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Simulation of solution of hyperbolic equation with Orlicz right side

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The paper is devoted to the construction of a model of a hyperbolic equation solution with a strictly Orlicz random right side and zero initial and boundary conditions. The model approximates the solution with a given level of reliability and accuracy in the uniform norm.

Key words and phrases: hyperbolic equation, simulation, Orlicz random field.

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Introduction

The methods of computer simulation of random processes and fields are widely used in various scientific fields, such as physics, radiophysics, in testing various mechanisms, etc. In this paper, a model is built that can be implemented on a computer, which approximates the solution of the boundary value problem (1)–(3) for a hyperbolic equation with a random right side with given reliability and accuracy in the uniform norm.

A centered strictly Orlicz random field [1,2] is considered as the right side $\xi(x,t)$ of equation (1) (see below). It is known that under certain conditions the solution of the boundary value problem can be represented as a series

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) \frac{1}{\mu_n} \int_0^t \sin \mu_n (t-u) \zeta_n(u)\, du,$$

where $X_n(x)$ are known functions, $\zeta_n(t) = \int_0^\pi \xi(x,t) X_n(x)\, dx$.

It is clear that it is possible to consider such model

$$u^N(x,t) = \sum_{n=1}^{N} X_n(x) \frac{1}{\mu_n} \int_0^t \sin \mu_n (t-u) \zeta_n(u) du$$

of the problem solution. But the random field $\xi(x,t)$ may not be exactly known. Therefore, initially, the right side $\hat{\xi}(x,t)$ of the equation is modeled with a certain accuracy, and then approximate values $\zeta_n(t)$ are found, namely $\hat{\zeta}_n(t) = \int_0^\pi \hat{\xi}(x,t) X_n(x) dx$. The solution model is then considered as the sum

$$\hat{u}^N(x,t) = \sum_{n=1}^{N} X_n(x) \frac{1}{\mu_n} \int_0^t \sin \mu_n (t-u) \hat{\zeta}_n(u) du.$$

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The problems of mathematical physics with random factors have been considered in papers [2–8]. In [2], the boundary-value problems for the hyperbolic equations with Orlicz initial conditions are considered. In the article [3], models for solutions of boundary problems for hyperbolic equations with sub-Gaussian random initial conditions are constructed. In [4], a model of solutions to the heat equation with initial conditions from the Orlicz space of random variables is built. The paper [8] is devoted to simulation of solution of hyperbolic equations with φ-sub-Gaussian right side. In papers [6,7], the suficient conditions for the existance of the solution for hyperbolic equations with homogeneous initial and boundary conditions and with Orlicz right side in form of uniformly convergent in probability series are found. In the paper [5], an estimate of supremum distribution of the problem solution is obtained. In this paper, we use these results for simulation of the solution.

1 Simulation of solution

Consider the following first boundary-value problem for nonhomogeneuos hyperbolic equation with zero initial and boundary conditions

$$\frac{\partial^2 u}{\partial x^2} - q(x)u - \frac{\partial^2 u}{\partial t^2} = -\xi(x,t), \quad x \in [0,\pi], \ \ t \in [0,T] \tag{1}$$

$$u|_{t=0} = 0, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \tag{2}$$

$$u|_{x=0} = 0, \qquad u|_{x=\pi} = 0, \tag{3}$$

where $T > 0$ is some constant, $q(x)$,$x \in [0,\pi]$, is continuously differentiable function such that $q(x) \geq 0$ and $\xi(x,t)$ for $x \in [0,\pi]$, $t \in [0,T]$ is sample continuous with probability one random field.

The coresponding Sturm-Liouville problem is written as follows

$$\frac{d^2 X}{dx^2} - qX + \lambda X = 0,$$
$$X(0) = X(\pi) = 0.$$

Let $X_n(x)$ be the ortonormal eigenfunctions and λ_n be the corresponding eigenvalues for this problem. We assume that λ_n are ordered in the ascending order. In view of the restrictions imposed on the function q, all eigenvalues are positive and zero is not an eigenvalue (see [9]).

Put $\mu_n = \sqrt{\lambda_n}$, $B(x,y,t,s) = \mathsf{E}\xi(x,t)\xi(y,s)$, $(x,y,t,s) \in [0,\pi]^2 \times [0,T]^2$. Assume that

$$B(0,y,t,s) = B(\pi,y,t,s) = 0 \quad \text{and} \quad B(x,0,t,s) = B(x,\pi,t,s) = 0$$

for $x \in [0,\pi]$, $y \in [0,\pi]$, $t \in [0,T]$, $s \in [0,T]$.

For every fixed pair $(t,s) \in [0,T]^2$, let us continue the function $B(x,y,t,s)$, considered as a function of the variables x,y, onto whole plane $\mathbb{R}^2$ in such a way that it becomes a periodic function with period 2π by variables x,y and that $B(-x,y,t,s) = -B(x,y,t,s) = B(x,-y,t,s)$. As a consequence of our assumptions such the continuation is possible.

Denote

$$\begin{aligned} B_{i,j}(x,y,t,s) &= \frac{\partial^{i+j}}{\partial x^i \partial y^j} B(x,y,t,s), \quad 0 \le i,j \le 2, \\ \Delta_{2,\delta_1,\delta_2} B(x,y,t,s) &= B(x+\delta_1, y+\delta_2, t,s) - B(x+\delta_1, y, t, s) - B(x, y+\delta_2, t, s) + B(x,y,t,s), \\ \Delta_{x,\delta} B(x,y,t,s) &= B(x+\delta, y, t, s) - B(x,y,t,s), \\ \Delta_{y,\delta} B(x,y,t,s) &= B(x, y+\delta, t, s) - B(x,y,t,s), \\ \widetilde{B}(x,y,t,s) &= B(x,y,t,t) - B(x,y,t,s) - B(x,y,s,t) + B(x,y,s,s). \end{aligned}$$

Theorem 1 ([6]). *Let in the equation (1) $\xi(x,t)$ be a centered strong Orlicz sample continuous with probability one random field from the space $L_U(\Omega)$, where U satisfies the g-condition (that is, $\exists z_0 \ge 0\ \exists K > 0\ \exists A > 0\ \forall x \ge z_0\ \forall y \ge z_0 \colon U(x)U(y) \le AU(Kxy)$). For $\lambda > 0$, let $\varphi(\lambda)$ be a continuous increasing function, such that $\varphi(\lambda) > 0$ for all $\lambda > 0$, $\varphi(\lambda) \to \infty$ as $\lambda \to \infty$ and $\frac{\lambda}{\varphi(\lambda)}$ is increasing for $\lambda > v_0$ for some constant $v_0 \ge 0$.*

Assume that for all $x \in [0,\pi]$, $y \in [0,\pi]$, $t \in [0,T]$, $s \in [0,T]$ the continuous derivative $\frac{\partial^4}{\partial x^2 \partial y^2} B(x,y,t,s)$ exsists and for some continuous functions $\tau(\delta_1,\delta_2)$, $\delta_1 \ge 0$, $\delta_2 \ge 0$, and $\tau(\delta)$, $\delta \ge 0$, such that $\tau(\delta_1,\delta_2) > 0$ for $\delta_1 > 0$, $\delta_2 > 0$, $\tau(0,\delta_2) = \tau(\delta_1,0) = 0$, $\tau(\delta_1,\delta_2)$ is monotonously increasing by δ_1 and by δ_2, $\tau(\delta) > 0$ for $\delta > 0$, $\tau(0) = 0$, $\tau(\delta)$ is monotonously increasing, the next conditions hold true:

– *the following series converge, i.e.*

$$\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{\tau\left(\frac{\pi}{k},\frac{\pi}{m}\right)}{km} \varphi(k^2)\varphi(m^2) < \infty, \qquad \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{\tau\left(\frac{\pi}{k}\right)}{km^2} \varphi(k^2)\varphi(m^2) < \infty,$$
$$\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{\tau\left(\frac{\pi}{m}\right)}{k^2 m} \varphi(k^2)\varphi(m^2) < \infty, \qquad \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{k^2 m^2} \varphi(k^2)\varphi(m^2) < \infty;$$

– *for all $\varepsilon > 0$ we have*

$$\int_0^{\varepsilon} U^{(-1)}\left(\left(\varphi^{(-1)}\left(\frac{1}{v}\right)\right)^2\right) dv < \infty; \tag{4}$$

– *for all $0 \le i,j \le 1$ there exist constants $C_{1,i,j} > 0$, $C_{2,i,j} > 0$, $C_{3,i,j} > 0$ such that*

$$\sup_{\substack{0\le t\le T\\ 0\le s\le T}} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left|\Delta_{2,\delta_1,\delta_2} B_{2i,2j}(x,y,t,s)\right| dx\, dy \le C_{1,i,j}\tau(\delta_1,\delta_2),$$
$$\sup_{\substack{0\le t\le T\\ 0\le s\le T}} \int_{0}^{\pi}\left(\int_{-\pi}^{\pi} \left|\Delta_{x,\delta} B_{2i,2j}(x,y,t,s)\right| dx\right) dy \le C_{2,i,j}\tau(\delta),$$
$$\sup_{\substack{0\le t\le T\\ 0\le s\le T}} \int_{0}^{\pi}\left(\int_{-\pi}^{\pi} \left|\Delta_{y,\delta} B_{2i,2j}(x,y,t,s)\right| dy\right) dx \le C_{3,i,j}\tau(\delta);$$

– *for all $0 \le i,j \le 1$ there exist constants $M_{i,j} > 0$ such that*

$$\left|\int_0^{\pi}\int_0^{\pi} \widetilde{B}_{2i,2j}(x,y,t,s)\, dx\, dy\right| \le \frac{M_{i,j}}{\varphi^2\left(\frac{1}{|t-s|} + v_0\right)}.$$

Then the series

$$u(x,t)=\sum_{n=1}^{\infty}X_n(x)\frac{1}{\mu_n}\int_0^t \sin\mu_n(t-u)\zeta_n(u)\,du, \tag{5}$$

where

$$\zeta_n(t)=\int_0^{\pi}\xi(x,t)X_n(x)\,dx,$$

uniformly converges in probability in the region $[0,\pi]\times[0,T]$*, where* $T>0$ *is some constant. Moreover, the series, obtained by diferentiation of* (5) *with respect to* x *one time and two times and with respect to* t *one time and two times, uniformly converge in probability and the problem* (1)–(3) *has a solution with probability one which is representable in the form* (5)*.*

Under the conditions of Theorem 1 we construct a model of solution of the problem (1)–(3), which approximate this solution with given reliability and accuracy in norm of the space $C([0,\pi]\times[0,T])$.

Let $\hat{\xi}(x,t)$ be a model of random field $\xi(x,t)$. Denote

$$\hat{\zeta}_n(t)=\int_0^{\pi}\hat{\xi}(x,t)X_n(x)\,dx.$$

The sum

$$\hat{u}^N(x,t)=\sum_{n=1}^{N}X_n(x)\frac{1}{\mu_n}\int_0^t \sin\mu_n(t-u)\hat{\zeta}_n(u)\,du \tag{6}$$

will be called the model of the process $u(x,t)$.

Definition 1. *The model* $\hat{u}^N(x,t)$ *approximate the solution of the problem* (1)–(3)*, which is representable in the form* (5) *with given reliability* $1-\omega$*, where* $\omega\in[0,1]$*, and accuracy* $s\geq 0$ *in norm of the space* $C([0,\pi]\times[0,T])$ *if*

$$\mathsf{P}\left\{\sup_{\substack{x\in[0,\pi]\\ t\in[0,T]}}\left|\hat{u}^N(x,t)-u(x,t)\right|>s\right\}\leq\omega.$$

Denote $\Delta_N(x,t)=u(x,t)-\hat{u}^N(x,t)=u_N(x,t)+V_N(x,t)$, where

$$u_N(x,t)=\sum_{n=N+1}^{\infty}X_n(x)\frac{1}{\mu_n}\int_0^t \sin\mu_n(t-u)\zeta_n(u)\,du,$$

$$V_N(x,t)=\sum_{n=1}^{N}X_n(x)\frac{1}{\mu_n}\int_0^t \sin\mu_n(t-u)\left(\zeta_n(u)-\hat{\zeta}_n(u)\right)du.$$

Theorem 2. *Let* $\xi(x,t)$ *be centered strong Orlicz sample continuous with probability one random field from the space* $L_U(\Omega)$*. Let for the problem* (1)–(3) *the conditions of Theorem 1 hold true. We assume that the model* $\hat{\xi}(x,t)$ *is such that*

$$\int_0^{\pi}\int_0^{T}\sqrt{\mathsf{E}\left(\xi(x,u)-\hat{\xi}(x,u)\right)^2}\,du\,dx<\Lambda.$$

Then random field $\hat{u}^N(x,t)$, determined in (6)*, is the model which approximate the random field $u(x,t)$ with reliability $1-\omega$ and accuracy s in norm of the space $C([0,\pi]\times[0,T])$, if ω, s and N are such that for some $\theta\in(0,1)$ the following condition*

$$\omega U\left(s\theta(1-\theta)\left(\int_0^{\theta w_0(N)}\chi_U\left(\left(\frac{\pi}{2}\left(\varphi^{(-1)}\left(u^{-1}C_\Delta(\widehat{D}_N+\Lambda B_N)\right)-v_0\right)+1\right)\right.\right.\right.$$
$$\left.\left.\left.\times\left(\frac{T}{2}\left(\varphi^{(-1)}\left(u^{-1}C_\Delta(\widehat{D}_N+\Lambda B_N)\right)-v_0\right)+1\right)\right)du\right)^{-1}\right)\geq 1$$

holds, where

$$w_0(N)=\frac{C_\Delta(\widehat{D}_N+\Lambda B_N)}{\varphi\left(\frac{1}{\max\{\pi,T\}}+v_0\right)},$$

$$\widehat{D}_N=T\max\{L,2C_X\}\left(\sum_{k=N+1}^{\infty}\sum_{m=N+1}^{\infty}\frac{1}{\mu_k\mu_m}\widehat{C}_{k,m}\varphi(\mu_k^2+v_0)\varphi(\mu_m^2+v_0)\right)^{1/2}$$
$$+C_X\left(\sum_{k=N+1}^{\infty}\sum_{m=N+1}^{\infty}\widehat{C}_{k,m}\frac{1}{\mu_k\mu_m}(4T^2\varphi(\mu_k+v_0)\varphi(\mu_m+v_0)\right.$$
$$+2T\max\{1,2T\}\varphi(\mu_k+v_0)\varphi(1+v_0)$$
$$\left.+2T\max\{1,2T\}\varphi(\mu_m+v_0)\varphi(1+v_0)+(\max\{1,2T\})^2\varphi^2(1+v_0))\right)^{1/2},$$

$$B_N=C_X\max\{L,2C_X\}\sum_{n=1}^{N}\frac{\varphi(\mu_n^2+v_0)}{\mu_n}$$
$$+C_X^2\left(\sum_{n=1}^{N}\sum_{k=1}^{N}\frac{1}{\mu_n\mu_k}(4\varphi(\mu_n+v_0)\varphi(\mu_k+v_0)+2\max\{1,2T\}\varphi(1+v_0)\varphi(\mu_n+v_0)\mu_k\right.$$
$$\left.+2\max\{1,2T\}\varphi(1+v_0)\mu_n\varphi(\mu_k+v_0)+(\max\{1,2T\})^2\varphi^2(1+v_0)\mu_n\mu_k)\right)^{1/2},$$

C_Δ is constant from the definition of strong Orlicz family of random variables (see [6]),

$$\widehat{C}_{k,m}=\sum_{i,j=0}^{1}\frac{C_q^{2-i-j}}{\mu_k^2\mu_m^2}\left(\frac{C_{1,i,j}\tau(\frac{\pi}{k},\frac{\pi}{m})}{8\pi}+\sqrt{\frac{2}{\pi}}\frac{LC_{2,i,j}\tau(\frac{\pi}{k})}{4m}+\sqrt{\frac{2}{\pi}}\frac{LC_{3,i,j}\tau(\frac{\pi}{m})}{4k}+\frac{C_{G,i,j}}{km}\right),$$

$$C_q=\sup_{0\leq x\leq\pi}|q(x)|,\qquad C_{G,i,j}=L^2\pi^2\max_{\substack{x,y\in[0,\pi]\\ t,s\in[0,T]}}|B_{2i,2j}(x,y,t,s)|,\qquad C_X=\sup_{\substack{x\in[0,\pi]\\ k\geq 1}}|X_k(x)|,$$

L is constant from the paper [7], i.e. such that $|X_k(x_1)-X_k(x_2)|\leq L\mu_k^2|x_1-x_2|$, $x_1,x_2\in[0,\pi]$.

Proof. Since $\Delta_N(x,t)$ is a centered strong Orlicz random field from the space $L_U(\Omega)$, according to [5, Theorem 1] we get that if there exists a continuous monotonously increasing function $\sigma(h)$, $h\geq 0$, such that $\sigma(0)=0$ and

$$\sup_{\substack{|x-y|\leq h\\ |t-s|\leq h}}\left(\mathsf{E}|\Delta_N(x,t)-\Delta_N(y,s)|^2\right)^{1/2}\leq\frac{\sigma(h)}{C_\Delta},$$

and for some $\varepsilon > 0$ the following condition

$$\int_0^{\varepsilon} U^{(-1)}\left(\left(\frac{\pi}{2\sigma^{(-1)}(u)}+1\right)\left(\frac{T}{2\sigma^{(-1)}(u)}+1\right)\right) du < \infty \tag{7}$$

holds, then for all $s > 0$ for all $0 < \theta < 1$ we have

$$\mathsf{P}\left\{ \sup_{\substack{x\in[0,\pi]\\ t\in[0,T]}} |\Delta_N(x,t)| > s \right\} \le \left(U\left(\frac{s}{B(\theta)}\right)\right)^{-1},$$

where

$$B(\theta) = \frac{1}{\theta(1-\theta)} \int_0^{\theta w_0} \chi_U\left(\left(\frac{\pi}{2\sigma^{(-1)}(u)}+1\right)\left(\frac{T}{2\sigma^{(-1)}(u)}+1\right)\right) du,$$

$$w_0 = \sigma\left(\max\{\pi, T\}\right), \qquad \chi_U(t) = \begin{cases} t, & t < U(z_0) \\ C_U U^{(-1)}(t), & t \ge U(z_0), \end{cases}$$

$C_U = K(1+U(z_0))\max\{1, A\}$, z_0, K, A are constants from definition of g-condition, and $\sigma^{(-1)}(u)$ is the inverse function of $\sigma(h)$.

It is easy to see that

$$\sup_{\substack{|x-y|\le h\\ |t-s|\le h}} \left(\mathsf{E}|\Delta_N(x,t)-\Delta_N(y,s)|^2\right)^{1/2}$$
$$= \sup_{\substack{|x-y|\le h\\ |t-s|\le h}} \left(\mathsf{E}|u_N(x,t)+V_N(x,t)-u_N(y,s)-V_N(y,s)|^2\right)^{1/2}$$
$$\le \sup_{\substack{|x-y|\le h\\ |t-s|\le h}} \left(\mathsf{E}|u_N(x,t)-u_N(y,s)|^2\right)^{1/2} + \sup_{\substack{|x-y|\le h\\ |t-s|\le h}} \left(\mathsf{E}|V_N(x,t)-V_N(y,s)|^2\right)^{1/2}.$$

According to [5, Lemma 1 and Remark 1], we get

$$\sup_{\substack{|x-y|\le h\\ |t-s|\le h}} (\mathsf{E}|u_N(x,t)-u_N(y,s)|^2)^{1/2} \le \frac{\widehat{D}_N}{\varphi\left(\frac{1}{h}+v_0\right)},$$

$$(\mathsf{E}|V_N(x,t)-V_N(y,s)|^2)^{1/2}$$
$$= \left(\mathsf{E}\left|\sum_{n=1}^{N} X_n(x)\frac{1}{\mu_n}\int_0^t \sin\mu_n(t-u)(\zeta_n(u)-\hat{\zeta}_n(u))\,du \right.\right.$$
$$\left.\left. - \sum_{n=1}^{N} X_n(y)\frac{1}{\mu_n}\int_0^s \sin\mu_n(s-u)(\zeta_n(u)-\hat{\zeta}_n(u))\,du\right|^2\right)^{1/2}$$
$$= \left(\mathsf{E}\left|\sum_{n=1}^{N} \frac{1}{\mu_n}(X_n(x)-X_n(y))\int_0^t (\zeta_n(u)-\hat{\zeta}_n(u))\sin\mu_n(t-u)\,du \right.\right.$$
$$+ \sum_{n=1}^{N}\frac{1}{\mu_n}X_n(y)\Bigg(\int_0^t (\zeta_n(u)-\hat{\zeta}_n(u))\sin\mu_n(t-u)\,du$$
$$\left.\left. - \int_0^s (\zeta_n(u)-\hat{\zeta}_n(u))\sin\mu_n(s-u)\,du\Bigg)\right|^2\right)^{1/2} \le A_1 + A_2,$$

where

$$A_1 = \left(\mathsf{E} \left| \sum_{n=1}^{N} \frac{1}{\mu_n} (X_n(x) - X_n(y)) \int_0^t (\zeta_n(u) - \hat{\zeta}_n(u)) \sin \mu_n (t-u)\, du \right|^2 \right)^{1/2},$$

$$A_2 = \left(\mathsf{E} \left| \sum_{n=1}^{N} \frac{1}{\mu_n} X_n(y) \left(\int_0^t (\zeta_n(u) - \hat{\zeta}_n(u)) \sin \mu_n (t-u)\, du \right.\right.\right.$$

$$\left.\left.\left. - \int_0^s (\zeta_n(u) - \hat{\zeta}_n(u)) \sin \mu_n (s-u)\, du \right) \right|^2 \right)^{1/2}.$$

Denote

$$Q_{n,k}(t,s) = \int_0^t \int_0^s \sin \mu_n(t-u) \sin \mu_k(s-v) \mathsf{E}(\zeta_n(u) - \hat{\zeta}_n(u))(\zeta_k(v) - \hat{\zeta}_k(v))\, dv\, du.$$

Then $A_1^2 \le \sum_{n=1}^{N} \sum_{k=1}^{N} \frac{1}{\mu_n \mu_k} |X_n(x) - X_n(y)||X_k(x) - X_k(y)||Q_{n,k}(t,t)|$.

According to [7, Lemmas 2, 3], we obtain

$$|X_n(x) - X_n(y)| \le \max\{L, 2C_X\} \frac{\varphi(\mu_n^2 + v_0)}{\varphi\left(\frac{1}{|x-y|} + v_0\right)}.$$

Moreover

$$\begin{aligned} |Q_{n,k}(t,t)| &\le \int_0^T \int_0^T \left| \mathsf{E}(\zeta_n(u) - \hat{\zeta}_n(u))(\zeta_k(v) - \hat{\zeta}_k(v)) \right| du\, dv \\ &\le \int_0^T \left(\mathsf{E}(\zeta_n(u) - \hat{\zeta}_n(u))^2 \right)^{1/2} du \int_0^T \left(\mathsf{E}(\zeta_k(v) - \hat{\zeta}_k(v))^2 \right)^{1/2} dv. \end{aligned}$$

Then

$$\begin{aligned} \mathsf{E}(\zeta_n(u) - \hat{\zeta}_n(u))^2 &= \mathsf{E} \left(\int_0^\pi X_n(x) (\xi(x,u) - \hat{\xi}(x,u))\, dx \right)^2 \\ &= \int_0^\pi \int_0^\pi X_n(x) X_n(y) \mathsf{E}(\xi(x,u) - \hat{\xi}(x,u))(\xi(y,u) - \hat{\xi}(y,u))\, dx\, dy \\ &\le C_X^2 \left(\int_0^\pi \left(\mathsf{E}(\xi(x,u) - \hat{\xi}(x,u))^2 \right)^{1/2} dx \right)^2. \end{aligned}$$

We have

$$|Q_{n,k}(t,t)| \le \left(C_X \int_0^\pi \int_0^T \sqrt{\mathsf{E}\big(\xi(x,u) - \hat{\xi}(x,u)\big)^2}\, du\, dx \right)^2 \le \Lambda^2 C_X^2.$$

Therefore

$$A_1^2 \le \Lambda^2 C_X^2 \big(\max\{L, 2C_X\} \big)^2 \sum_{n=1}^{N} \sum_{k=1}^{N} \frac{1}{\mu_n \mu_k} \varphi(\mu_n^2 + v_0) \varphi(\mu_k^2 + v_0) \left(\varphi\Big(\frac{1}{|x-y|} + v_0 \Big) \right)^{-2}.$$

Namely

$$A_1 \le \Lambda C_X \max\{L, 2C_X\} \sum_{n=1}^{N} \frac{\varphi(\mu_n^2 + v_0)}{\mu_n} \left(\varphi\Big(\frac{1}{|x-y|} + v_0 \Big) \right)^{-1}.$$

Further, we have

$$A_2^2 \le \sum_{n=1}^{N}\sum_{k=1}^{N} \frac{1}{\mu_n\mu_k}|X_n(y)X_k(y)| \times \left| \mathsf{E}\left(\int_0^t (\zeta_n(u)-\hat{\zeta}_n(u))\sin\mu_n(t-u)\,du - \int_0^s (\zeta_n(u)-\hat{\zeta}_n(u))\sin\mu_n(s-u)\,du\right) \times \left(\int_0^t (\zeta_k(v)-\hat{\zeta}_k(v))\sin\mu_k(t-v)\,dv - \int_0^s (\zeta_k(v)-\hat{\zeta}_k(v))\sin\mu_k(s-v)\,dv\right)\right|.$$

Let for certainty $s \le t$. Then

$$\int_0^t (\zeta_n(u)-\hat{\zeta}_n(u))\sin\mu_n(t-u)\,du - \int_0^s (\zeta_n(u)-\hat{\zeta}_n(u))\sin\mu_n(s-u)\,du = \int_0^s (\zeta_n(u)-\hat{\zeta}_n(u))\left(\sin\mu_n(t-u)-\sin\mu_n(s-u)\right)du + \int_s^t (\zeta_n(u)-\hat{\zeta}_n(u))\sin\mu_n(t-u)du.$$

Therefore $A_2^2 \le C_X^2 \sum_{n=1}^{N}\sum_{k=1}^{N} \frac{1}{\mu_n\mu_k} q_{n,k}(t,s)$, where

$$\begin{aligned} q_{n,k}(t,s) = &\left|\mathsf{E}\Big(\int_0^s (\zeta_n(u)-\hat{\zeta}_n(u))(\sin\mu_n(t-u)-\sin\mu_n(s-u))\,du + \int_s^t (\zeta_n(u)-\hat{\zeta}_n(u))\sin\mu_n(t-u)\,du\Big) \times \Big(\int_0^s (\zeta_k(v)-\hat{\zeta}_k(v))(\sin\mu_k(t-v)-\sin\mu_k(s-v))\,dv + \int_s^t (\zeta_k(v)-\hat{\zeta}_k(v))\sin\mu_k(t-v)\,dv\Big)\right| \\ \le &\int_0^s\int_0^s |\mathsf{E}(\zeta_n(u)-\hat{\zeta}_n(u))(\zeta_k(v)-\hat{\zeta}_k(v))| \times |\sin\mu_n(t-u)-\sin\mu_n(s-u)|\,|\sin\mu_k(t-v)-\sin\mu_k(s-v)|\,dv\,du \\ &+ \int_0^s\int_s^t |\mathsf{E}(\zeta_n(u)-\hat{\zeta}_n(u))(\zeta_k(v)-\hat{\zeta}_k(v))| \times |\sin\mu_n(t-u)-\sin\mu_n(s-u)|\,|\sin\mu_k(t-v)|\,dv\,du \\ &+ \int_s^t\int_0^s |\mathsf{E}(\zeta_n(u)-\hat{\zeta}_n(u))(\zeta_k(v)-\hat{\zeta}_k(v))| \times |\sin\mu_n(t-u)|\,|\sin\mu_k(t-v)-\sin\mu_k(s-v)|\,dv\,du \\ &+ \int_s^t\int_s^t |\mathsf{E}(\zeta_n(u)-\hat{\zeta}_n(u))(\zeta_k(v)-\hat{\zeta}_k(v))|\,|\sin\mu_n(t-u)|\,|\sin\mu_k(t-v)|\,dv\,du. \end{aligned}$$

According to [7, Lemma 3] with $Z_1(u) = u$, $u \in [0,T]$, $\lambda = 1$, $C = 1$, $B = T$ for $u \in [s,t]$ we have

$$|\sin\mu_n(t-u)| \le \mu_n|t-s| \le \max\{1,2T\}\mu_n \frac{\varphi(1+v_0)}{\varphi\left(\frac{1}{|t-s|}+v_0\right)}.$$

Moreover with $Z_{\mu_n}(t) = \sin \mu_n(t-u)$, $B = 1$, $\lambda = \mu_n$, $C = 1$ we get

$$|\sin \mu_n(t-u) - \sin \mu_n(s-u)| \leq 2\frac{\varphi(\mu_n + v_0)}{\varphi\left(\frac{1}{|t-s|} + v_0\right)}.$$

Then

$$\begin{aligned} q_{n,k}(t,s) \leq \Lambda^2 C_X^2 \big(4\varphi(\mu_n + v_0)\varphi(\mu_k + v_0) &+ 2\max\{1, 2T\}\varphi(\mu_n + v_0)\mu_k\varphi(1+v_0) \\ &+ 2\max\{1, 2T\}\mu_n\varphi(\mu_k + v_0)\varphi(1+v_0) \\ &+ (\max\{1, 2T\})^2\mu_n\mu_k\varphi^2(1+v_0)\big)\frac{1}{\varphi^2\left(\frac{1}{|t-s|} + v_0\right)}. \end{aligned}$$

It is clear that this inequality also holds for $t \leq s$. So,

$$\begin{aligned} A_2 \leq \Lambda C_X^2 \Bigg(\sum_{n=1}^{N}\sum_{k=1}^{N} \frac{1}{\mu_n\mu_k} \big(4\varphi(\mu_n + v_0)\varphi(\mu_k + v_0) \\ &+ 2\max\{1, 2T\}\varphi(\mu_n + v_0)\mu_k\varphi(1+v_0) \\ &+ 2\max\{1, 2T\}\mu_n\varphi(\mu_k + v_0)\varphi(1+v_0) \\ &+ (\max\{1, 2T\})^2\mu_n\mu_k\varphi^2(1+v_0)\big)\Bigg)^{1/2}\frac{1}{\varphi\left(\frac{1}{|t-s|} + v_0\right)}. \end{aligned}$$

Therefore, we get

$$\sup_{\substack{|x-y|\leq h\\|t-s|\leq h}} \left(\mathsf{E}|V_N(x,t) - V_N(y,s)|^2\right)^{1/2} \leq \frac{\Lambda B_N}{\varphi\left(\frac{1}{h} + v_0\right)},$$

where

$$\begin{aligned} B_N =\, & C_X \max\{L, 2C_X\} \sum_{n=1}^{N} \frac{\varphi(\mu_n^2 + v_0)}{\mu_n} \\ &+ C_X^2 \Bigg(\sum_{n=1}^{N}\sum_{k=1}^{N} \frac{1}{\mu_n\mu_k} \big(4\varphi(\mu_n + v_0)\varphi(\mu_k + v_0) + 2\max\{1, 2T\}\varphi(\mu_n + v_0)\mu_k\varphi(1+v_0) \\ &\quad + 2\max\{1, 2T\}\mu_n\varphi(\mu_k + v_0)\varphi(1+v_0) + (\max\{1, 2T\})^2\mu_n\mu_k\varphi^2(1+v_0)\big)\Bigg)^{1/2}. \end{aligned}$$

Finally,

$$\sup_{\substack{|x-y|\leq h\\|t-s|\leq h}} \left(\mathsf{E}|\Delta_N(x,t) - \Delta_N(y,s)|^2\right)^{1/2} \leq \frac{\widehat{D}_N + \Lambda B_N}{\varphi\left(\frac{1}{h} + v_0\right)},$$

or

$$\sup_{\substack{|x-y|\leq h\\|t-s|\leq h}} \left(\mathsf{E}|\Delta_N(x,t) - \Delta_N(y,s)|^2\right)^{1/2} \leq \frac{\sigma(h)}{C_\Delta}, \qquad \text{where} \quad \sigma(h) = \frac{C_\Delta(\widehat{D}_N + \Lambda B_N)}{\varphi\left(\frac{1}{h} + v_0\right)}, \;\; h > 0.$$

Then

$$\sigma^{(-1)}(u) = \frac{1}{\varphi^{(-1)}\left(\frac{C_\Delta(\widehat{D}_N+\Lambda B_N)}{u}\right) - v_0}, \quad u > 0,$$

$$w_0 = \sigma\left(\max\{\pi, T\}\right) = \frac{C_\Delta(\widehat{D}_N + \Lambda B_N)}{\varphi\left(\frac{1}{\max\{\pi,T\}} + v_0\right)} = w_0(N),$$

$$B(\theta) = \frac{1}{\theta(1-\theta)} \int_0^{\theta w_0(N)} \chi U\Bigg(\left(\frac{\pi}{2}\left(\varphi^{(-1)}\left(u^{-1}C_\Delta(\widehat{D}_N + \Lambda B_N)\right) - v_0\right) + 1\right) \times \left(\frac{T}{2}\left(\varphi^{(-1)}\left(u^{-1}C_\Delta(\widehat{D}_N + \Lambda B_N)\right) - v_0\right) + 1\right)\Bigg)\, du.$$

So, inequality (7) holds because the condition (4) is fulfilled.

Therefore, in order to obtain a solution model with $1-\omega$ reliability at s accuracy, it is necessary that for some $\theta \in (0,1)$ the condition

$$\Bigg(U\Bigg(s\theta(1-\theta)\Bigg(\int_0^{\theta w_0(N)} \chi U\Bigg(\left(\frac{\pi}{2}\left(\varphi^{(-1)}\left(u^{-1}C_\Delta(\widehat{D}_N + \Lambda B_N)\right) - v_0\right) + 1\right) \times \left(\frac{T}{2}\left(\varphi^{(-1)}\left(u^{-1}C_\Delta(\widehat{D}_N + \Lambda B_N)\right) - v_0\right) + 1\right)\Bigg) du\Bigg)^{-1}\Bigg)\Bigg)^{-1} \le \omega$$

is fulfilled. □

2 Example

Let us consider the following problem of the vibrations of a homogeneous string

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = -\xi(x,t), \quad x \in [0,\pi], \;\; t \in [0,T],$$

$$u|_{t=0} = 0, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0,$$

$$u|_{x=0} = 0, \qquad u|_{x=\pi} = 0.$$

The solution to this problem is expressed in the next form

$$u(x,t) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \frac{1}{n} \sin nx \int_0^t \zeta_n(u) \sin n(t-u)\, du,$$

where

$$\zeta_n(t) = \sqrt{\frac{2}{\pi}} \int_0^\pi \xi(x,t) \sin nx\, dx,$$

namely $\mu_n = n$, $X_n(x) = \sqrt{\frac{2}{\pi}} \sin nx$.

Let $\xi(x,t) = \sum_{k=1}^{\infty} \eta_k \frac{1}{k^4} \sin kx \cos kt$, where $0 \le x \le \pi$, $0 \le t \le T$, $\eta_k \in L_5(\Omega)$, be such independent identically distributed strictly Orlicz random variables, for which $\mathsf{E}\eta_k^2 = 1$, $\mathsf{E}\eta_k = 0$, that is $U(x) = |x|^5$.

Let us assume $\varphi(\lambda) = \lambda^{\frac{3}{7}}$, $\tau(\delta_1, \delta_2) = \delta_1\delta_2$, $\tau(\delta) = \delta$. Then, from the example provided in [5], it follows that the conditions of Theorem 1 are satisfied.

Let

$$\hat{\xi}(x,t) = \hat{\xi}_M(x,t) = \sum_{k=1}^{M} \eta_k \frac{1}{k^4} \sin kx \cos kt.$$

Then

$$\mathsf{E}\left(\xi(x,t) - \hat{\xi}_M(x,t)\right)^2 = \sum_{k=M+1}^{\infty} \frac{1}{k^2} \sin^2 kx \cos^2 kt \leq \sum_{k=M+1}^{\infty} \frac{1}{k^2}.$$

So, for a given Λ, the number M is chosen in such a way that the condition

$$\pi T \left(\sum_{k=M+1}^{\infty} \frac{1}{k^2} \right)^{1/2} < \Lambda$$

holds true.

In this case,

$$C_X = L = \sqrt{\frac{2}{\pi}}, \;\; C_q = 0, \;\; C_{1,1,1} = 4\pi^2 \sum_{k=1}^{\infty} \frac{1}{k^2}, \;\; C_{2,1,1} = C_{3,1,1} = 2\pi^2 \sum_{k=1}^{\infty} \frac{1}{k^2}, \;\; C_{G,1,1} = 2\pi \sum_{k=1}^{\infty} \frac{1}{k^4}$$

and

$$\widehat{C}_{k,m} = \frac{1}{k^3 m^3} \left(\left(\frac{1}{2}\pi^3 + 2\pi^2 \right) \sum_{k=1}^{\infty} \frac{1}{k^2} + 2\pi \sum_{k=1}^{\infty} \frac{1}{k^4} \right),$$

$$B_N = \frac{4}{\pi} \sum_{n=1}^{N} \frac{1}{n^{1/7}} + \frac{2}{\pi} \left(\sum_{n=1}^{N} \sum_{k=1}^{N} \frac{1}{nk} \big(4n^{3/7}k^{3/7} + 2\max\{1,2T\} n^{3/7} k \right.$$
$$\left. + 2\max\{1,2T\} n k^{3/7} + (\max\{1,2T\})^2 nk\big) \right)^{1/2},$$

$$\widehat{D}_N = T\sqrt{\frac{2}{\pi}} \left(\sum_{k=N+1}^{\infty} \sum_{m=N+1}^{\infty} \frac{\widehat{C}_{k,m}}{k^{1/7} m^{1/7}} \right)^{1/2} + \sqrt{\frac{2}{\pi}} \left(\sum_{k=N+1}^{\infty} \sum_{m=N+1}^{\infty} \frac{\widehat{C}_{k,m}}{km} \Big(4T^2 k^{3/7} m^{3/7} \right.$$
$$\left. + 2T\max\{1,2T\} k^{3/7} + 2T \max\{1,2T\} m^{3/7} + (\max\{1,2T\})^2 \Big) \right)^{1/2},$$

$w_0(N) = C_\Delta(\widehat{D}_N + \Lambda B_N)(\max\{\pi, T\})^{3/7}$, $\varphi^{(-1)}(\lambda) = \lambda^{\frac{7}{3}}$, $\chi_U(t) = t^{-5}$. Thus, the desired model is the one in which N and Λ are chosen such that for some $\theta \in (0,1)$ we have

$$\omega\Bigg(s\theta(1-\theta) \Bigg(\int_0^{\theta w_0(N)} \left(\frac{\pi}{2} \left(u^{-1} C_\Delta(\widehat{D}_N + \Lambda B_N) \right)^{7/3} + 1 \right)^{-5}$$
$$\times \left(\frac{T}{2} \left(u^{-1} C_\Delta(\widehat{D}_N + \Lambda B_N) \right)^{7/3} + 1 \right)^{-5} du \Bigg)^{-1} \Bigg)^5 \geq 1.$$

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Робота присвячена побудові моделі розв'язку гіперболічного рівняння з випадковою строго орлічевою правою частиною та нульовими початковими і крайовими умовами, яка наближає цей розв'язок в рівномірній нормі із заданою точністю та надійністю.

Ключові слова і фрази: гіперболічне рівняння, моделювання, орлічеве випадкове поле.