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On some approximate properties of biharmonic Poisson integrals in the integral metric

Zhyhallo K.M., Kharkevych Yu.I.

This paper is devoted to solving one of the extremal problems in the theory of approximation of functional classes by linear methods of summation of the Fourier series in the integral metric, namely, approximation of classes $L^{\psi}_{\beta,1}$ by biharmonic Poisson integrals. As a result of the research, we have found the asymptotic equalities for the approximation values of classes of (ψ,β)-differentiable functions by biharmonic Poisson integrals, that is, have found solutions of the Kolmogorov-Nikol'skii problem for biharmonic Poisson integrals on classes $L^{\psi}_{\beta,1}$ in the integral metric.

Key words and phrases: (ψ,β)-derivative, Kolmogorov-Nikol'skii problem, biharmonic Poisson integral, integral metric.

Lesya Ukrainka Volyn National University, 13 Voli ave., 43025, Lutsk, Ukraine
E-mail: zhyhallo@modulsoft.eu (Zhyhallo K.M.), kharkevich.juriy@gmail.com (Kharkevych Yu.I.)

Let L_1 be the space of summable 2π-periodic continuous functions f with norm

$$\|f\|_{L_1} = \|f\|_1 = \int_{-\pi}^{\pi} |f(x)| dx.$$

Let L_∞ be the space of measurable and essentially bounded 2π-periodic functions with norm $\|f\|_C = \operatorname*{ess\,sup}_x |f(x)|$.

If $f \in L_1$, then according to [1], the quantity

$$B_\rho(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)P(\rho,t)dt, \quad 0 \le \rho < 1, \tag{1}$$

is called the biharmonic Poisson integral of the function f, where

$$P(\rho,t) = \frac{\left(1-\rho^2\right)^2(1-\rho\cos t)}{\left(1-2\rho\cos t+\rho^2\right)^2} \tag{2}$$

is its kernel. Having expanded the right-hand side of (2) into the Fourier series and having put at the same time $\rho = e^{-\frac{1}{\delta}}$, $\delta > 0$, everywhere and hereafter (see, e.g., [2]), it is convenient to write the biharmonic Poisson integral (1) as

$$B_\rho(f;x) := B_\delta(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\left\{\frac{1}{2} + \sum_{k=1}^{\infty}\left(1+\frac{k}{2}\left(1-e^{-\frac{2}{\delta}}\right)\right)e^{-\frac{k}{\delta}}\cos kt\right\}dt. \tag{3}$$

O.I. Stepanets [3, Ch. III] proposed the classification of periodic functions based on the concept of (ψ,β)-derivatives. Let $\psi(k)$ be an arbitrary fixed function of the natural argument

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and β be a fixed real number. If the series

$$\sum_{k=1}^{\infty}\frac{1}{\psi\left(k\right)}\left(a_k\cos\left(kx+\frac{\pi\beta}{2}\right)+b_k\sin\left(kx+\frac{\pi\beta}{2}\right)\right)$$

is the Fourier series of some function $\varphi\in L_1$, then this function is called the (ψ,β)-derivative of the function f and is denoted by $f_\beta^\psi(x)$, namely $\varphi(x)=f_\beta^\psi(x)$. The set of all functions f satisfying this condition is denoted by L_β^ψ. If $f\in L_\beta^\psi$ and $f_\beta^\psi\in\mathfrak{N}$, where $\mathfrak{N}$ is some subset of functions from L_1, then we write that $f\in L_\beta^\psi\mathfrak{N}$. Further, we assume that $L_\beta^\psi\cap C=C_\beta^\psi$ and $L_\beta^\psi\mathfrak{N}\cap C=C_\beta^\psi\mathfrak{N}$. Let us consider the unit balls

$$U_\infty^0=\left\{\varphi\in L_\infty:\|\varphi\|_\infty\le 1,\ \varphi\perp 1\right\},\qquad U_1^0=\left\{\varphi\in L_1:\|\varphi\|_1\le 1,\ \varphi\perp 1\right\}$$

in spaces L_∞ and L_1, respectively. Then the sets $L_\beta^\psi U_1^0$, $C_\beta^\psi U_\infty^0$ are denoted by $L_{\beta,1}^\psi$ and $C_{\beta,\infty}^\psi$, respectively.

If $\psi(k)=k^{-r}$, $r>0$, then according to [4, Ch. I], $C_\beta^\psi=W_\beta^r$ and $f_\beta^\psi=f_\beta^{(r)}$ is the (r,β)-derivative in the sense of Weyl-Nagy. If $r\in\mathbb{N}$ and $r=\beta$, then $W_\beta^r=W^r$.

The set of positive continuous and convex downward functions $\psi(u)$, $u\ge 1$, that satisfy the condition $\lim\limits_{u\to\infty}\psi\left(u\right)=0$ is denoted by $\mathfrak{M}$ [3, p. 159]. Let $\mathfrak{M}_0$ be the subset of the functions from $\mathfrak{M}$ defined as follows

$$\mathfrak{M}_0=\left\{\psi\in\mathfrak{M}:\ 0<\frac{t}{\eta(t)-t}\le K\ \ \forall t\ge 1\right\},\tag{4}$$

where $\eta(t)=\eta\left(\psi;t\right)=\psi^{-1}\left(\frac{1}{2}\psi(t)\right)$ and the function ψ^{-1} is the inverse function to ψ. We should note that the constant K in (4) can depend on ψ.

Following O.I. Stepanets [5], the problem of finding the asymptotic equalities for the quantities

$$\mathcal{E}\left(\mathfrak{A};B_\delta\right)_X=\sup_{f\in\mathfrak{A}}\left\|f\left(\cdot\right)-B_\delta\left(f;\cdot\right)\right\|_X\tag{5}$$

as $\delta\to\infty$ is called the Kolmogorov-Nikol'skii problem for the biharmonic Poisson integral $B_\delta\left(f;\cdot\right)$ and the class of functions $\mathfrak{A}$ in the metric of space X.

The problem of type (5) for the biharmonic Poisson integral was first solved by S. Kaniev [1]. Later, his research was continued in the papers [6–11] for various classes of functions in the uniform metric. The Kolmogorov-Nikol'skii problem (5) on the classes of (ψ,β)-differentiable functions for the biharmonic Poisson integral in the integral metric has not been solved yet. That is why the purpose of this paper is to study the approximate properties of the biharmonic Poisson integral on the classes $L_{\beta,1}^\psi$ for an arbitrary real value of β in the integral metric.

The following statement is true.

Theorem 1. *Let $\psi\in\mathfrak{M}_0$, the function $g(u)=u^2\psi(u)$ is convex downward for all $u\in[1,\infty)$ and*

$$\lim_{\delta\to\infty}\int_1^\delta\frac{g(u)}{u}du=\infty,\tag{6}$$

$$\lim_{u\to\infty}g(u)=K<\infty.\tag{7}$$

Then for $\delta\to\infty$ and $\sin\frac{\beta\pi}{2}\neq 0$ the following asymptotic equality holds

$$\mathcal{E}\left(L_{\beta,1}^\psi;B_\delta\right)_1=\frac{1}{\pi}\left|\sin\frac{\beta\pi}{2}\right|\frac{1}{\delta^2}\int_1^\delta\frac{g(u)}{u}du+O\left(\frac{1}{\delta^2}\right).\tag{8}$$

Proof. Similarly to [12] (formula (7)), it is convenient to introduce the summing function

$$\tau(u) = \tau_\delta(u;\psi) = \begin{cases} \left[1 - \left(1 + \frac{\delta}{2}\left(1 - e^{-\frac{2}{\delta}}\right) u\right) e^{-u}\right] \frac{\psi(1)}{\psi(\delta)}, & 0 \le u \le \frac{1}{\delta}, \\ \left[1 - \left(1 + \frac{\delta}{2}\left(1 - e^{-\frac{2}{\delta}}\right) u\right) e^{-u}\right] \frac{\psi(\delta u)}{\psi(\delta)}, & u \ge \frac{1}{\delta}, \end{cases} \tag{9}$$

for the biharmonic Poisson integral (2) defined by relation (3). According to [13], the Fourier transform

$$\widehat{\tau}(t) = \widehat{\tau}_\delta(t) = \frac{1}{\pi}\int_0^\infty \tau(u)\cos\left(ut + \frac{\beta\pi}{2}\right) du \tag{10}$$

of the function (9) is summable on the whole real axis. That is, the integral

$$\int_{-\infty}^{\infty} \left| \frac{1}{\pi}\int_0^\infty \tau(u)\cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt \tag{11}$$

is convergent.

Taking into account summability on the $\mathbb{R}$ of the function (10), let us further consider the function

$$I^\psi_{\beta,1}(\delta, x) = \int_{-\infty}^{\infty} f^\psi_\beta\left(x + \frac{t}{\delta}\right)\widehat{\tau}(t)dt, \quad \delta > 0, \ x \in \mathbb{R}, \tag{12}$$

for all $f \in L^\psi_{\beta,1}$, where the improper integral should be understood as the limit of integrals over symmetric intervals that have the ability to expand.

Since the periodicity of the function (12) is obvious, then using the generalized Minkowski inequality [14, p. 299], we obtain

$$\begin{aligned} \int_{-\pi}^{\pi} \left| I^\psi_{\beta,1}(\delta, x)\right| dx &\le \int_{-\infty}^{\infty}\int_{-\pi}^{\pi} \left|\widehat{\tau}(t)\right| \left| f^\psi_\beta\left(x + \frac{t}{\delta}\right)\right| dx\, dt \\ &= \int_{-\infty}^{\infty} \left|\widehat{\tau}(t)\right| \int_{-\pi}^{\pi} \left| f^\psi_\beta\left(x + \frac{t}{\delta}\right)\right| dx\, dt < \infty. \end{aligned}$$

Thus, we have shown the convergence of the function $I^\psi_{\beta,1}(\delta, x)$. Using the methods from [3], we can find the Fourier coefficients of the function $I^\psi_{\beta,1}(\delta, x)$, namely

$$a_k\left(I^\psi_{\beta,1}(\delta, x)\right) = \frac{1}{\psi(k)}\tau\left(\frac{k}{\delta}\right) a_k(f), \qquad b_k\left(I^\psi_{\beta,1}(\delta, x)\right) = \frac{1}{\psi(k)}\tau\left(\frac{k}{\delta}\right) b_k(f),$$

where the function $\tau\left(\frac{k}{\delta}\right)$ is given by relation (9) and coefficients $a_k(f)$ and $b_k(f)$ are the Fourier coefficients of the function $f \in L^\psi_{\beta,1}$. Thus, according to the notations of the paper [15], the Fourier series of the function $I^\psi_{\beta,1}(\delta, x)$ takes the form

$$S\left[I^\psi_{\beta,1}(\delta, x)\right] = S\left[\int_{-\infty}^{\infty} f^\psi_\beta\left(x + \frac{t}{\delta}\right)\widehat{\tau}(t)dt\right] = \sum_{k=0}^{\infty}\tau\left(\frac{k}{\delta}\right)\frac{1}{\psi(k)}\left(a_k(f)\cos kx + b_k(f)\sin kx\right).$$

As before, the function $\psi(u)$ is continuous and defined for all $u \ge 1$. At points $u = k$ this function takes the value $\psi(k)$. According to the above equality, the function $\tau(u)$ given by the relation (9), satisfies the condition

$$S\left[\int_{-\infty}^{\infty} f^\psi_\beta\left(x + \frac{t}{\delta}\right)\widehat{\tau}(t)dt\right] = \frac{1}{\psi(\delta)} S\left[f(x) - P_\delta(f;x)\right]$$

that is valid for all $f \in L^\psi_{\beta,1}$.

It follows that if the function $\tau(u)$ is given by the relation (9) and its Fourier transform is summable on the whole real axis (according to [13]), then for all $f \in L^{\psi}_{\beta,1}$ and almost all x the equality

$$\left\|f(x) - B_\delta(f;x)\right\|_1 = \left\|\psi(\delta)\int_{-\infty}^{\infty} f^{\psi}_{\beta}\left(x+\frac{t}{\delta}\right)\widehat{\tau_\delta}(t)dt\right\|_1 \tag{13}$$

holds.

Taking into account the equalities (3) and (13), similarly to the considerations of the paper [16], we can show that

$$\mathcal{E}\left(L^{\psi}_{\beta,1};B_\delta\right)_1 = \mathcal{E}\left(C^{\psi}_{\beta,\infty};B_\delta\right)_C + O\big(\psi(\delta)\big)\int_{|t|\geq\frac{\delta\pi}{2}}\left|\int_0^{\infty}\tau(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt, \tag{14}$$

where

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty};B_\delta\right)_C = \sup_{f\in C^{\psi}_{\beta,\infty}}\left\|f(x)-B_\delta(f;x)\right\|_C.$$

Using [13, formula (5)] to the first term of the right-hand side of (14), we obtain

$$\begin{aligned}\mathcal{E}\left(L^{\psi}_{\beta,1};B_\delta\right)_1 =& \frac{\psi(\delta)}{\pi}\int_{-\infty}^{\infty}\left|\int_0^{\infty}\tau(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt\\ &+O\bigg(\psi(\delta)\int_{|t|\geq\frac{\delta\pi}{2}}\left|\int_0^{\infty}\tau(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt\bigg),\quad \delta\to\infty.\end{aligned} \tag{15}$$

According to [13, equality (63)] and the notation of the function $g(u)$ from the condition of Theorem 1, we have the estimate for the first integral in the right-hand side of (15), namely

$$\begin{aligned}&\int_{-\infty}^{\infty}\left|\int_0^{\infty}\tau(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt\\ &\qquad=\left|\sin\frac{\beta\pi}{2}\right|\left(\frac{1}{\delta^2\psi(\delta)}\int_1^{\delta}\frac{g(u)}{u}du+\frac{2}{\psi(\delta)}\int_0^{\infty}\frac{g(u)}{u^3}du\right)\\ &\qquad+O\left(1+\frac{1}{\delta^2\psi(\delta)}\int_1^{\delta}\frac{g(u)}{u^2}du\right).\end{aligned} \tag{16}$$

As for the estimate of the second term in the right-hand side of (15), we can repeat the considerations given in [13]. Hence,

$$\psi(\delta)\int_{|t|\geq\frac{\delta\pi}{2}}\left|\int_0^{\infty}\tau(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt = O\left(\frac{1}{\delta^2}+\frac{1}{\delta^3}\int_1^{\delta}\frac{g(u)}{u}du\right),\quad \delta\to\infty. \tag{17}$$

By the condition of Theorem 1, the function $g(u)$ is convex downward for all $u\geq 1$, then it is obvious that

$$\frac{1}{\psi(\delta)}\int_{\delta}^{\infty}\frac{g(u)}{u^3}du = \frac{1}{\psi(\delta)}\int_{\delta}^{\infty}\frac{u^2\psi(u)}{u^3}du \leq \delta^2\int_{\delta}^{\infty}\frac{du}{u^3} = O(1) \tag{18}$$

and

$$\int_1^{\delta}\frac{g(u)}{u^2}du \leq \psi(1)\int_1^{\delta}\frac{du}{u^2} = O(1). \tag{19}$$

Using to the right-sides of the equalities (16) and (17), the relations (18), (19), (7) and (6), from (15) we obtain the validity of the formula (8). Thus, the theorem is proved. □

Remark 1. *We should note that Theorem 1 holds when the functions $\psi \in \mathfrak{M}_0$ for $u \geq 1$ have the form*

$$\psi(u) = \frac{e^{-u} + K}{u^2}, \quad K = const > 0,$$

or

$$\psi(u) = \frac{\arctan u}{u^2},$$

or

$$\psi(u) = \frac{\big(\ln(u+K)\big)^{\gamma}}{u^2}, \quad -1 \leq \gamma \leq 0, \quad K = const > 0.$$

In particular, when $\psi(u) = \frac{1}{u^2}$ we have

$$\mathcal{E}(W^2_{\beta,1}; B_\delta)_1 = \frac{1}{\pi}\left|\sin\frac{\beta\pi}{2}\right| \frac{\ln\delta}{\delta^2} + O\left(\frac{1}{\delta^2}\right), \quad \delta \to \infty, \tag{20}$$

where $W^2_{\beta,1}$ is the class of 2π-periodic functions that have an absolutely continuous derivative and $\left\|f''(x)\right\|_1 \leq 1$.

By considerations that are similar to the proof of Theorem 1, we can show that the following theorem holds.

Theorem 2. *Let $\psi \in \mathfrak{M}_0$, the function $g(u) = u^2\psi(u)$ is convex upward or downward for all $u \in [1, \infty)$ and*

$$\lim_{\delta\to\infty} \frac{1}{\delta^2\psi(\delta)} \int_1^\delta \frac{g(u)}{u} du = \infty, \qquad \lim_{u\to\infty} g(u) = \infty.$$

Then for $\delta \to \infty$ and $\sin\frac{\beta\pi}{2} \neq 0$ the asymptotic equality

$$\mathcal{E}\left(L^\psi_{\beta,1}; B_\delta\right)_1 = \frac{1}{\pi}\left|\sin\frac{\beta\pi}{2}\right| \frac{1}{\delta^2} \int_1^\delta \frac{g(u)}{u} du + O\big(\psi(\delta)\big)$$

holds.

Remark 2. *We should note that conditions of Theorem 2 are satisfied, e.g., for functions of the form*

$$\psi(u) = \frac{\big(\ln(u+K)\big)^{\gamma}}{u^2}, \quad K = const > 0, \quad \gamma > 0.$$

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Робота присвячена розв'язанню однієї з екстремальних задач теорії наближення функціональних класів лінійними методами підсумовування рядів Фур'є в інтегральній метриці, а саме, наближенню класів $L^{\psi}_{\beta,1}$ бігармонічними інтегралами Пуассона. У результаті проведених досліджень вдалося знайти асимптотичні рівності для величин наближення класів (ψ,β)-диференційовних функцій бігармонійними інтегралами Пуассона, тобто знайти розв'язки задачі Колмогорова-Нікольського для бігармонічних інтегралів Пуассона на класах $L^{\psi}_{\beta,1}$ в інтегральній метриці.

Ключові слова і фрази: (ψ,β)-похідна, задача Колмогорова-Нікольського, бігармонічний інтеграл Пуассона, інтегральна метрика.