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Remarks to the growth of the maximum modulus of Dirichlet series

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For a Dirichlet series $F(s) = \sum\limits_{n=0}^{\infty} a_n \exp\{s\lambda_n\}$, $s = \sigma + it$, with the abscissa of absolute convergence $\sigma_a = A \in (-\infty, +\infty]$, let $M(\sigma, F) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$ for $\sigma < A$. By L we denote a class of continuous non-negative on $(-\infty, +\infty)$ functions α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ and $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$. It is proved, for example, that if $A = +\infty$, $p > 1$, $q > 1$, $\alpha \in L$, $\beta \in L$, $\ln \beta(x + O(1)) = (1 + o(1)) \ln \beta(x)$ as $x \to +\infty$ and $\varlimsup\limits_{n\to\infty} \frac{\ln \ln \alpha(\lambda_n)}{\ln \beta\left(\frac{1}{\lambda_n} \ln \frac{1}{|a_n|}\right)} = \eta^* > 0$, then $\varlimsup\limits_{\sigma\to+\infty} \frac{\alpha(\ln M(\beta^{-1}(q\beta(\sigma)),F))}{\alpha^p(\ln M(\sigma,F))} = +\infty$ for each $q > p^{1/\eta^*}$. Similar result is obtained for Dirichlet series with zero abscissa absolute convergence.

Key words and phrases: Dirichlet series, maximum modulus, generalized order.

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1 Introduction

Let (λ_n) be an increasing to $+\infty$ sequence of positive numbers, $\lambda_0 = 0$, and

$$F(s) = \sum_{n=0}^{\infty} a_n \exp\{s\lambda_n\}, \quad s = \sigma + it, \tag{1}$$

be a Dirichlet series with the abscissa of absolute convergence $\sigma_a = A \in (-\infty, +\infty]$. We put $M(\sigma, F) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$ for $\sigma < A$.

By L we denote a class of continuous non-negative on $(-\infty, +\infty)$ functions α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ and $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_{si}$, if $\alpha \in L$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x \to +\infty$ for each $c \in (0, +\infty)$, i.e. α is a slowly increasing function.

We remark that for every slowly increasing function α there exists a function α_1 such that

$$\alpha_1(x) = (1 + o(1))\alpha(x) \qquad \text{and} \qquad \frac{x\alpha_1'(x)}{\alpha_1(x)} \to 0 \quad \text{as} \quad x \to +\infty.$$

That is why we consider the function $\alpha \in L_{si}$ satisfying the condition $\frac{x\alpha'(x)}{\alpha(x)} \to 0$ as $x \to +\infty$. We remark also that $L_{si} \subset L^0$.

Using the properties of positive continuous functions increasing on $(-\infty, +\infty)$, the following theorems are proved in [1].

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Theorem A. *Let* $A=+\infty$, $p>1$ *and* $\alpha\in L$. *If*

$$\varlimsup_{n\to\infty}\frac{\ln\ln\alpha(\lambda_n)}{\ln\left(\frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right)}=\eta^*>0,$$

then

$$\varlimsup_{\sigma\to+\infty}\frac{\alpha(\ln M(q\sigma,F))}{\alpha^p(\ln M(\sigma,F))}=+\infty$$

for each $q>p^{1/\eta^*}$. *If the function* α *is continuously differentiable,*

$$\varkappa_n:=\frac{\ln|a_n|-\ln|a_{n+1}|}{\lambda_{n+1}-\lambda_n}\nearrow+\infty,$$

$\alpha(\lambda_{n+1})=(1+o(1))\alpha(\lambda_n)$, $\ln n=O(\lambda_{n+1})$ *as* $n\to\infty$ *and*

$$\varliminf_{n\to\infty}\frac{\ln\ln\alpha(\lambda_n)}{\ln\left(\frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right)}=\eta_*<+\infty,$$

then

$$\varliminf_{\sigma\to+\infty}\frac{\alpha(\ln M(q\sigma,F))}{\alpha^p(\ln M(\sigma,F))}=0$$

for each $q<p^{1/\eta_*}$.

Theorem B. *Let* $A=0$, $p>1$, $\alpha(e^x)\in L^0$ *and* $\ln\ln\alpha(x)=o(\ln x)$ *as* $x\to+\infty$. *If*

$$\varlimsup_{n\to\infty}\frac{\ln\ln\alpha(\lambda_n)}{\ln\frac{\lambda_n}{\ln^+|a_n|}}=\eta^*>0,$$

then

$$\varlimsup_{\sigma\uparrow 0}\frac{\alpha(\ln M(\sigma/q,F))}{\alpha^p(\ln M(\sigma,F))}=+\infty$$

for each $q>p^{1/\eta^*}$. *If the function* α *is continuously differentiable,* $\ln\alpha(x)\in L_{si}$, $\varkappa_n\nearrow 0$, $\ln\ln\alpha(\lambda_{n+1})=(1+o(1))\ln\ln\alpha(\lambda_n)$ *as* $n\to\infty$, $\varlimsup\limits_{n\to\infty}\frac{\ln\ln n}{\ln\lambda_n}<1$ *and*

$$\varliminf_{n\to\infty}\frac{\ln\ln\alpha(\lambda_n)}{\ln\frac{\lambda_n}{\ln^+|a_n|}}=\eta_*<+\infty,$$

then

$$\varliminf_{\sigma\uparrow 0}\frac{\alpha(\ln M(\sigma/q,F))}{\alpha^p(\ln M(\sigma,F))}=0$$

for each $q<p^{1/\eta_*}$.

Here we will study the behavior of

$$\frac{\alpha(\ln M(\beta^{-1}(q\beta(\sigma)),F))}{\alpha^p(\ln M(\sigma,F))}$$

provided $A=+\infty$ and

$$\frac{1}{\alpha^p(\ln M(\sigma,F))}\alpha\left(\ln M\left(-\frac{1}{\beta^{-1}(q\beta(1/|\sigma|))},F\right)\right)$$

provided $A=0$, where $\alpha\in L$ and $\beta\in L$.

Remark that for $\beta(x)=x$ from here we obtain relations considered in Theorems A and B. Choosing $\beta(x)=e^x$, from the first relation we get

$$\frac{\alpha(\ln M(\sigma+h,F))}{\alpha^p(\ln M(\sigma,F))}$$

with $h=\ln q$.

2 Entire Dirichlet series

Let us start with the following generalization of Theorem A.

Theorem 1. *Let* $A=+\infty$, $p>1$, $q>1$, $\alpha\in L$ *and* $\beta\in L$. *If* $\ln\beta(x+O(1))=(1+o(1))\ln\beta(x)$ *as* $x\to+\infty$ *and*

$$\overline{\lim_{n\to\infty}}\frac{\ln\ln\alpha(\lambda_n)}{\ln\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right)}=\eta^*>0, \tag{2}$$

then for each $q>p^{1/\eta^*}$ *we get*

$$\overline{\lim_{\sigma\to+\infty}}\frac{\alpha(\ln M(\beta^{-1}(q\beta(\sigma)),F))}{\alpha^p(\ln M(\sigma,F))}=+\infty. \tag{3}$$

If continuously differentiable functions $\alpha\in L$ *and* $\beta\in L$ *satisfy the conditions* $\ln\ln\alpha\in L_{si}$, $\ln\beta\in L^0$,

$$\frac{d\beta^{-1}(\ln^c\alpha(x))}{d\ln x}=O(1),\quad x\to+\infty, \tag{4}$$

$\varkappa_n\nearrow+\infty$, $\ln\ln\alpha(\lambda_{n+1})=(1+o(1))\ln\ln\alpha(\lambda_n)$, $\ln n=o(\lambda_n\exp\{\beta^{-1}(\ln^c\alpha(\lambda_n))\})$ *as* $n\to\infty$ *for each* $c\in(0,+\infty)$ *and*

$$\underline{\lim_{n\to\infty}}\frac{\ln\ln\alpha(\lambda_n)}{\ln\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right)}=\eta_*<+\infty, \tag{5}$$

then for each $q<p^{1/\eta_*}$ *we obtain*

$$\underline{\lim_{\sigma\to+\infty}}\frac{\alpha(\ln M(\beta^{-1}(q\beta(\sigma)),F))}{\alpha^p(\ln M(\sigma,F))}=0. \tag{6}$$

Proof. Let $q>1$, $p>1$, $\alpha\in L$ and Φ be a positive continuous function on $(-\infty,+\infty)$ increasing to $+\infty$. By [1, Theorem 1], if

$$\overline{\lim_{\sigma\to+\infty}}\frac{\alpha(\Phi(q\sigma))}{\alpha^p(\Phi(\sigma))}<\infty,$$

then

$$\overline{\lim_{\sigma\to+\infty}}\frac{\ln\ln\alpha(\Phi(\sigma))}{\ln\sigma}\le\frac{\ln p}{\ln q},$$

and if

$$\underline{\lim_{\sigma\to+\infty}}\frac{\alpha(\Phi(q\sigma))}{\alpha^p(\Phi(\sigma))}>0,$$

then

$$\underline{\lim_{\sigma\to+\infty}}\frac{\ln\ln\alpha(\Phi(\sigma))}{\ln\sigma}\ge\frac{\ln p}{\ln q}.$$

Now suppose that Φ_1 is a positive continuous function on $(-\infty,+\infty)$ increasing to $+\infty$, $\alpha\in L$, $\beta\in L$ and $q>1$. We put $\Phi(x)=\Phi_1(\beta^{-1}(x))$, and let $\beta(\sigma)=x$. Then

$$\overline{\lim_{\sigma\to+\infty}}\frac{\alpha(\Phi_1(\beta^{-1}(q\beta(\sigma))))}{\alpha^p(\Phi_1(\sigma))}=\overline{\lim_{\sigma\to+\infty}}\frac{\alpha(\Phi_1(\beta^{-1}(q\beta(\sigma))))}{\alpha^p(\Phi_1(\beta^{-1}(\beta(\sigma))))}=\overline{\lim_{x\to+\infty}}\frac{\alpha(\Phi(qx))}{\alpha^p(\Phi(x))},$$

$$\overline{\lim_{\sigma\to+\infty}}\frac{\ln\ln\alpha(\Phi(\sigma))}{\ln\sigma}=\overline{\lim_{\sigma\to+\infty}}\frac{\ln\ln\alpha(\Phi_1(\beta^{-1}(\sigma)))}{\ln\sigma}=\overline{\lim_{x\to+\infty}}\frac{\ln\ln\alpha(\Phi_1(x))}{\ln\beta(x)}$$

and

$$\varliminf_{\sigma\to+\infty}\frac{\ln\ln\alpha(\Phi(\sigma))}{\ln\sigma}=\varliminf_{\sigma\to+\infty}\frac{\ln\ln\alpha(\Phi_1(\sigma))}{\ln\beta(\sigma)}.$$

Therefore, the above result from [1] implies such statement: *if*

$$\varlimsup_{\sigma\to+\infty}\frac{\alpha(\Phi_1(\beta^{-1}(q\beta(\sigma))))}{\alpha^p(\Phi_1(\sigma))}<+\infty,$$

then

$$\varlimsup_{\sigma\to+\infty}\frac{\ln\ln\alpha(\Phi_1(\sigma))}{\ln\beta(\sigma)}\le\frac{\ln p}{\ln q},$$

and if

$$\varliminf_{\sigma\to+\infty}\frac{\alpha(\Phi_1(\beta^{-1}(q\beta(\sigma))))}{\alpha^p(\Phi_1(\sigma))}>0,$$

then

$$\varliminf_{\sigma\to+\infty}\frac{\ln\ln\alpha(\Phi_1(\sigma))}{\ln\beta(\sigma)}\ge\frac{\ln p}{\ln q}.$$

We choose $\Phi_1(\sigma)=\ln M(\sigma,F)$. Then from here it follows that if

$$\varlimsup_{\sigma\to+\infty}\frac{\ln\ln\alpha(\ln M(\sigma,F))}{\ln\beta(\sigma)}>\frac{\ln p}{\ln q},\tag{7}$$

then (3) holds, and if

$$\varliminf_{\sigma\to+\infty}\frac{\ln\ln\alpha(\ln M(\sigma,F))}{\ln\beta(\sigma)}<\frac{\ln p}{\ln q},\tag{8}$$

then (6) holds.

In view of (2) for every $\eta\in(0,\eta^*)$ there exists an increasing sequence (n_k) such that $\ln|a_{n_k}|\ge-\lambda_{n_k}\beta^{-1}\big(\ln^{1/\eta}\alpha(\lambda_{n_k})\big)$ for all k. Choosing $\sigma_k=\beta^{-1}\big(\ln^{1/\eta}\alpha(\lambda_{n_k})\big)+1$, in view of Cauchy inequality we have

$$\begin{aligned}\ln M(\sigma_k,F)&\ge\ln|a_{n_k}|+\sigma_k\lambda_{n_k}\\&\ge-\lambda_{n_k}\beta^{-1}\big(\ln^{1/\eta}\alpha(\lambda_{n_k})\big)+\lambda_{n_k}\big(\beta^{-1}(\ln^{1/\eta}\alpha(\lambda_{n_k}))+1\big)\\&=\lambda_{n_k}=\alpha^{-1}\left(\exp\{\beta^{\eta}(\sigma_k-1)\}\right),\end{aligned}$$

i.e.

$$\ln\ln\alpha(\ln M(\sigma_k,F))\ge\eta\ln\beta(\sigma_k-1),$$

whence in view of condition $\ln\beta(x+O(1))=(1+o(1))\ln\beta(x)$ as $x\to+\infty$ and of the arbitrariness of η, we get

$$\varlimsup_{\sigma\to+\infty}\frac{\ln\ln\alpha(\ln M(\sigma,F))}{\ln\beta(\sigma)}\ge\varlimsup_{k\to\infty}\frac{\ln\ln\alpha(\ln M(\sigma_k,F))}{\ln\beta(\sigma_k)}\ge\eta^*.\tag{9}$$

Therefore, if $q>p^{1/\eta^*}$, that is $\eta^*>\dfrac{\ln q}{\ln p}$, then (9) implies (7) and, thus, implies (3). The first part of Theorem 1 is proved.

For the proof of second part we use a formula for finding the lower generalized order of entire Dirichlet series. For this purpose choose

$$\alpha_0(x)\equiv\ln\ln\alpha(x),\qquad\beta_0(x)\equiv\ln\beta(x),\quad x\ge x_0.$$

Then $\alpha_0 \in L_{si}$, $\beta_0 \in L^0$, condition (4) is equivalent to the condition

$$\frac{d\beta_0^{-1}(c\alpha_0(x))}{d\ln x} = O(1), \qquad x \to +\infty$$

for each $c \in (0, +\infty)$, $\alpha_0(\lambda_{n+1}) = (1 + o(1))\alpha_0(\lambda_n)$ and $\ln n = o\big(\lambda_n \beta_0^{-1}(c\alpha_0(\lambda_n))\big)$ as $n \to \infty$ for each $c \in (0, +\infty)$.

In [2], it is proven that if these conditions hold and $\varkappa_n \nearrow +\infty$ as $n \to \infty$, then for the lower generalized order $\lambda_{\alpha_0,\beta_0}[F]$ we have

$$\lambda_{\alpha_0,\beta_0}[F] := \underline{\lim}_{\sigma \to +\infty} \frac{\alpha_0(\ln M(\sigma, F))}{\beta_0(\sigma)} = \underline{\lim}_{n \to \infty} \frac{\alpha_0(\lambda_n)}{\beta_0\left(\frac{1}{\lambda_n} \ln \frac{1}{|a_n|}\right)}.$$

Thus, equality (5) is equivalent to the equality

$$\underline{\lim}_{\sigma \to +\infty} \frac{\ln \ln \alpha(\ln M(\sigma, F))}{\ln \beta(\sigma)} = \eta_* < +\infty. \tag{10}$$

If now $q < p^{1/\eta_*}$, then (10) implies (8) and, thus, implies (6). $\square$

From the proof of Theorem 1 it follows that if

$$0 < \underline{\lim}_{\sigma \to +\infty} \frac{\alpha(\ln M(\beta^{-1}(q\beta(\sigma)), F))}{\alpha^p(\ln M(\sigma, F))} \le \overline{\lim}_{\sigma \to +\infty} \frac{\alpha(\ln M(\beta^{-1}(q\beta(\sigma)), F))}{\alpha^p(\ln M(\sigma, F))} < +\infty,$$

then there exists

$$\lim_{\sigma \to +\infty} \frac{\ln \ln \alpha(\ln M(\sigma, F))}{\ln \beta(\sigma)} = \frac{\ln p}{\ln q}.$$

For $q = e^h$, $\alpha(x) \equiv x$ and $\beta(x) \equiv e^x$, Theorem 1 implies the following statement.

Corollary 1. *Let $A = +\infty$ and $p > 1$. If*

$$\overline{\lim}_{n \to \infty} \frac{\lambda_n \ln \ln \lambda_n}{-\ln |a_n|} = \eta^* > 0 \qquad \text{and} \qquad h > \frac{\ln p}{\eta^*},$$

then

$$\overline{\lim}_{\sigma \to +\infty} \frac{\ln M(\sigma + h, F)}{\ln^p M(\sigma, F)} = +\infty.$$

If $\varkappa_n \nearrow +\infty$, $\ln \ln \lambda_{n+1} = (1 + o(1)) \ln \ln \lambda_n$, $\ln n = o(\lambda_n \ln \ln \lambda_n)$ as $n \to \infty$,

$$\underline{\lim}_{n \to \infty} \frac{\lambda_n \ln \ln \lambda_n}{-\ln |a_n|} = \eta_* < +\infty \qquad \text{and} \qquad h < \frac{\ln p}{\eta_*},$$

then

$$\underline{\lim}_{\sigma \to +\infty} \frac{\ln M(\sigma + h, F)}{\ln^p M(\sigma, F)} = 0.$$

The following statement complements Corollary 1.

Proposition 1. *If $A = +\infty$ and*

$$\overline{\lim}_{n \to \infty} \frac{\lambda_n \ln \lambda_n}{-\ln |a_n|} = \eta^* > 0,$$

then for every $h > 0$ we get

$$\tau := \overline{\lim}_{\sigma \to +\infty} \frac{\ln M(\sigma + h, F)}{\ln M(\sigma, F)} \ge e^{h\eta^*}. \tag{11}$$

Proof. For every $\eta\in(0,\eta^*)$ there exists a sequence (n_k) such that $\ln|a_{n_k}|\ge -\frac{1}{\eta}\lambda_{n_k}\ln\lambda_{n_k}$. Choosing $\sigma_k=\frac{1}{\eta}\ln\lambda_{n_k}+1$, we get $\ln M(\sigma_k,F)\ge\ln|a_{n_k}|+\sigma_k\lambda_{n_k}=\lambda_{n_k}=e^{\eta(\sigma_k-1)}$, i.e.

$$\frac{\ln\ln M(\sigma_k,F)}{\sigma_k}\ge(1+o(1))\eta\quad\text{as}\quad k\to\infty,$$

whence

$$\varlimsup_{\sigma\to+\infty}\frac{\ln\ln M(\sigma,F)}{\sigma}\ge\eta^*. \tag{12}$$

It remains to prove that (12) implies (11). Suppose, on the contrary, that $\tau<e^{h\eta^*}$. Then $\ln M(\sigma+h,F)\le\tau_1\ln M(\sigma,F)$ for every $\tau_1\in(\tau,e^{h\eta^*})$ and all $\sigma\ge\sigma_0=\sigma_0(\tau_1)$.

We put $\sigma_j=\sigma_0+jh$ and suppose that $\sigma_j\le\sigma\le\sigma_{j+1}$. Then

$$\ln M(\sigma,F)\le\ln M(\sigma_{j+1},F)\le\ln M(\sigma_j+h,F)\le\tau_1\ln M(\sigma_j,F)\le\cdots\le\tau_1^{j+1}\ln M(\sigma_0,F).$$

Since

$$j=\frac{\sigma_j-\sigma_0}{h}\le\frac{\sigma-\sigma_0}{h},$$

we have

$$\ln\ln M(\sigma,F)\le\left(\frac{\sigma-\sigma_0}{h}+1\right)\ln\tau_1+\ln\ln M(\sigma_0,F),$$

i.e. in view of (12) and of the arbitrariness of τ_1, we obtain

$$\eta^*\le\varlimsup_{\sigma\to+\infty}\frac{\ln\ln M(\sigma,F)}{\sigma}\le\frac{\ln\tau}{h},$$

what is impossible. □

Remark that the estimate (11) is sharp. Indeed, for an entire Dirichlet series

$$F(s)=\exp\{e^{s\varrho}\}=\sum_{n=0}^{\infty}\frac{1}{n!}e^{ns\varrho}$$

we have $\eta^*=\varrho$, $\ln M(\sigma,F)=e^{\varrho\sigma}$ and $\tau=e^{h\varrho}=e^{h\eta^*}$.

3 Dirichlet series with zero abscissa absolute convergence

The following analog of Theorem 1 is true.

Theorem 2. *Let $A=0$, $p>1$, $q>1$, $\alpha(e^x)\in L^0$, $\ln\beta\in L_{si}$ and*

$$\ln\beta^{-1}(\ln^c\alpha(x))=o(\ln x)\quad(x\to+\infty)$$

for each $c\in(0,+\infty)$. If

$$\varlimsup_{n\to\infty}\frac{\ln\ln\alpha(\lambda_n)}{\ln\beta\left(\dfrac{\lambda_n}{\ln^+|a_n|}\right)}=\eta^*>0, \tag{13}$$

then for each $q>p^{1/\eta^*}$ we get

$$\varlimsup_{\sigma\uparrow 0}\frac{\alpha\left(\ln M\left(-\dfrac{1}{\beta^{-1}(q\beta(1/|\sigma|))},F\right)\right)}{\alpha^p(\ln M(\sigma),F)}=+\infty. \tag{14}$$

If

$$\frac{x}{\beta^{-1}(\ln^c \alpha(x))} \uparrow +\infty, \qquad \ln\ln\alpha\left(\frac{x}{\beta^{-1}(\ln^c \alpha(x))}\right) = (1+o(1))\ln\ln\alpha(x) \quad \text{as} \quad x \to +\infty \tag{15}$$

for each $c \in (0, +\infty)$, $\varkappa_n \nearrow 0$, $\ln\ln\alpha(\lambda_{n+1}) = (1+o(1))\ln\ln\alpha(\lambda_n)$, $\ln\ln\alpha(\lambda_n) = o\Big(\ln\beta\Big(\frac{\lambda_n}{\ln n}\Big)\Big)$ *as* $n \to \infty$ *and*

$$\underline{\lim_{n\to\infty}} \frac{\ln\ln\alpha(\lambda_n)}{\ln\beta\left(\dfrac{\lambda_n}{\ln^+|a_n|}\right)} = \eta_* < +\infty, \tag{16}$$

then for each $q < p^{1/\eta_*}$ *we have*

$$\underline{\lim_{\sigma\uparrow 0}} \frac{\alpha\left(\ln M\left(-\dfrac{1}{\beta^{-1}(q\beta(1/|\sigma|))}, F\right)\right)}{\alpha^p(\ln M(\sigma), F)} = 0. \tag{17}$$

Proof. Let Φ_1 be a positive continuous function on $(-\infty, 0)$ increasing to $+\infty$, $\alpha \in L$ and $\beta \in L$. We put

$$\Phi(x) = \Phi_1\left(-\frac{1}{\beta^{-1}(x)}\right),$$

and let $\beta(1/|\sigma|) = x$. Then $x \uparrow +\infty$ as $\sigma \uparrow 0$, $\Phi(x) \uparrow +\infty$ as $x \uparrow +\infty$ and

$$\overline{\lim_{\sigma\uparrow 0}} \frac{\alpha\Big(\Phi_1\Big(-\frac{1}{\beta^{-1}(q\beta(1/|\sigma|))}\Big)\Big)}{\alpha^p(\Phi_1(\sigma))} = \overline{\lim_{\sigma\uparrow 0}} \frac{\alpha\Big(\Phi_1\Big(-\frac{1}{\beta^{-1}(q\beta(1/|\sigma|))}\Big)\Big)}{\alpha^p\Big(\Phi_1\Big(-\frac{1}{\beta^{-1}(\beta(1/|\sigma|))}\Big)\Big)} = \overline{\lim_{x\to+\infty}} \frac{\alpha(\Phi(qx))}{\alpha^p(\Phi(x))}.$$

Similarly,

$$\underline{\lim_{\sigma\uparrow 0}} \frac{\alpha\Big(\Phi_1\Big(-\frac{1}{\beta^{-1}(q\beta(1/|\sigma|))}\Big)\Big)}{\alpha^p(\Phi_1(\sigma))} = \underline{\lim_{x\to+\infty}} \frac{\alpha(\Phi(qx))}{\alpha^p(\Phi(x))}.$$

Further,

$$\overline{\lim_{x\to+\infty}} \frac{\ln\ln\alpha(\Phi(x))}{\ln x} = \overline{\lim_{x\to+\infty}} \frac{\ln\ln\alpha\big(\Phi_1\big(-1/\beta^{-1}(x)\big)\big)}{\ln x} = \overline{\lim_{\sigma\uparrow 0}} \frac{\ln\ln\alpha(\Phi_1(\sigma))}{\ln\beta(1/|\sigma|)}.$$

Therefore, as in the proof of Theorem 1, we get the following statement: *if*

$$\overline{\lim_{\sigma\uparrow 0}} \frac{\alpha\Big(\Phi_1\Big(-\frac{1}{\beta^{-1}(q\beta(1/|\sigma|))}\Big)\Big)}{\alpha^p(\Phi_1(\sigma))} < +\infty,$$

then

$$\overline{\lim_{\sigma\uparrow 0}} \frac{\ln\ln\alpha(\Phi_1(\sigma))}{\ln\beta(1/|\sigma|)} \le \frac{\ln p}{\ln q},$$

and if

$$\underline{\lim_{\sigma\uparrow 0}} \frac{\alpha\Big(\Phi_1\Big(-\frac{1}{\beta^{-1}(q\beta(1/|\sigma|))}\Big)\Big)}{\alpha^p(\Phi_1(\sigma))} > 0,$$

then

$$\underline{\lim_{\sigma\uparrow 0}} \frac{\ln\ln\alpha(\Phi_1(\sigma))}{\ln\beta(1/|\sigma|)} \ge \frac{\ln p}{\ln q}.$$

We choose $\Phi_1(\sigma) = \ln M(\sigma, F)$. Then from here it follows that if

$$\varlimsup_{\sigma\uparrow 0} \frac{\ln \ln \alpha(\ln M(\sigma, F))}{\ln \beta(1/|\sigma|)} > \frac{\ln p}{\ln q}, \tag{18}$$

then (14) holds, and if

$$\varliminf_{\sigma\uparrow 0} \frac{\ln \ln \alpha(\ln M(\sigma, F))}{\ln \beta(1/|\sigma|)} < \frac{\ln p}{\ln q}, \tag{19}$$

then (17) holds.

In view of (13) for every $\eta \in (0, \eta^*)$ there exists an increasing sequence (n_k) such that $\ln |a_{n_k}| \geq \frac{\lambda_{n_k}}{\beta^{-1}(\ln^{1/\eta} \alpha(\lambda_{n_k}))}$ for all k. Choosing $\sigma_k = -\frac{1}{2\beta^{-1}(\ln^{1/\eta} \alpha(\lambda_{n_k}))}$, in view of Cauchy inequality and of condition $\ln \beta^{-1}(\ln^c \alpha(x)) = o(\ln x)$ as $x \to +\infty$ for each $c \in (0, +\infty)$ we have

$$\begin{aligned}\ln M(\sigma_k, F) &\geq \ln |a_{n_k}| + \sigma_k\lambda_{n_k} \geq \lambda_{n_k}\left(\frac{1}{\beta^{-1}(\ln^{1/\eta} \alpha(\lambda_{n_k}))} - |\sigma_k|\right) = \lambda_{n_k}|\sigma_k| \\ &= \alpha^{-1}\left(\exp\left\{\beta^{\eta}\left(\frac{1}{2|\sigma_k|}\right)\right\}\right)|\sigma_k| = \exp\left\{\ln \alpha^{-1}\left(\exp\left\{\beta^{\eta}\left(\frac{1}{2|\sigma_k|}\right)\right\}\right) - \ln \frac{1}{|\sigma_k|}\right\} \\ &= \exp\left\{(1+o(1)) \ln \alpha^{-1}\left(\exp\left\{\beta^{\eta}\left(\frac{1}{2|\sigma_k|}\right)\right\}\right)\right\}, \qquad k \to \infty.\end{aligned}$$

Using that $\alpha(e^x) \in L^0$ and $\ln \beta \in L_{si}$, we get $\ln \ln \alpha(\ln M(\sigma_k, F)) \geq (1 + o(1))\eta \ln \beta(1/|\sigma_k|)$, as $k \to \infty$, whence in view of the arbitrariness of η we obtain

$$\varlimsup_{\sigma\uparrow 0} \frac{\ln \ln \alpha(\ln M(\sigma, F))}{\ln \beta(1/|\sigma|)} \geq \varlimsup_{k\to\infty} \frac{\ln \ln \alpha(\ln M(\sigma_k, F))}{\ln \beta(1/|\sigma_k|)} \geq \eta^*. \tag{20}$$

Therefore, if $q > p^{1/\eta^*}$, that is $\eta^* > \dfrac{\ln q}{\ln p}$, then (20) implies (18) and, thus, implies (14). The first part of Theorem 2 is proved.

For the proof of second part as above we choose $\alpha_0(x) \equiv \ln \ln \alpha(x)$ and $\beta_0(x) \equiv \ln \beta(x)$ for $x \geq x_0$. Then $\alpha_0 \in L_{si}$, $\beta_0 \in L_{si}$, conditions (15) are equivalent to the conditions

$$\frac{x}{\beta_0^{-1}(c\,\alpha_0(x))} \uparrow +\infty, \qquad \alpha_0\left(\frac{x}{\beta_0^{-1}(c\,\alpha_0(x))}\right) = (1 + o(1))\alpha_0(x) \quad \text{as} \quad x \to +\infty \tag{21}$$

for each $c \in (0, +\infty)$, $\alpha_0(\lambda_{n+1}) = (1 + o(1))\alpha_0(\lambda_n)$ and $\alpha(\lambda_n) = o\left(\beta_0\left(\frac{\lambda_n}{\ln n}\right)\right)$ as $n \to \infty$.

In [3], it is proven that if these conditions hold and $\kappa_n \nearrow 0$ as $n \to \infty$ then for the lower generalized order $\lambda^0_{\alpha_0,\beta_0}[F]$ of a Dirichlet series (1) with null abscissa of absolute convergence we have

$$\lambda^0_{\alpha_0,\beta_0}[F] := \varliminf_{\sigma\uparrow 0} \frac{\alpha_0(\ln M(\sigma, F))}{\beta_0(1/|\sigma|)} = \varliminf_{n\to\infty} \frac{\alpha_0(\lambda_n)}{\beta_0\left(\frac{\lambda_n}{\ln^+ |a_n|}\right)}.$$

Thus, equality (16) is equivalent to the equality

$$\varliminf_{\sigma\uparrow 0} \frac{\ln \ln \alpha(\ln M(\sigma, F))}{\ln \beta(1/|\sigma|)} = \eta_* < +\infty. \tag{22}$$

If now $q < p^{1/\eta_*}$, then (22) implies (19) and, thus, implies (17). The proof of Theorem 2 is complete. □

Choosing $\alpha(x) = \beta(x) = \ln x$ for $x \geq x_0$, from Theorem 2 we get the following assertion.

Corollary 2. *Let* $A = 0$ *and* $p > 1$. *If*

$$\varlimsup_{n\to\infty} \frac{\ln\ln\ln \lambda_n}{\ln\ln\left(\frac{\lambda_n}{\ln^+ |a_n|}\right)} = \eta^* > 0,$$

then

$$\varlimsup_{\sigma\uparrow 0} \frac{\ln\left(\ln M\left(-|\sigma|^q, F\right)\right)}{\ln^p(\ln M(\sigma, F))} = +\infty$$

for each $q > p^{1/\eta^*}$.

If $\varkappa_n \nearrow 0$, $\ln\ln\ln \lambda_{n+1} = (1+o(1))\ln\ln\ln \lambda_n$, $\varlimsup\limits_{n\to\infty} \dfrac{\ln\ln n}{\ln\ln \lambda_n} < 1$ *and*

$$\varliminf_{n\to\infty} \frac{\ln\ln\ln \lambda_n}{\ln\ln\left(\frac{\lambda_n}{\ln^+ |a_n|}\right)} = \eta_* < +\infty,$$

then

$$\varliminf_{\sigma\uparrow 0} \frac{\ln\left(\ln M\left(-|\sigma|^q, F\right)\right)}{\ln^p(\ln M(\sigma, F))} = 0$$

for each $q < p^{1/\eta_*}$.

Remark. From the proofs of Theorems 1 and 2 it is clear that inequalities (9) and (20) play an important role. They can also be obtained using the results of article [4].

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Для ряду Діріхле $F(s) = \sum\limits_{n=0}^{\infty} a_n \exp\{s\lambda_n\}$, $s = \sigma + it$, з абсцисою абсолютної збіжності $\sigma_a = A \in (-\infty, +\infty]$ нехай $M(\sigma, F) = \sup\{|F(\sigma+it)| : t \in \mathbb{R}\}$ для $\sigma < A$. Через L позначимо клас таких неперервних невід′ємних на $(-\infty, +\infty)$ функцій α, що $\alpha(x) = \alpha(x_0) \geq 0$, $x \leq x_0$ і $\alpha(x) \uparrow +\infty$ при $x_0 \leq x \to +\infty$. Доведено, наприклад, що якщо $A = +\infty$, $p > 1$, $q > 1$, $\alpha \in L$, $\beta \in L$, $\ln \beta(x + O(1)) = (1+o(1))\ln \beta(x)$ при $x \to +\infty$ і $\varlimsup\limits_{n\to\infty} \frac{\ln\ln \alpha(\lambda_n)}{\ln \beta\left(\frac{1}{\lambda_n} \ln \frac{1}{|a_n|}\right)} = \eta^* > 0$, то $\varlimsup\limits_{\sigma\to+\infty} \frac{\alpha(\ln M(\beta^{-1}(q\beta(\sigma)), F))}{\alpha^p(\ln M(\sigma, F))} = +\infty$ для кожного $q > p^{1/\eta^*}$. Подібний результат отримано для рядів Діріхле з нульовою абсцисою абсолютної збіжності.

Ключові слова і фрази: ряд Діріхле, максимум модуля, узагальнений порядок.