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Titchmarsh-Weyl theory for impulsive Hahn-Dirac systems

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In this article, the Titchmarsh-Weyl theory of the impulsive Hahn-Dirac system is studied. The limit-circle and limit-point classifications of the singular point $\zeta = \infty$ of the impulsive Hahn-Dirac system are obtained. Additionally, it will demonstrate that the limit-circle situation for the impulsive Hahn-Dirac system does not exist.

Key words and phrases: difference equation, discontinuous equation, Hahn-Dirac system, Titchmarsh-Weyl theory.

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Introduction

Sturm-Liouville problems have a long history in the theory of differential equations. These types of equations have been studied quite intensively under various conditions. One of these conditions is impulsive boundary conditions. Impulsive equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. Hence there are many studies on this subject in the literature (for example, see [5–9, 14–16, 19–22, 26]).

In 1910, H. Weyl published his famous article [25]. In this article, he examined singular Sturm-Liouville problems. He introduced the concepts of limit circle/limit point into the literature. Later, E.C. Titchmarsh [23] and R. McLeod [18] extended this theory to classical Dirac systems.

In the 1940s, W. Hahn [11] made a derivative definition and gathered two important operators under a single structure. These operators are the q-difference operator and the forward difference operator. Thus, it was possible to examine q-difference equations and advanced difference equations under Hahn difference equations. In 2018, M.H. Annaby et al. [4] studied the classical Hahn-Sturm-Liouville problems using this definition. In 2020, F. Hira [13] discussed the Hahn-Dirac system and examined the existence and uniqueness of solutions of such equations. B.P. Allahverdiev and H. Tuna [2] investigated the basic properties of the eigenvalue and eigenvector by showing the self-adjointness of the operator corresponding to the Hahn-Dirac system. For detailed information on Dirac systems, see [1, 2, 10, 13, 17, 24].

In this study, singular Hahn-Dirac systems will be examined under impulsive conditions. For such systems, the Titchmarsh-Weyl theory will be established.

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1 Preliminaries

In this context, we provide a concise overview of the Hahn calculus [3, 4, 11, 12]. Let $q \in (0,1)$, $\sigma_0 := \sigma/(1-q)$, $\sigma > 0$, and let $\Psi : J \subset \mathbb{R} \to \mathbb{R}$ be a function such that $\sigma_0 \in J$.

Definition 1 ([11,12]). *The Hahn derivative $D_{\sigma,q}\Psi$ is defined by*

$$D_{\sigma,q}\Psi(\zeta) = \begin{cases} \frac{\Psi(\sigma+q\zeta)-\Psi(\zeta)}{\sigma+(q-1)\zeta}, & \zeta \neq \sigma_0, \\ \Psi'(\sigma_0), & \zeta = \sigma_0. \end{cases}$$

$D_{\sigma,q}$ unifies two well-known operators. When $q \to 1$, we get the forward difference operator defined as

$$\Delta_\sigma f(\zeta) := \frac{f(\sigma+\zeta) - f(\zeta)}{(\sigma+\zeta)-\zeta}, \quad \zeta \in \mathbb{R}.$$

When $\sigma \to 0$, we get the Jackson q-difference operator defined as

$$D_q f(\zeta) := \frac{f(q\zeta) - f(\zeta)}{(q\zeta)-\zeta}, \quad \zeta \neq 0.$$

Furthermore, under appropriate conditions, we obtain

$$\lim_{\substack{q\to 1 \\ \omega \to 0}} D_{\omega,q} f(\zeta) = f'(\zeta).$$

Definition 2 ([3]). *Let $a, b, \sigma_0 \in J$. The Hahn integral (σ, q-integral) is given by*

$$\int_a^b \Psi(\zeta)\, d_{\sigma,q}\zeta := \int_{\sigma_0}^b \Psi(\zeta)\, d_{\sigma,q}\zeta - \int_{\sigma_0}^a \Psi(\zeta)\, d_{\sigma,q}\zeta,$$

where

$$\int_{\sigma_0}^\zeta \Psi(\gamma)\, d_{\sigma,q}\gamma := ((1-q)\zeta - \sigma) \sum_{n=0}^{\infty} q^n \Psi\left(\sigma \frac{1-q^n}{1-q} + \zeta q^n\right), \quad \zeta \in J,$$

provided that the series converges at $\zeta = a$ and $\zeta = b$.

Definition 3 ([4]). *The σ, q-trigonometric functions are defined by the formulas*

$$C_{\sigma,q}(z;\lambda) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n^2} (\lambda (z(1-q) - \sigma))^{2n}}{(q;q)_{2n}},$$

$$S_{\sigma,q}(z;\lambda) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)} (\lambda (z(1-q) - \sigma))^{2n+1}}{(q;q)_{2n+1}},$$

where

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1} \left(1 - aq^k\right).$$

2 Main Results

In this section, we introduce the limit-circle and limit-point classification of the singular point $\zeta = \infty$ of the impulsive Hahn-Dirac system.

Consider the impulsive boundary-value problem

$$\begin{cases} -\frac{1}{q}D_{-\frac{\sigma}{q},\frac{1}{q}}u_2 + p(\zeta)u_1 = \lambda u_1, \\ D_{\sigma,q}u_1 + r(\zeta)u_2 = \lambda u_2, \end{cases} \quad \zeta \in I, \tag{1}$$

$$\cos\alpha u_1(\sigma_0) + \sin\alpha u_2(\sigma_0) = 0, \tag{2}$$

$$u_1(d-) - k_1 u_1(d+) = 0, \tag{3}$$

$$u_2\left(\varrho^{-1}(d-)\right) - k_2 u_2\left(\varrho^{-1}(d+)\right) = 0, \tag{4}$$

$$\cos\beta u_1(a) + \sin\beta u_2\left(\varrho^{-1}(a)\right) = 0, \tag{5}$$

where $I_1 = [\sigma_0, d),\ I_2 = (d, a],\ \sigma_0 < d < a < \infty,\ I = I_1 \cup I_2$,

$$u(\zeta) = \begin{pmatrix} u_1(\zeta) \\ u_2(\zeta) \end{pmatrix}, \tag{6}$$

$\alpha, \beta \in \mathbb{R} := (-\infty, \infty)$, and $\lambda \in \mathbb{C}$.

Our basic assumptions throughout the paper are the following:

(A_1) Let $q \in (0,1)$, $\sigma_0 := \sigma/(1-q)$, $\sigma > 0$, and $k_1 k_2 = \delta > 0$.

(A_2) $p, r : I \to \mathbb{R}$ are continuous functions on $[\sigma_0, d) \cup (d, \varrho^{-1}(a)]$, $\varrho(\zeta) = q\zeta + \sigma$ and have finite limits $p(d\pm)$, $r(d\pm)$.

A similar problem has been investigated in [1] without impulsive conditions.

The space $\mathcal{H} := L^2_{\sigma,q}(I; \mathbb{C}^2)$ $(\mathcal{H}_1 := L^2_{\sigma,q}(I_1 \cup (d,\infty); \mathbb{C}^2))$ is a Hilbert space with the inner product

$$\langle u, v\rangle_{\mathcal{H}} := \int_{\sigma_0}^{d} (u,v)_{\mathbb{C}^2} d_{\sigma,q}\zeta + \delta \int_{d}^{a} (u,v)_{\mathbb{C}^2} d_{\sigma,q}\zeta,$$

$$\left(\langle u, v\rangle_{\mathcal{H}_1} := \int_{\sigma_0}^{d} (u,v)_{\mathbb{C}^2} d_{\sigma,q}\zeta + \delta \int_{d}^{\infty} (u,v)_{\mathbb{C}^2} d_{\sigma,q}\zeta\right),$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \text{ and } v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad u, v \in \mathcal{H}\ (\mathcal{H}_1).$$

Theorem 1. *For $\lambda \in \mathbb{C}$, the following problem*

$$\begin{cases} -\frac{1}{q}D_{-\frac{\sigma}{q},\frac{1}{q}}u_2 + p(\zeta)u_1 = \lambda u_1, \\ D_{\sigma,q}u_1 + r(\zeta)u_2 = \lambda u_2, \end{cases}$$

$$u_1(\sigma_0) = c_1,\ u_2(\sigma_0) = c_2,\ c_1, c_2 \in \mathbb{R},$$

$$u_1(d-) - k_1 u_1(d+) = 0,$$
$$u_2\left(\varrho^{-1}(d-)\right) - k_2 u_2\left(\varrho^{-1}(d+)\right) = 0,$$

has a unique solution ψ in $\mathcal{H}_1$.

Proof. From [13], we infer that the following problem

$$\begin{cases} -\frac{1}{q}D_{-\frac{\sigma}{q},\frac{1}{q}}u_2 + p(\zeta)u_1 = \lambda u_1, \\ D_{\sigma,q}u_1 + r(\zeta)u_2 = \lambda u_2, \end{cases} \quad \zeta \in (\sigma_0, d),$$

$$u_1(\sigma_0) = c_1,\ u_2(\sigma_0) = c_2,$$

has a unique solution

$$\psi_1 = \begin{pmatrix} \psi_{11} \\ \psi_{12} \end{pmatrix}.$$

Let us consider the following problem

$$\begin{cases} -\frac{1}{q}D_{-\frac{\sigma}{q},\frac{1}{q}}u_2 + p(\zeta)u_1 = \lambda u_1, \\ D_{\sigma,q}u_1 + r(\zeta)u_2 = \lambda u_2, \end{cases} \quad \zeta \in (d, \infty), \tag{7}$$

$$u_1(d+) = \frac{1}{k_1}\psi_{11}(d-), \tag{8}$$

$$u_2\left(\varrho^{-1}(d+)\right) = \frac{1}{k_2}\psi_{12}\left(\varrho^{-1}(d-)\right). \tag{9}$$

Set

$$U_n(\zeta,\lambda) = U_0(\zeta,\lambda) + q\int_d^\zeta \begin{bmatrix} \varphi_2(\zeta,\lambda)\varphi_1^T(\varrho(\gamma),\lambda) \\ \\ -\varphi_1(\zeta,\lambda)\varphi_2^T(\varrho(\gamma),\lambda) \end{bmatrix} M(\varrho(\gamma))U_{n-1}(\varrho(\gamma),\lambda)\,d_{\sigma,q}\gamma,$$

where

$$U_0(\zeta,\lambda) = \frac{1}{k_1}\psi_{11}(d-,\lambda) + \frac{1}{k_2}(\zeta-d)\psi_{12}\left(\varrho^{-1}(d-),\lambda\right), \quad \zeta \in (d,\infty),$$

$$\varphi_1(\zeta,\lambda) = \begin{pmatrix} \varphi_{11}(\zeta,\lambda) \\ \varphi_{12}(\zeta,\lambda) \end{pmatrix} = \begin{pmatrix} C_{\sigma,q}(\zeta;\lambda) \\ \frac{S_{\sigma,q}(\zeta;\lambda)}{\lambda} \end{pmatrix},$$

$$\varphi_2(\zeta,\lambda) = \begin{pmatrix} \varphi_{21}(\zeta,\lambda) \\ \varphi_{22}(\zeta,\lambda) \end{pmatrix} = \begin{pmatrix} -\frac{S_{\sigma,q}(\zeta;\lambda)}{\lambda} \\ C_{\sigma,q}(\zeta;\lambda) \end{pmatrix},$$

$$M = \begin{pmatrix} p & 0 \\ 0 & r \end{pmatrix},$$

and the notation ^{T} means the operation of matrix transpose.

Here φ_1 and φ_2 are the fundamental solutions of equation (7) for

$$M = \begin{pmatrix} p & 0 \\ 0 & r \end{pmatrix} = 0.$$

Let $\lambda \in \mathbb{C}$ be fixed. There exist positive numbers $N(\lambda)$ and A, such that

$$\left|\varphi_{ij}(\zeta,\lambda)\right| \le \frac{\sqrt{N(\lambda)}}{2}, \quad i,j = 1,2,$$

$$\|M(\zeta,\lambda)\| \le A,\ \|U_0(\zeta,\lambda)\| \le \widetilde{N(\lambda)}, \quad \zeta \in (d,\infty).$$

Then, we see that

$$\|U_1(\zeta,\lambda)-U_0(\zeta,\lambda)\| \leq \left| \begin{array}{l} q\int_d^\zeta \left[\begin{array}{c} C_{\sigma,q}(\zeta;\lambda)\frac{S_{\sigma,q}(\varrho(\gamma);\lambda)}{\lambda} \\ -\frac{S_{\sigma,q}(\zeta;\lambda)}{\lambda}C_{\sigma,q}(\varrho(\gamma);\lambda) \end{array} \right] p(\varrho(\gamma))\,U_{01}(\varrho(\gamma),\lambda)\,d_{\sigma,q}\gamma \\ -q\int_d^\zeta \left[\begin{array}{c} C_{\sigma,q}(\zeta;\lambda)\,C_{\sigma,q}(\varrho(\gamma);\lambda) \\ +\frac{S_{\sigma,q}(\zeta;\lambda)}{\lambda}\frac{S_{\sigma,q}(\varrho(\gamma);\lambda)}{\lambda} \end{array} \right] r(\varrho(\gamma))\,U_{02}(\varrho(\gamma),\lambda)\,d_{\sigma,q}\gamma \end{array} \right|$$

$$\leq 2qN(\lambda)\,A\widetilde{N(\lambda)}\left|\int_d^\zeta d_{\sigma,q}\gamma\right| \leq 2qN(\lambda)\,A\widetilde{N(\lambda)}\frac{(\zeta(1-q)-\sigma)}{(1-q)}.$$

Likewise, we deduce that

$$\|U_3(\zeta,\lambda)-U_2(\zeta,\lambda)\| \leq 2^2q^2\widetilde{N(\lambda)}\frac{A^2N^2(\lambda)(\zeta(1-q)-\sigma)^2}{(1-q)(1-q^2)}.$$

Then we have

$$\|U_{n+1}(\zeta,\lambda)-U_n(\zeta,\lambda)\| \leq 2^nq^n\widetilde{N(\lambda)}\frac{(AN(\lambda)(\zeta(1-q)-\sigma))^n}{(q;q)_n}, \quad n=1,2,3,\dots. \tag{10}$$

By Weierstrass's test, we conclude that the series

$$\sum_{n=1}^{\infty} 2^n q^{\frac{n(n+1)}{2}}\widetilde{N(\lambda)}\frac{(AN(\lambda)(\zeta(1-q)-\sigma))^n}{(q;q)_n}$$

is uniformly convergent. Consequently, the series

$$U_1(\zeta,\lambda)+\sum_{n=1}^{\infty}\{U_{n+1}(\zeta,\lambda)-U_n(\zeta,\lambda)\} \tag{11}$$

is uniformly convergent with respect to ζ on (d,∞). Hence $\lim_{n\to\infty} U_n(\zeta,\lambda)=\psi_2(\zeta,\lambda)$, i.e.

$$\psi_2(\zeta,\lambda)=U_1(\zeta,\lambda)+\sum_{n=1}^{\infty}\{U_{n+1}(\zeta,\lambda)-U_n(\zeta,\lambda)\}.$$

Our next goal is to prove that ψ_2 satisfies (7). We have

$$D_{\sigma,q}U_{n+1}(\zeta,\lambda)-D_{\sigma,q}U_n(\zeta,\lambda)=q\int_d^\zeta\left[D_{\sigma,q}\varphi_2(\zeta,\lambda)\varphi_1^T(\varrho(\gamma),\lambda)-D_{\sigma,q}\varphi_1(\zeta,\lambda)\varphi_2^T(\varrho(\gamma),\lambda)\right]$$
$$\times M(\varrho(\gamma))\left[U_n(\varrho(\gamma),\lambda)-U_{n-1}(\varrho(\gamma),\lambda)\right]d_{\sigma,q}\gamma.$$

By assumption (10), we infer that the series

$$\sum_{n=1}^{\infty}\left\{D_{\sigma,q}U_{n+1}(\zeta,\lambda)-D_{\sigma,q}U_n(\zeta,\lambda)\right\}$$

is uniformly convergent with respect to ζ on (d,∞). Due to

$$D_{\sigma,q}C_{\sigma,q}(\zeta;\lambda)=(-r(\zeta)+\lambda)\frac{S_{\sigma,q}(\zeta;\lambda)}{\lambda}, \qquad D_{\sigma,q}\frac{S_{\sigma,q}(\zeta;\lambda)}{\lambda}=(-r(\zeta)+\lambda)\,C_{\sigma,q}(\zeta;\lambda),$$

we find

$$\begin{aligned} D_{\sigma,q}\psi_{21}(\zeta,\lambda) &= \sum_{n=1}^{\infty}\left\{D_{\sigma,q}U_{n+1,1}(\zeta,\lambda) - D_{\sigma,q}U_{n,1}(\zeta,\lambda)\right\} \\ &= (-r(\zeta)+\lambda)\sum_{n=1}^{\infty}\left\{U_{n,1}(\zeta,\lambda) - U_{n-1,1}(\zeta,\lambda)\right\} \\ &= (-r(\zeta)+\lambda)\,\psi_{22}(\zeta,\lambda). \end{aligned}$$

The validity of the other equation in (7) is proved similarly. Moreover, ψ_2 satisfies (8)–(9). Consequently, we conclude that

$$\psi(\zeta,\lambda) = \begin{cases} \psi_1(\zeta,\lambda), & \zeta \in I_1, \\ \psi_2(\zeta,\lambda), & \zeta \in (d,\infty), \end{cases} \tag{12}$$

satisfies (2)–(4). □

Likewise, we can obtain the following theorem.

Theorem 2. *For any $\lambda \in \mathbb{C}$, equation* (1) *has a solution*

$$\chi(\zeta,\lambda) = \begin{cases} \chi_1(\zeta,\lambda), & \zeta \in I_1, \\ \chi_2(\zeta,\lambda), & \zeta \in (d,\infty), \end{cases} \tag{13}$$

satisfying conditions (3)–(5).

Write

$$D_{\max} = \left\{ u \in H : \begin{array}{c} u_1 \text{ and } u_2 \text{ are continuous on } [\sigma_0,d)\cup(d,\varrho^{-1}(a)] \\ \text{one-sided limits } u_1(d\pm) \text{ and } u_2\left(\varrho^{-1}(d\pm)\right) \text{ exist and finite,} \\ u_1(d-) - k_1u_1(d+) = 0, \\ u_2\left(\varrho^{-1}(d-)\right) - k_2u_2\left(\varrho^{-1}(d+)\right) = 0 \text{ and } \Gamma u \in H \end{array} \right\},$$

where

$$\Gamma u := \begin{cases} -\frac{1}{q}D_{-\frac{\sigma}{q},\frac{1}{q}}u_2 + p(\zeta)\,u_1, \\ D_{\sigma,q}u_1 + r(\zeta)\,u_2, \end{cases} \quad \text{and} \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Then the *maximal operator* $\mathcal{L}_{\max}$ on is defined by $\mathcal{L}_{\max}u = \Gamma u$, $u \in D_{\max}$.

Green's formula is defined as

$$\begin{aligned} &\int_{\sigma_0}^{a}\left[(\Gamma u, v)_{\mathbb{C}^2} - (u,\Gamma v)_{\mathbb{C}^2}\right]d_{\sigma,q}\zeta \\ &\qquad = W_{\sigma,q}(u,\overline{v})(a) - W_{\sigma,q}(u,\overline{v})(d+) + W_{\sigma,q}(u,\overline{v})(d-) - W_{\sigma,q}(u,\overline{v})(\sigma_0), \end{aligned} \tag{14}$$

where $W_{\sigma,q}(u,\overline{v})(\zeta) := u_1(\zeta)\overline{v_2(\varrho^{-1}(\zeta))} - u_2(\varrho^{-1}(\zeta))\overline{v_1(\zeta)}$, and

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},\ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad u, v \in D_{\max}.$$

Let ψ and χ be the solutions of (1) satisfying

$$\begin{aligned} &\psi_1(\sigma_0,\lambda) = -\sin\alpha, \quad \psi_2(\sigma_0,\lambda) = \cos\alpha, \\ &\chi_1(\sigma_0,\lambda) = \cos\alpha, \quad \chi_2(\sigma_0,\lambda) = \sin\alpha, \end{aligned} \tag{15}$$

$$\begin{aligned} &\psi_1(d-) - k_1\psi_1(d+) = 0, \quad \psi_2\left(\varrho^{-1}(d-)\right) - k_2\psi_2\left(\varrho^{-1}(d+)\right) = 0, \\ &\chi_1(d-) - k_1\chi_1(d+) = 0, \quad \chi_2\left(\varrho^{-1}(d-)\right) - k_2\chi_2\left(\varrho^{-1}(d+)\right) = 0, \end{aligned} \tag{16}$$

where $\alpha \in \mathbb{R}$. Then every solution Φ of equation (1) is of the form $\Phi = \chi + m\psi$ for some m, which will depend on λ. Thus Φ satisfies the following condition

$$\left(\chi_1(a,\lambda) + m\psi_1(a,\lambda)\right)\cos\beta + \left(\chi_2\left(\varrho^{-1}(a),\lambda\right) + m\psi_2\left(\varrho^{-1}(a),\lambda\right)\right)\sin\beta = 0. \tag{17}$$

Clearly, m must satisfy

$$m = -\frac{\chi_1(a,\lambda)\cot\beta + \chi_2\left(\varrho^{-1}(a),\lambda\right)}{\psi_1(a,\lambda)\cot\beta + \psi_2\left(\varrho^{-1}(a),\lambda\right)}. \tag{18}$$

As a, λ, β vary, m becomes a function of these arguments and due to χ and ψ are entire in λ, m is meromorphic in λ. Writing $z = \cot\beta$, yields

$$m = -\frac{\chi_1(a,\lambda) z + \chi_2\left(\varrho^{-1}(a),\lambda\right)}{\psi_1(a,\lambda) z + \psi_2\left(\varrho^{-1}(a),\lambda\right)}. \tag{19}$$

On the other hand, we see that

$$\chi_1(a,\lambda)\psi_2\left(\varrho^{-1}(a),\lambda\right) - \psi_1(a,\lambda)\chi_2\left(\varrho^{-1}(a),\lambda\right) = W_{\sigma,q}(\chi,\psi), \quad (a,\lambda) \neq 0.$$

From the properties of the mapping (19), we conclude that the real axis of the z-plane has as its image a circle $\Lambda(a)$ in the m-plane.

Theorem 3. *i) The center of $\Lambda(a)$ is*

$$\mu_a(\lambda) = -\frac{W_{\sigma,q}\left(\chi,\overline{\psi}\right)(a,\lambda)}{W_{\sigma,q}\left(\psi,\overline{\psi}\right)(a,\lambda)}. \tag{20}$$

ii) The radius of $\Lambda(a)$ is

$$\tau_a(\lambda) = \frac{1}{\left|W_{\sigma,q}\left(\psi,\overline{\psi}\right)(a,\lambda)\right|}. \tag{21}$$

Proof. *i)* Since

$$m(\lambda, z') = \infty \iff z' = -\frac{\psi_2\left(\varrho^{-1}(a),\lambda\right)}{\psi_1(a,\lambda)},$$

we obtain

$$\begin{aligned} \mu_a(\lambda) &= m\left(\lambda, -\overline{\frac{\psi_2\left(\varrho^{-1}(a),\lambda\right)}{\psi_1(a,\lambda)}}\right) = -\frac{\chi_1(a,\lambda)\left(-\overline{\frac{\psi_2\left(\varrho^{-1}(a),\lambda\right)}{\psi_1(a,\lambda)}}\right) + \chi_2\left(\varrho^{-1}(a),\lambda\right)}{\psi_1(a,\lambda)\left(-\overline{\frac{\psi_2\left(\varrho^{-1}(a),\lambda\right)}{\psi_1(a,\lambda)}}\right) + \psi_2\left(\varrho^{-1}(a),\lambda\right)} \\ &= -\frac{\chi_1(a,\lambda)\psi_2(\varrho^{-1}(a),\overline{\lambda}) - \psi_1(a,\overline{\lambda})\chi_2\left(\varrho^{-1}(a),\lambda\right)}{\psi_1(a,\lambda)\psi_2(\varrho^{-1}(a),\overline{\lambda}) - \psi_1(a,\overline{\lambda})\psi_2\left(\varrho^{-1}(a),\lambda\right)} = -\frac{W_{\sigma,q}(\chi,\overline{\psi})(a,\lambda)}{W_{\sigma,q}\left(\psi,\overline{\psi}\right)(a,\lambda))}. \end{aligned}$$

ii) The radius of $\Lambda(a)$ is

$$\begin{aligned} \tau_a(\lambda) &= \left|\frac{\chi_2(a,\lambda)}{\psi_2(a,\lambda)} - \frac{W_{\sigma,q}\left(\chi,\overline{\psi}\right)(a,\lambda)}{W_{\sigma,q}\left(\psi,\overline{\psi}\right)(a,\lambda)}\right| = \left|\frac{\psi_2(a,\overline{\lambda})\,W_{\sigma,q}(\chi,\psi)(a,\lambda)}{\psi_2(a,\lambda)\,W_{\sigma,q}(\psi,\overline{\psi})(a,\lambda)}\right| \\ &= \left|\frac{W_{\sigma,q}(\chi,\psi)(a,\lambda)}{W_{\sigma,q}\left(\psi,\overline{\psi}\right)(a,\lambda)}\right| = \frac{1}{\left|W_{\sigma,q}\left(\psi,\overline{\psi}\right)(a,\lambda)\right|}. \end{aligned}$$

$\square$

Theorem 4. *Let $\lambda \in \mathbb{C}$. If $\operatorname{Im}\lambda > 0$, then to the upper z-half-plane there corresponds the exterior of the circle $\Lambda(a)$.*

Proof. By (14), we infer that

$$\int_{\sigma_0}^{x} (\Gamma\psi,\chi)_{\mathbb{C}^2} d_{\sigma,q}\zeta - \int_{\sigma_0}^{x} (\psi,\Gamma\chi)_{\mathbb{C}^2} d_{\sigma,q}\zeta = \left(\lambda-\overline{\lambda}\right)\int_{\sigma_0}^{x} (\psi,\chi)_{\mathbb{C}^2} d_{\sigma,q}\zeta$$
$$= W_{\sigma,q}\left(\psi,\overline{\chi}\right)(x,\lambda) - W_{\sigma,q}\left(\psi,\overline{\chi}\right)(\sigma_0,\lambda),$$

since $\Gamma\psi = \lambda\psi$ and $\Gamma\chi = \lambda\chi$. Writing $\chi = \psi$ gives

$$\int_{\sigma_0}^{x} \|\psi\|^2_{\mathbb{C}^2} d_{\sigma,q}\zeta = \frac{1}{2i\operatorname{Im}\lambda}\left(W_{\sigma,q}\left(\psi,\overline{\psi}\right)(x,\lambda) - W_{\sigma,q}\left(\psi,\overline{\psi}\right)(\sigma_0,\lambda)\right), \tag{22}$$

where $x \in I_1$ and

$$\int_{\sigma_0}^{d} \|\psi\|^2_{\mathbb{C}^2} d_{\sigma,q}\zeta + \delta\int_{d}^{x} \|\psi\|^2_{\mathbb{C}^2} d_{\sigma,q}\zeta = \frac{1}{2i\operatorname{Im}\lambda}\left(W_{\sigma,q}\left(\psi,\overline{\psi\ x\in I_2}\right)(x,\lambda) - W_{\sigma,q}\left(\psi,\overline{\psi}\right)(\sigma_0,\lambda)\right),$$

where $x \in I_2$. Therefore, (22) yields

$$\operatorname{Im}\left\{\frac{\psi_2\left(\varrho^{-1}(a),\lambda\right)}{\psi_1(a,\lambda)}\right\} = \frac{1}{2}i\left\{-\frac{\psi_2\left(\varrho^{-1}(a),\lambda\right)}{\psi_1(a,\lambda)} + \frac{\psi_2\left(\varrho^{-1}(a),\overline{\lambda}\right)}{\psi_1\left(a,\overline{\lambda}\right)}\right\}$$
$$= \frac{1}{2}i\frac{W_{\sigma,q}\left(\overline{\psi},\psi\right)(a,\lambda)}{|\psi_1(a,\lambda)|^2} = -\frac{1}{2}i\frac{W_{\sigma,q}\left(\psi,\overline{\psi}\right)(a,\lambda)}{|\psi_1(a,\lambda)|^2}$$
$$= \frac{\operatorname{Im}\lambda}{|\psi_1(a,\lambda)|^2}\left(\int_{\sigma_0}^{d} \|\psi\|^2_{\mathbb{C}^2} d_{\sigma,q}\zeta + \delta\int_{d}^{a} \|\psi\|^2_{\mathbb{C}^2} d_{\sigma,q}\zeta\right) > 0.$$

□

Theorem 5. *Let $\lambda \in \mathbb{C}$, where $\operatorname{Im}\lambda \neq 0$. The function $\Phi = \chi + m\psi$ satisfies* (17) *if and only if m lies on the circle $\Lambda(a)$, whose equation is*

$$\lim_{a\to\infty} W_{\sigma,q}\left(\Phi,\overline{\Phi}\right)(a,\lambda) = 0. \tag{23}$$

As $a \to \infty$ either $\Lambda(a) \to \Lambda(\infty)$, a limit circle, or $\Lambda(a) \to \varsigma(\infty)$, a limit point. Every solution of equation (1) *is in $\mathcal{H}_1$ in the former case, and only one linearly independent solution is in $\mathcal{H}_1$ in the latter case. A point is on $\Lambda(\infty)$ if and only if* (23) *holds.*

Proof. We have the equalities

$$W_{\sigma,q}(\chi+m\psi,\overline{\chi+m\psi})(\sigma_0,\lambda) = W_{\sigma,q}\left(\chi,\overline{\chi}\right)(\sigma_0,\lambda) + mW_q\left(\psi,\overline{\chi}\right)(\sigma_0,\lambda)$$
$$+ \overline{m}W_{\sigma,q}\left(\chi,\overline{\psi}\right)(\sigma_0,\lambda) + |m|^2 W_{\sigma,q}\left(\psi,\overline{\psi}\right)(\sigma_0,\lambda)$$
$$= -m + \overline{m} = -2i\operatorname{Im} m.$$

From (22), we conclude that

$$\int_{\sigma_0}^{d} \|\chi+m\psi\|^2_{\mathbb{C}^2} d_{\sigma,q}\zeta + \delta\int_{d}^{a} \|\chi+m\psi\|^2_{\mathbb{C}^2} d_{\sigma,q}\zeta$$
$$= \frac{1}{2\operatorname{Im}\lambda i}\left[W_{\sigma,q}(\chi+m\psi,\overline{\chi+m\psi})(a,\lambda) + 2i\operatorname{Im} m\right]. \tag{24}$$

By Theorem 4, we see that if $\operatorname{Im} z < 0$, then m is inside $\Lambda(a)$ for $\operatorname{Im}\lambda > 0$. It follows from (19) that

$$z = -\frac{\chi_2\left(\varrho^{-1}(a),\lambda\right) + m\psi_2\left(\varrho^{-1}(a),\lambda\right)}{\chi_1(a,\lambda) + m\psi_1(a,\lambda)}.$$

Hence

$$i(z-\overline{z}) = i\left\{\begin{array}{c} -\frac{\chi_2\left(\varrho^{-1}(a),\lambda\right)+m\psi_2\left(\varrho^{-1}(a),\lambda\right)}{\chi_1(a,\lambda)+m\psi_1(a,\lambda)} \\ \\ +\frac{\overline{\chi_2\left(\varrho^{-1}(a),\lambda\right)}+\overline{m\psi_2\left(\varrho^{-1}(a),\lambda\right)}}{\overline{\chi_1(a,\lambda)}+\overline{m\psi_1(a,\lambda)}} \end{array}\right\} = i\frac{W_{\sigma,q}\left(\chi+m\psi,\overline{\chi+m\psi}\right)(a,\lambda)}{\left|\chi_1(a,\lambda)+m\psi_1(a,\lambda)\right|^2}.$$

Therefore, we get

$$\operatorname{Im} z < 0 \iff iW_q\left(\chi+m\psi,\overline{\chi+m\psi}\right)(a,\lambda) > 0. \tag{25}$$

By virtue of (24) and (25), we arrive at

$$\int_{\sigma_0}^{d} \|\Phi\|^2_{\mathbb{C}^2}\, d_{\sigma,q}\zeta + \delta\int_{d}^{a} \|\Phi\|^2_{\mathbb{C}^2}\, d_{\sigma,q}\zeta < \frac{\operatorname{Im} m}{\operatorname{Im}\lambda}. \tag{26}$$

On the other hand, $\operatorname{Im} z = 0$ is equivalent to that m is on the circle $\Lambda(a)$. Hence, we find $W_{\sigma,q}\left(\Phi,\overline{\Phi}\right)(a,\lambda) = 0$, and therefore

$$\int_{\sigma_0}^{d} \|\Phi\|^2_{\mathbb{C}^2}\, d_{\sigma,q}\zeta + \delta\int_{d}^{a} \|\Phi\|^2_{\mathbb{C}^2}\, d_{\sigma,q}\zeta = \frac{\operatorname{Im} m}{\operatorname{Im}\lambda}.$$

Moreover, if m is inside $\Lambda(a)$ and $\sigma_0 < a' < a$, then

$$\int_{\sigma_0}^{d} \|\Phi\|^2_{\mathbb{C}^2}\, d_{\sigma,q}\zeta + \delta\int_{d}^{a'} \|\Phi\|^2_{\mathbb{C}^2}\, d_{\sigma,q}\zeta < \int_{\sigma_0}^{d} \|\Phi\|^2_{\mathbb{C}^2}\, d_{\sigma,q}\zeta + \delta\int_{d}^{a} \|\Phi\|^2_{\mathbb{C}^2}\, d_{\sigma,q}\zeta < \frac{\operatorname{Im} m}{\operatorname{Im}\lambda}.$$

In other words, the circles $\Lambda(a)$ are nested.

If $\Lambda(a) \to \Lambda(\infty)$ as $a \to \infty$, then $\tau_\infty > 0$, so $\psi \in \mathcal{H}_1$ by (21). Let η be a point lying on $\Lambda(\infty)$. By (26), we see that

$$\int_{\sigma_0}^{d} \|\chi+\eta\psi\|^2_{\mathbb{C}^2}\, d_{\sigma,q}\zeta + \delta\int_{d}^{a} \|\chi+\eta\psi\|^2_{\mathbb{C}^2}\, d_{\sigma,q}\zeta < \frac{\operatorname{Im}\eta}{\operatorname{Im}\lambda},$$

and therefore $\chi+\eta\psi \in \mathcal{H}_1$. If $\Lambda(a) \to \Lambda(\infty)$ as $a \to \infty$ and $\operatorname{Im}\lambda \neq 0$, then all solutions of (1) belong to $\mathcal{H}_1$ since ψ and $\chi+\eta\psi$ are in $\mathcal{H}_1$. If $\Lambda(a) \to \varsigma(\infty)$ as $a \to \infty$, then $\lim_{a\to\infty} \tau_a(\lambda) = 0$. From (21), we deduce that $\psi \notin \mathcal{H}_1$. Hence, there is exactly one linearly independent solution from $\mathcal{H}_1$. Furthermore, m is on $\Lambda(\infty)$ if and only if

$$\int_{\sigma_0}^{d} \|\Phi\|^2_{\mathbb{C}^2}\, d_{\sigma,q}\zeta + \delta\int_{d}^{\infty} \|\Phi\|^2_{\mathbb{C}^2}\, d_{\sigma,q}\zeta = \frac{\operatorname{Im} m}{\operatorname{Im}\lambda}. \tag{27}$$

Using (24), (26), (27), we find that m is on $\Lambda(\infty)$ is equivalent to $\lim_{a\to\infty} W_{\sigma,q}(\Phi,\overline{\Phi})(a,\lambda) = 0$. □

Now, let us investigate whether the limit-circle situation occurs in the impulsive Hahn-Dirac system since the limit-circle situation does not occur in the classical Dirac system (see [17,24]).

Theorem 6. *For the impulsive Hahn-Dirac system* (1)*, there is only one linearly independent solution of space* $\mathcal{H}_1$.

Proof. Let

$$\Omega = \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} \quad \text{and} \quad \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

be two solutions of $\Gamma u = 0$, such that $W_{\sigma,q}(\Omega, \Psi)(\zeta) = \kappa \neq 0$, where $\zeta \in (d, \infty)$. Hence

$$\begin{aligned} |\kappa| &= \left|W_{\sigma,q}(\Omega,\Psi)(\zeta)\right| = \left|\Omega_1(\zeta)\Psi_2\left(\varrho^{-1}(\zeta)\right) - \Psi_1(\zeta)\Omega_2\left(\varrho^{-1}(\zeta)\right)\right| \\ &= \left|(\Theta(\zeta), \Upsilon(\zeta))_{\mathbb{C}^2}\right| \leq \|\Theta(\zeta)\|_{\mathbb{C}^2}\|\Upsilon(\zeta)\|_{\mathbb{C}^2} \leq \frac{1}{2}(\|\Theta(\zeta)\|^2_{\mathbb{C}^2} + \|\Upsilon(\zeta)\|^2_{\mathbb{C}^2}), \end{aligned}$$

where

$$\Theta(\zeta) = \begin{pmatrix} \Omega_1(\zeta) \\ \Omega_2(\varrho^{-1}(\zeta)) \end{pmatrix}, \qquad \Upsilon(\zeta) = \begin{pmatrix} \Psi_2(\varrho^{-1}(\zeta)) \\ -\Psi_1(\zeta) \end{pmatrix}.$$

From this, we infer that Θ and Υ (also Ω and Ψ) cannot both lie right in $\mathcal{H}_1$, which proves the theorem. □

Conclusion

In this paper, we have explored the impulsive Hahn-Dirac system. We began by examining the uniqueness and existence of solutions relevant to the problem. Limit-point and limit-circle singularities are defined. Finally, it is proved that the limit-circle situation occurs in the impulsive Hahn-Dirac system.

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У цій статті досліджується теорія Тітчмарша-Вейля для імпульсної системи Гана-Дірака. Отримано класифікації типу граничного кола та граничної точки для особливої точки $\zeta = \infty$ імпульсної системи Гана-Дірака. Крім того, показано, що випадок граничного кола для імпульсної системи Гана-Дірака не реалізується.

Ключові слова і фрази: різницеве рівняння, рівняння з розривами, система Гана-Дірака, теорія Тітчмарша-Вейля.