-n}^0(\mathcal{D}), \mathbf{H}_{q-m}(\mathcal{S})), \quad m = 0, 1, \dots, n-1.$$

In particular, for $u = \sum_{k \in \mathbb{Z}} u_k(t) z^k \in \mathbf{H}_q^n(\mathcal{D})$, we have $Bu = \sum_{k \in \mathbb{Z}} k u_k(t) z^k \in \mathbf{H}_{q-1}^n(\mathcal{D})$, $Lu = \sum_{k \in \mathbb{Z}} L(d/dt, k) u_k(t) z^k \in \mathbf{H}_{q-n}^0(\mathcal{D})$, $M_m u = \sum_{k \in \mathbb{Z}} (\mu u_k^{(m)}(0) - u_k^{(m)}(T)) z^k \in \mathbf{H}_{q-m}(\mathcal{S})$.

Definition. By a solution of the problem (1), (2) we mean a function $u = u(t, z)$, that belongs to the space $\mathbf{H}_q^n(\mathcal{D})$, where q is a fixed real number, and satisfies equation (1) and conditions (2), i.e. $\langle Lu(t, \cdot), P \rangle = \langle f(t, \cdot), P \rangle$, $\langle M_m u, P \rangle = 0$ for every $P \in \mathbf{W}$, where $t \in [0, T]$, $m = 0, 1, \dots, n-1$.

Hence, we obtain the necessary condition $f \in \mathbf{H}_{q-n}^0(\mathcal{D})$ for the solvability of the problem.

According to the definition, the solution of the problem (1), (2) can be represented by the series

$$u(t, z) = \sum_{k \in \mathbb{Z}} u_k(t) z^k, \quad (3)$$

where the coefficients $u_k(t)$ are unknown functions. The function $u_k = u_k(t)$ for any $k \in \mathbb{Z}$ is the solution of the corresponding problem for the ordinary differential equation

$$L\left(\frac{d}{dt}, k\right) u_k(t) = \sum_{j=0}^n b_j(k) u_k^{(n-j)} = f_k(t), \quad (4)$$

$$\mu u_k^{(m)}|_{t=0} - u_k^{(m)}|_{t=T} = 0, \quad m = 0, 1, \dots, n-1, \quad (5)$$

where $b_j(k) = \sum_{s_1=0}^j a_{n-j, s_1} k^{s_1}$ are no higher than j degree polynomials, $f_k(t)$ are Fourier coefficients of the function f , i.e. $f = \sum_{k \in \mathbb{Z}} f_k(t) z^k$

The solution of the problem (4), (5) can be represented by the formula

$$u_k(t) = \int_0^T G_k(t, \tau) f_k(\tau) d\tau, \quad (6)$$

where $G_k(t, \tau)$ is the corresponding Green's function, provided that it exists.

Let us normalize the coefficients $b_1(k), \dots, b_n(k)$ of equation (4) and represent them by the product $b_j(k) = \tilde{k}^j \tilde{b}_j(k)$. The functions $\tilde{b}_j(k)$, like the coefficients $b_j(k)$, depend linearly on the parameters $a_{n-j,0}, a_{n-j,1}, \dots, a_{n-j,j}$.

If the coefficients $a_{s_0, s_1} \in \mathbb{C}$ of the equation (1) are considered in a circle of a certain radius A with the center at the origin of the complex plane, then the functions $\tilde{b}_j(k)$ are uniformly bounded by k and parameters a_{s_0, s_1} . Since the inequality

$$|\tilde{b}_j(k)| \leq \sum_{s_1=0}^j |a_{n-j, s_1}| \frac{|k|^{s_1}}{\tilde{k}^j} \leq \max_{s_1=0,1,\dots,j} |a_{n-j, s_1}| \sum_{s_1=0}^j \frac{|k|^{s_1}}{\tilde{k}^j}$$

holds, we obtain the following estimates

$$|\tilde{b}_j(0)| = |a_{n-j,0}| \leq A, \quad |\tilde{b}_j(\pm 1)| \leq (j+1)2^{-j/2} A \leq \frac{3}{2} A,$$

$$|\tilde{b}_j(k)| \leq \frac{A}{\tilde{k}^j} \frac{|k|^{j+1}}{|k|-1} < \frac{A|k|}{|k|-1}, \quad k \notin \{-1, 0, 1\},$$

i.e. $|\tilde{b}_j(k)| < 2A$ for all $k \in \mathbb{Z}$. The characteristic polynomial of the equation (4) satisfies the relation $L(\lambda, k) = \tilde{k}^n P_k(\lambda)$, where

$$P_k(\lambda) = \lambda^n + \sum_{j=1}^n \tilde{b}_j(k) \lambda^{n-j} = \prod_{j=1}^n (\lambda - \lambda_j(k)).$$

For all roots (counting multiplicities) $\lambda_1(k), \dots, \lambda_n(k)$ of the polynomial, the inequalities

$$|\lambda_j(k)| \leq 1 + \max(|\tilde{b}_1|, \dots, |\tilde{b}_n|) \leq 1 + 2A \quad (7)$$

hold (see [14]).

Let us denote by $\mathbf{K}$ the set of those integers k for which the polynomial $P_k(\lambda)$ has a multiple root. From formulas (3) and (6) we derive a formal representation of the solution of the problem (1), (2), namely

$$u(t, z) = \sum_{k \in \mathbb{Z}} z^k \int_0^T G_k(t, \tau) f_k(\tau) d\tau. \quad (8)$$

The functions $G_k(t, \tau)$, defined on the square $K_T = [0, T]^2$, for $k \in \mathbb{Z} \setminus \mathbf{K}$, are determined by the formulas

$$G_k(t, \tau) = g_k(t, \tau) + \frac{1}{2\tilde{k}^{n-1}} \sum_{j=1}^n \frac{e^{\tilde{k}\lambda_j(k)(t-\tau)} (\mu + e^{\tilde{k}\lambda_j(k)T})}{P'_k(\lambda_j(k)) (\mu - e^{\tilde{k}\lambda_j(k)T})}, \quad (9)$$

where $P'_k(\lambda_j(k)) = \prod_{q=1, q \neq j}^n (\lambda_j(k) - \lambda_q(k)) \neq 0$ and

$$g_k(t, \tau) = \frac{\operatorname{sgn}(t - \tau)}{2\tilde{k}^{n-1}} \sum_{j=1}^n \frac{e^{\tilde{k}\lambda_j(k)(t-\tau)}}{P'_k(\lambda_j(k))}.$$

The functions $G_k(t, \tau)$ exist if and only if the inequality $\mu - e^{\tilde{k}\lambda_l(k)T} \neq 0$ holds for all $k \in \mathbb{Z}$ and $l = 1, \dots, n$. From this condition it follows that $\ln \mu \neq \tilde{k}\lambda_l(k)T + i2\pi m$ or $\lambda_l(k) \neq \frac{\ln \mu - i2\pi m}{\tilde{k}T}$ for arbitrary $m \in \mathbb{Z}$ and $l = 1, \dots, n$.

In the opposite case, when $\mu = e^{\tilde{k}\lambda_l(k)T}$ for some l , there exists an integer $m \in \mathbb{Z}$ such that the root $\lambda_l(k)$ is given by the formula $\lambda_l(k) = \frac{\ln \mu - i2\pi m}{\tilde{k}T}$. Therefore, the following equality $\frac{(\ln \mu - i2\pi m)^n}{T^n \tilde{k}^n} + \sum_{j=1}^n \tilde{b}_j(k) \frac{(\ln \mu - i2\pi m)^{n-j}}{T^{n-j} \tilde{k}^{n-j}} = 0$ or equivalent equality

$$(\ln \mu - i2\pi m)^n + \sum_{j=1}^n b_j(k) T^j (\ln \mu - i2\pi m)^{n-j} = 0 \quad (10)$$

hold.

Theorem 1. For the uniqueness of the solution of the problem (1), (2) in the space $\mathbf{H}_q^n(\mathcal{D})$ it is necessary and sufficient that the equation (10) has no solutions in the integers m and k .

Proof. Necessity. Assume that the homogeneous problem (1), (2) admits at most one solution in the space $\mathbf{H}_q^n(\mathcal{D})$. If a solution of the problem (1), (2) exists, then all functions $u_k(t)$ are uniquely determined. That is, the homogeneous problem (4), (5) has a unique (trivial) solution in the space $\mathbb{C}^n[0, T]$ for all $k \in \mathbb{Z}$. The corresponding system of linear algebraic equations for $k \in \mathbb{Z} \setminus \mathbf{K}$ is nondegenerate and its determinant $\Delta(k) \cdot \prod_{l=1}^n (\mu - e^{\tilde{k}\lambda_l(k)T}) \neq 0$, where $\Delta(k) = \prod_{1 \leq r < q \leq n} (\lambda_q(k) - \lambda_r(k))$ is the Vandermonde determinant constructed from $\lambda_1(k), \dots, \lambda_n(k)$, i.e. $\mu \neq e^{\tilde{k}\lambda_l(k)T}$ for $l = 1, \dots, n$. Similar inequalities were obtained for $k \in \mathbf{K}$.

Sufficiency. Let us proceed by contradiction. Suppose that equation (10) admits a solution for some k^*, m^* . Then we may assume that $\lambda_1(k^*) = \frac{\ln \mu - i2\pi m^*}{\tilde{k}^* T}$, and thus the homogeneous problem (4), (5) admits a solution $e^{\tilde{k}^* \lambda_1(k^*) t} = e^{(\ln \mu - i2\pi m^*) t / T}$. It follows that the homogeneous problem (1), (2) has many nonzero solutions in the space $\mathbf{H}_q^n(\mathcal{D})$. $\square$

It will be shown that the set $\mathbf{K}$ is finite for almost all coefficients a_{s_0, s_1} belonging to the circle of radius A . Therefore, we shall evaluate the functions G_k for $k \in \mathbb{Z} \setminus \mathbf{K}$.

Let us rewrite formula (9) in the following form

$$\begin{aligned} G_k(t, \tau) &= \frac{1}{2\tilde{k}^{n-1}} \sum_{j=1}^n \frac{e^{\tilde{k}\lambda_j(k)(t-\tau)}}{P'_k(\lambda_j(k))} \left(\operatorname{sgn}(t-\tau) + \frac{\mu + e^{\tilde{k}\lambda_j(k)T}}{\mu - e^{\tilde{k}\lambda_j(k)T}} \right) \\ &= \frac{1}{2\tilde{k}^{n-1}} \sum_{j=1}^n \frac{(\operatorname{sgn}(t-\tau)(\mu - e^{\tilde{k}\lambda_j(k)T}) + \mu + e^{\tilde{k}\lambda_j(k)T}) e^{\tilde{k}\lambda_j(k)(t-\tau)}}{P'_k(\lambda_j(k))(\mu - e^{\tilde{k}\lambda_j(k)T})} \\ &= \frac{1}{2\tilde{k}^{n-1}} \sum_{j=1}^n \frac{\Delta_j(k) (\operatorname{sgn}(t-\tau)(\mu - e^{\tilde{k}\lambda_j(k)T}) + \mu + e^{\tilde{k}\lambda_j(k)T}) e^{\tilde{k}\lambda_j(k)(t-\tau)}}{\Delta(k)(\mu - e^{\tilde{k}\lambda_j(k)T})} \\ &= \begin{cases} \frac{\mu}{\tilde{k}^{n-1}} \sum_{j=1}^n \frac{\Delta_j(k) e^{\tilde{k}\lambda_j(k)(t-\tau)}}{\Delta(k)(\mu - e^{\tilde{k}\lambda_j(k)T})}, & t > \tau, \\ \frac{1}{\tilde{k}^{n-1}} \sum_{j=1}^n \frac{\Delta_j(k) e^{\tilde{k}\lambda_j(k)(t+T-\tau)}}{\Delta(k)(\mu - e^{\tilde{k}\lambda_j(k)T})}, & t < \tau. \end{cases} \end{aligned}$$

Since the determinants $\Delta_j(k) = \prod_{\substack{1 \leq r < q \leq n \\ q, r \neq j}} (\lambda_q(k) - \lambda_r(k))$ are of the order $n-1$, with bounded elements that are powers of the values $\lambda_1, \dots, \lambda_n$, the following estimate

$$\Delta_j(k) \leq (n-1)!(1+2A)^{(n-1)(n-2)/2} \quad (11)$$

holds.

Let us compute the derivative of order r with respect to t of the Green's function, where $r = 0, 1, \dots, n-1$. We obtain

$$\frac{\partial^r G_k(t, \tau)}{\partial t^r} = \begin{cases} \frac{\mu}{\tilde{k}^{n-1-r}} \sum_{j=1}^n \frac{\Delta_j(k) \lambda_j^r(k) e^{\tilde{k}\lambda_j(k)(t-\tau)}}{\Delta(k)(\mu - e^{\tilde{k}\lambda_j(k)T})}, & t > \tau, \\ \frac{1}{\tilde{k}^{n-1-r}} \sum_{j=1}^n \frac{\Delta_j(k) \lambda_j^r(k) e^{\tilde{k}\lambda_j(k)(t+T-\tau)}}{\Delta(k)(\mu - e^{\tilde{k}\lambda_j(k)T})}, & t < \tau. \end{cases}$$

Now we consider the denominator $\Delta^2(k)$, which represents the discriminant $D(k)$ of the polynomial $P_k(\lambda)$ and for which the following two representations hold:

$$\Delta^2(k) = D(k) = \prod_{1 \leq r < q \leq n} (\lambda_q(k) - \lambda_r(k))^2 = \tilde{k}^{-n(n-1)} \prod_{1 \leq r < q \leq n} (\tilde{k}\lambda_q(k) - \tilde{k}\lambda_r(k))^2,$$

$$D(k) = \pm \begin{vmatrix} 1 & \tilde{b}_1(k) & \dots & \tilde{b}_{n-1}(k) & \tilde{b}_n(k) & 0 & \dots & 0 \\ 0 & 1 & \dots & \tilde{b}_{n-2}(k) & \tilde{b}_{n-1}(k) & \tilde{b}_n(k) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \tilde{b}_1(k) & \tilde{b}_2(k) & \tilde{b}_3(k) & \dots & \tilde{b}_n(k) \\ n & (n-1)\tilde{b}_1(k) & \dots & \tilde{b}_{n-1}(k) & 0 & 0 & \dots & 0 \\ 0 & n & \dots & 2\tilde{b}_{n-2}(k) & \tilde{b}_{n-1}(k) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & n & (n-1)\tilde{b}_1(k) & (n-2)\tilde{b}_2(k) & \dots & \tilde{b}_{n-1}(k) \end{vmatrix}.$$

The sign preceding the determinant is determined by the formula $(-1)^{(n-1)n/2}$.

Let us express the discriminant $D(k)$ as a polynomial

$$\begin{aligned} D(k) &= D_0 \left(\frac{k}{\bar{k}} \right)^{n(n-1)} + \frac{D_1}{\bar{k}} \left(\frac{k}{\bar{k}} \right)^{n(n-1)-1} + \frac{D_2}{\bar{k}^2} \left(\frac{k}{\bar{k}} \right)^{n(n-1)-2} + \dots + \frac{D_{n(n-1)}}{\bar{k}^{n(n-1)}} \\ &= \left(\frac{k}{\bar{k}} \right)^{n(n-1)} \left(D_0 + \frac{D_1}{k} + \frac{D_2}{k^2} + \dots + \frac{D_{n(n-1)}}{k^{n(n-1)}} \right), \end{aligned}$$

where $D_0, D_1, D_2, \dots, D_{n(n-1)}$ are complex coefficients that depend polynomially on the quantities a_{s_0, s_1} . In particular, D_0 is the discriminant of the polynomial $\lambda^n + \sum_{j=1}^n a_{n-j, j} \lambda^{n-j}$, which is constructed from the leading-order terms of equation (1):

$$D_0 = (-1)^{\frac{(n-1)n}{2}} \begin{vmatrix} 1 & a_{n-1,1} & \dots & a_{1,n-1} & a_{0,n} & 0 & \dots & 0 \\ 0 & 1 & \dots & a_{2,n-2} & a_{1,n-1} & a_{0,n} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{n-1,1} & a_{n-2,2} & a_{n-3,3} & \dots & a_{0,n} \\ n & (n-1)a_{n-1,1} & \dots & a_{1,n-1} & 0 & 0 & \dots & 0 \\ 0 & n & \dots & 2a_{2,n-2} & a_{1,n-1} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & n & (n-1)a_{n-1,1} & (n-2)a_{n-2,2} & \dots & a_{1,n-1} \end{vmatrix},$$

$D_{n(n-1)}$ is the discriminant of a polynomial $\lambda^n + \sum_{j=1}^n a_{n-j,0} \lambda^{n-j}$:

$$D_{n(n-1)} = (-1)^{\frac{(n-1)n}{2}} \begin{vmatrix} 1 & a_{n-1,0} & \dots & a_{1,0} & a_{0,0} & 0 & \dots & 0 \\ 0 & 1 & \dots & a_{2,0} & a_{1,0} & a_{0,0} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{n-1,0} & a_{n-2,0} & a_{n-3,0} & \dots & a_{0,0} \\ n & (n-1)a_{n-1,0} & \dots & a_{1,0} & 0 & 0 & \dots & 0 \\ 0 & n & \dots & 2a_{2,0} & a_{1,0} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & n & (n-1)a_{n-1,0} & (n-2)a_{n-2,0} & \dots & a_{1,0} \end{vmatrix}.$$

Assuming $D_0 \neq 0$, we factorize the discriminant $D(k)$ for $k \neq 0$ as follows

$$\begin{aligned} D(k) &= \frac{D_0}{2} \left(\frac{k}{\bar{k}} \right)^{n(n-1)} \left(2 + \frac{2D_1}{D_0 k} + \frac{2D_2}{D_0 k^2} + \dots + \frac{2D_{n(n-1)}}{D_0 k^{n(n-1)}} \right) \\ &= \frac{D_0}{2} \left(\frac{k}{\bar{k}} \right)^{n(n-1)} \left(2 + \frac{2}{k D_0} \left(D_1 + \frac{D_2}{k} + \dots + \frac{D_{n(n-1)}}{k^{n(n-1)-1}} \right) \right). \end{aligned}$$

The inequality $|D(k)| \geq \frac{|D_0|}{2} \left(\frac{|k|}{\tilde{D}_0}\right)^{n(n-1)}$ for $|k| \geq \frac{\tilde{D}_0}{|D_0|}$, where $\tilde{D}_0 = 2(|D_1| + \dots + |D_{n(n-1)}|)$, follows directly from the preceding formula.

Let us apply the inequality $\frac{|k|}{\tilde{D}_0} \geq \frac{1}{\sqrt{2}}$, $k \in \mathbb{Z}^p \setminus \{0\}$, to estimate the modulus of $D(k)$ from below, namely

$$|D(k)| \geq \frac{|D_0|}{2} \cdot \left(\frac{1}{\sqrt{2}}\right)^{n(n-1)} = (\sqrt{2})^{-n(n-1)-2} |D_0|, \quad |k| \geq \frac{\tilde{D}_0}{|D_0|}. \quad (12)$$

This estimate is sharp with respect to k for $|k| \geq \frac{\tilde{D}_0}{|D_0|}$, since the upper bound, which follows from the image of the discriminant $D(k)$, takes the form $|D(k)| \leq \frac{3}{2}|D_0|$.

Estimate (12) also implies the finiteness of the set $\mathbf{K}$. Specifically, only those $k \in \mathbb{Z}$ satisfying the condition $|k| < \frac{\tilde{D}_0}{|D_0|}$ can belong to this set.

Now, we proceed to estimate the fractions $\frac{e^{\tilde{k}\lambda_l(k)t}}{\mu - e^{\tilde{k}\lambda_l(k)T}}$ from above. To do this, we apply the estimate $|e^{\tilde{k}\lambda_l(k)t}| = e^{\tilde{k}\operatorname{Re} \lambda_l(k)t} \leq \max \{1, e^{\tilde{k}\operatorname{Re} \lambda_l(k)T}\}$ and prove the asymptotics $\tilde{k}|\operatorname{Re} \lambda_j| \rightarrow \infty$ as $|k| \rightarrow \infty$.

Using the identity $2\operatorname{Re} \lambda_j(k) = \lambda_j(k) + \bar{\lambda}_j(k) = \lambda_j(k) - (-\bar{\lambda}_j(k))$, we note that the values $-\bar{\lambda}_1(k), \dots, -\bar{\lambda}_n(k)$ are the roots of $P_{1k}(\lambda) = \prod_{j=1}^n (\lambda + \bar{\lambda}_j(k)) = \lambda^n + \sum_{j=1}^n (-1)^{-j} \tilde{b}_j(k) \lambda^{n-j}$. Therefore, we can conclude that the quantities $2\operatorname{Re} \lambda_j(k)$ appear as pairwise differences in $R(k) = \prod_{j=1}^n \prod_{l=1}^n (\lambda_j(k) - (-\bar{\lambda}_l(k)))$, which is the resultant of the polynomials P_k and P_{1k} . It can be expressed as the following determinant

$$R(k) = \begin{vmatrix} 1 & \tilde{b}_1(k) & \dots & \tilde{b}_{n-1}(k) & \tilde{b}_n(k) & 0 & \dots & 0 \\ 0 & 1 & \dots & \tilde{b}_{n-2}(k) & \tilde{b}_{n-1}(k) & \tilde{b}_n(k) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \tilde{b}_1(k) & \tilde{b}_2(k) & \tilde{b}_3(k) & \dots & \tilde{b}_n(k) \\ 1 & -\tilde{b}_1(k) & \dots & (-1)^{n-1} \tilde{b}_{n-1}(k) & (-1)^n \tilde{b}_n(k) & 0 & \dots & 0 \\ 0 & 1 & \dots & (-1)^{n-2} \tilde{b}_{n-2}(k) & (-1)^{n-1} \tilde{b}_{n-1}(k) & (-1)^n \tilde{b}_n(k) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -\tilde{b}_1(k) & -\tilde{b}_2(k) & -\tilde{b}_3(k) & \dots & (-1)^n \tilde{b}_n(k) \end{vmatrix}.$$

Let us estimate the modulus of this resultant from above. As the result, we obtain $|R(k)| \leq 2^{n^2} (1 + 2A)^{n^2-1} |\operatorname{Re} \lambda_j|$ for an arbitrary $j = 1, \dots, n$.

We represent the resultant in the form $R(k) = \left(\frac{k}{\tilde{k}}\right)^{n^2} \left(R_0 + \frac{R_1}{k} + \frac{R_2}{k^2} + \dots + \frac{R_{n^2}}{k^{n^2}}\right)$, $k \neq 0$, where R_0 is equal to the determinant

$$R_0 = \begin{vmatrix} 1 & a_{n-1,1} & \dots & a_{1,n-1} & a_{0,n} & 0 & \dots & 0 \\ 0 & 1 & \dots & a_{2,n-2} & a_{1,n-1} & a_{0,n} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{n-1,1} & a_{n-2,2} & a_{n-3,3} & \dots & 0 \\ 1 & -\bar{a}_{n-1,1} & \dots & (-1)^{n-1} \bar{a}_{1,n-1} & (-1)^n \bar{a}_{0,n} & 0 & \dots & 0 \\ 0 & 1 & \dots & (-1)^{n-2} \bar{a}_{2,n-2} & (-1)^{n-1} \bar{a}_{1,n-1} & (-1)^n \bar{a}_{0,n} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -\bar{a}_{n-1,1} & -\bar{a}_{n-2,2} & -\bar{a}_{n-3,3} & \dots & (-1)^n \bar{a}_{0,n} \end{vmatrix},$$

and in the case $R_0 \neq 0$, the resultant can be expressed as

$$R(k) = \frac{R_0}{2} \left(\frac{k}{\tilde{k}} \right)^{n^2} \left(2 + \frac{2}{kR_0} \left(R_1 + \frac{R_2}{k} + \dots + \frac{R_{n^2}}{k^{n^2-1}} \right) \right).$$

If $k \in \mathbb{Z}$ and $|k| \geq \frac{\tilde{R}_0}{|R_0|}$, where $\tilde{R}_0 = 2(|R_1| + |R_2| + \dots + |R_{n^2}|)$, then the following inequality $|R(k)| \geq \frac{|R_0|}{2} \left(\frac{|k|}{\tilde{k}} \right)^{n^2} \geq (\sqrt{2})^{-n^2-2} |R_0|$ holds.

The formula $\tilde{k} |\operatorname{Re} \lambda_j(k)| \rightarrow \infty$ follows from the above inequality, since $\tilde{k} \rightarrow \infty$ as $|k| \rightarrow \infty$. Indeed, $\tilde{k} |\operatorname{Re} \lambda_j(k)| \geq \tilde{k} \cdot 2^{-n^2} (1+2A)^{1-n^2} |R(k)| \geq \tilde{k} \cdot (\sqrt{2})^{-3n^2-2} (1+2A)^{1-n^2} |R_0| \rightarrow \infty$.

We take into account the sign of $\operatorname{Re} \lambda_j(k)$. If $\operatorname{Re} \lambda_j(k) > 0$, then the following uniform estimate holds on the interval $[0, T]$

$$\left| \frac{e^{\tilde{k}\lambda_l(k)t}}{\mu - e^{\tilde{k}\lambda_l(k)T}} \right| \leq \frac{e^{\tilde{k}\operatorname{Re} \lambda_l(k)T}}{|\mu - e^{\tilde{k}\lambda_l(k)T}|} = \frac{1}{|\mu e^{-\tilde{k}\lambda_l(k)T} - 1|} \leq 2$$

provided that $\tilde{k} \geq \frac{M_1}{|R_0|}$ and $|k| \geq \frac{\tilde{R}_0}{|R_0|}$, where $M_1 = \frac{\ln(2|\mu|)}{T} (\sqrt{2})^{3n^2+2} (1+2A)^{n^2-1}$.

Similarly, if $\operatorname{Re} \lambda_j(k) < 0$, then the following uniform estimate holds

$$\left| \frac{e^{\tilde{k}\lambda_l(k)t}}{\mu - e^{\tilde{k}\lambda_l(k)T}} \right| \leq \frac{1}{|\mu - e^{\tilde{k}\lambda_l(k)T}|} \leq \frac{2}{|\mu|}$$

provided that $\tilde{k} \geq \frac{M_2}{|R_0|}$ and $|k| \geq \frac{\tilde{R}_0}{|R_0|}$, where $M_2 = \frac{\ln(2/|\mu|)}{T} (\sqrt{2})^{3n^2+2} (1+2A)^{n^2-1}$.

Therefore, for $\tilde{k} \geq \max \frac{(M_1, M_2)}{|R_0|} = \frac{(\sqrt{2})^{3n^2+2} (1+2A)^{n^2-1}}{T|R_0|} \ln 2|\mu|$ and $|k| \geq \frac{\tilde{R}_0}{|R_0|}$, the following estimate holds

$$\left| \frac{e^{\tilde{k}\lambda_l(k)t}}{\mu - e^{\tilde{k}\lambda_l(k)T}} \right| \leq 2 \max \left(1, \frac{1}{|\mu|} \right). \quad (13)$$

In the case of a vector-valued variable z the left side of inequality (13) cannot be bounded from above by a constant. This phenomenon arises due to the occurrence of small denominators in the expression.

From inequalities (7), (11), (12), (13) it follows that for almost all vectors composed of the coefficients a_{s_0, s_1} of the equation and the parameter μ the following estimate holds

$$\left| \frac{\partial^r G_k(t, \tau)}{\partial t^r} \right| \leq \hat{C} \tilde{k}^{\sigma+r}, \quad \sigma \geq 1-n, \quad (14)$$

where $\hat{C} = \begin{cases} \sqrt{2}^{n(n-1)+4} (n-1)! (1+2A)^{(n-1)(n-2)/2+r} |\mu| \max(1, 1/|\mu|), & t > \tau, \\ \sqrt{2}^{n(n-1)+4} (n-1)! (1+2A)^{(n-1)(n-2)/2+r} \max(1, 1/|\mu|), & t < \tau. \end{cases}$

From the formula (6) and the inequality (14) we obtain the estimate

$$|u_k^{(r)}(t)| \leq \hat{C} \tilde{k}^{\sigma+r} \int_0^T |f_k(t)| dt, \quad k \in \mathbb{Z} \setminus \mathbf{K}_0, \quad r = 0, 1, \dots, n,$$

where $\mathbf{K}_0$ consists of all integers k , for which at least one of the following inequalities holds:

$\tilde{k} \leq \max \frac{(M_1, M_2)}{|R_0|}$ or $|k| \leq \max \left(\frac{\tilde{D}_0}{|D_0|}, \frac{\tilde{R}_0}{|R_0|} \right)$. The set $\mathbf{K}_0$ is finite, and it contains $\mathbf{K}$ as a subset.

By squaring both sides of the above inequality, we obtain the following estimate

$$|u_k^{(r)}(t)|^2 \leq \tilde{C}_1 \tilde{k}^{2\sigma+2r} \int_0^T |f_k(t)|^2 dt, \quad (15)$$

where the positive constant $\tilde{C}_1$ depends on the parameter μ and the quantities A and n .

Now we can formulate the next theorem.

Theorem 2. *Let the condition $D_0 R_0 \neq 0$ holds and for any $k \in \mathbf{K}_0$ the equation (10) has no integer solutions for m . Then, for an arbitrary function $f \in H_{q+\sigma}^0(\mathcal{D})$, where $\sigma \geq 1 - n$, the problem (1), (2) admits a unique solution belonging to the space $\mathbf{H}_q^n(\mathcal{D})$.*

Proof. Let us write the estimate for the n th derivative of the function $u_k(t)$ for $k \in \mathbb{Z} \setminus K_0$, i.e.

$$|u_k^{(n)}(t)|^2 \leq |f_k(t)|^2 + \sum_{j=1}^n |\tilde{b}_j(k)|^2 |u_k^{(n-j)}(t)|^2 \leq |f_k(t)|^2 + 4A^2 \tilde{C} \tilde{k}^{2\sigma+2n} \int_0^T |f_k(t)|^2 dt. \quad (16)$$

Then, by formulas (8), (15) and (16), we obtain the following inequality

$$\|u\|_{H_q^n(\mathcal{D})}^2 \leq C \|f\|_{H_{q+\sigma}^0(\mathcal{D})}^2,$$

where $C > 0$ is a constant depending only on the parameters of the problem. This completes the proof of the theorem. $\square$

The set of vectors composed of the coefficients a_{s_0, s_1} of equation (1), for which the conditions of Theorem 2 are not satisfied, has measure zero. This exceptional set can be represented as a finite union of portions of algebraic surfaces defined by the equations (10), $D_0 = 0$ and $R_0 = 0$.

Conclusions

In the paper, we considered a nonlocal two-point boundary value problem for a nonhomogeneous equation with partial derivatives with constant coefficients. The differential equation involved the generalized differentiation operator $B = z \frac{\partial}{\partial z}$ instead of the standard operator $\frac{\partial}{\partial z}$.

A scale of functional spaces $\{\mathbf{H}_q^n(\mathcal{D})\}_{q \in \mathbb{R}}$ is introduced, and sufficient conditions for the existence and necessary and sufficient conditions for the uniqueness of the solution of the problem in the space $\mathbf{H}_q^n(\mathcal{D})$ are established.

The well-posedness of the nonlocal boundary value problem is demonstrated for a broad class of linear nonhomogeneous partial differential equations in one complex spatial variable.

References

- [1] Fardigola L.V. *On a two-point nonlocal boundary value problem in a layer for an equation with variable coefficients.* Siberian Math. J. 1997, **38** (2), 367–379.
- [2] Il'kiv V.S., Ptashnyk B.Yo. *Representation and Investigation of Solutions of a Nonlocal Boundary-Value Problem for a System of Partial Differential Equations.* Ukrain. Mat. Zh. 1996, **48** (2), 184–194. (in Ukrainian)
- [3] Il'kiv V.S., Strap N.I. *Nonlocal boundary value problem for partial differential equation in a multidimensional complex domain.* Nauk. Visn. Uzhgorod. Univ., Ser. Mat. Inform. 2013, **24** (1), 60–72. (in Ukrainian)
- [4] Il'kiv V.S., Strap N.I. *Nonlocal boundary-value problem for a differential-operator equation with weak nonlinearity in the spaces of Dirichlet-Taylor series with fixed spectrum.* J. Math. Sci. 2018, **231** (4), 572–585. doi:10.1007/s10958-018-3835-8 (translation of Mat. Metody Fiz.-Mekh. Polya 2016, **59** (2), 77–85. (in Ukrainian))

- [5] Il'kiv V.S., Strap N.I., Volyanska I.I. *Solvability Conditions for the Nonlocal Boundary-Value Problem for a Differential-Operator Equation with Weak Nonlinearity in the Refined Sobolev Scale of Spaces of Functions of Many Real Variables*. Ukrainian Math. J. 2020, **72** (4), 515–535. doi:10.1007/s11253-020-01798-7 (translation of Ukrain. Mat. Zh. 2020, **72** (4), 452–466. (in Ukrainian))
- [6] Il'kiv V.S., Strap N.I., Volyanska I.I. *Conditions of solvability of the nonlocal boundary-value problem for a differential-operator equation with weak nonlinearity*. J. Math. Sci. 2021, **256** (6), 753–769. doi:10.1007/s10958-021-05458-4 (translation of Ukr. Mat. Visnyk 2021, **18** (1), 37–59. (in Ukrainian))
- [7] Il'kiv V.S., Strap N.I., Volyanska I.I. *Nonlocal Boundary-Value Problem for an Equation with Differentiation Operator $z \frac{\partial}{\partial z}$ in a Refined Sobolev Scale*. J. Math. Sci. 2023, **273** (6), 885–900. doi:10.1007/s10958-023-06552-5 (translation of Mat. Metody Fiz.-Mekh. Polya 2020, **63** (4), 5–16. (in Ukrainian))
- [8] Kalenyuk P.I., Kohut I.V., Nytrebych Z.M. *Problem with nonlocal two-point condition in time variable for homogeneous partial differential equation of infinite order in spatial variables*. Mat. Metody Fiz.-Mekh. Polya 2008, **51** (4), 17–26. (in Ukrainian)
- [9] Kalenyuk P.I., Kohut I.V., Nytrebych Z.M. *Problem with nonlocal two-point condition in time for a homogeneous partial differential equation of infinite order with respect to space variables*. J. Math. Sci. 2010, **167** (1), 1–15. doi:10.1007/s10958-010-9898-9 (translation of Mat. Metody Fiz.-Mekh. Polya 2008, **51** (4), 17–26. (in Ukrainian))
- [10] Matiichuk M.I. *Parabolic and Elliptic Boundary-Value Problems with Singularities*. Prut, Chernivtsi, 2003. (in Ukrainian)
- [11] Ptashnyk B.Yo., Il'kiv V.S., Kmit I.Ya., Polishchuk V.M. *Nonlocal boundary value problems for partial differential equations*. Naukova Dumka, Kyiv, 2002. (in Ukrainian)
- [12] Savka I.Ya. *Nonlocal problem with dependent coefficients in conditions for the second-order equation in time variable*. Carpathian Math. Publ. 2010, **2** (2), 101–110.
- [13] Volyanska I.I., Il'kiv V.S., Symotyuk M.M. *Nonlocal boundary-value problem for a second-order partial differential equation in an unbounded strip*. Ukrainian Math. J. 2019, **70** (10), 1585–1593. doi:10.1007/s11253-019-01591-1 (translation of Ukrain. Mat. Zh. 2018, **70** (10), 1374–1381. (in Ukrainian))
- [14] van der Waerden B.L. *Algebra*. Ungar, New York, 1970.

Received 10.09.2023

Revised 22.05.2026

Ільків В.С., Страп Н.І., Волянська І.І. *Нелокальна задача для неоднорідного рівняння з оператором диференціювання $z \frac{\partial}{\partial z}$ та однорідними крайовими умовами* // Карпатські матем. публ. — 2026. — Т.18, №2. — С. 468–477.

Досліджено нелокальну задачу з однорідними крайовими умовами для неоднорідного диференціального рівняння з частинними похідними, що містить узагальнений оператор диференціювання $z \frac{\partial}{\partial z}$, який діє на функцію комплексної змінної z зі скалярними коефіцієнтами. Доведено теореми єдиності та існування розв'язку задачі у просторі $H_q^n(\mathcal{D})$. Встановлено коректність задачі за Адамаром. Це відрізняє дану задачу від некоректних за Адамаром задач з багатьма просторовими комплексними змінними, розв'язність яких пов'язана з проблемою малих знаменників.

Ключові слова і фрази: крайова задача, неоднорідне рівняння з частинними похідними, нелокальна крайова умова, функція комплексної змінної.