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Some new identities for Schur polynomials

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The Schur polynomials play a central role in combinatorics, representation theory, and symmetric functions. Classical identities such as the Cauchy and Littlewood formulas establish fundamental relationships between these polynomials. In this article, we present novel generalizations of these identities for bounded and unbounded cases. Specifically, we prove that a family of Schur polynomial expansions, parameterized by arbitrary polynomial sequences, satisfies a determinant-based factorization property. This result extends the classical Cauchy identities and provides a unifying framework for bounded analogues. Additionally, we derive new bounded identities for Schur polynomials, which refine Macdonald's earlier results and incorporate constraints on partitions. These findings include determinant representations that encapsulate both classical and bounded settings, enabling further generalizations. Applications of these results to combinatorial structures and the theory of symmetric functions are also discussed.

Key words and phrases: Schur polynomial, Cauchy identity, Littlewood identity, bounded Macdonald identity.

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Introduction

Let $\mathcal{P}_n$ be the set of all partitions with a maximum length of n. A partition $\lambda = (\lambda_1, \dots, \lambda_n)$ is a sequence of nonnegative integers, which are called parts, and are ordered as $\lambda_i \geq \lambda_{i+1}$. The sum of the entries is denoted $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_n$. We introduce a partial ordering $\leq$ in $\mathcal{P}_n$ by defining $\lambda \leq \mu$ if $\lambda_i \leq \mu_i$ for all i.

The Schur polynomial corresponding to $\lambda \in \mathcal{P}_n$ is defined as the following polynomial in variables $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$:

$$s_\lambda(\boldsymbol{x}) = \frac{\det\left(x_j^{\lambda_i+n-i}\right)}{\det\left(x_j^{n-i}\right)} = \frac{\begin{vmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \dots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \dots & x_n^{\lambda_2+n-2} \\ \vdots & \vdots & \dots & \vdots \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \dots & x_n^{\lambda_n} \end{vmatrix}}{\begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \dots & x_n^{n-2} \\ \vdots & \vdots & \dots & \vdots \\ 1 & 1 & \dots & 1 \end{vmatrix}}.$$

The total degree of the Schur polynomial $s_\lambda(\boldsymbol{x})$ equals $|\lambda|$.

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The Schur polynomials are invariants of the symmetric group S_n. One can read more about symmetric functions and symmetric polynomials in books [1,2].

The classical Cauchy identities for Schur polynomials state that

$$\sum_{\lambda\in\mathcal{P}_n} s_\lambda(\boldsymbol{x})s_\lambda(\boldsymbol{y}) = \prod_{i=1}^{n}\prod_{j=1}^{m}\frac{1}{1-x_iy_j},$$

$$\sum_{\lambda\in\mathcal{P}_n} s_\lambda(\boldsymbol{x})s_{\lambda'}(\boldsymbol{y}) = \prod_{i=1}^{n}\prod_{j=1}^{m}(1+x_iy_j),$$

where $\boldsymbol{y} = (y_1, y_2, \ldots, y_m)$ and λ' is the conjugate partition (see [2] for the definition of the conjugate partition).

There are many different generalizations of these identities (see, for example, [3–6,8]).

The aim of the paper is to prove some variants of the Cauchy identities. Firstly, we prove that for an arbitrary family of polynomials $f_i = f_i(\boldsymbol{y}), i = 0, 1, \ldots$, the following identity

$$\sum_{\lambda\in\mathcal{P}_n} s_\lambda(\boldsymbol{x}) \begin{vmatrix} f_{\lambda_1} & f_{\lambda_1+1} & f_{\lambda_1+2} & \cdots & f_{\lambda_1+n-1} \\ f_{\lambda_2-1} & f_{\lambda_2} & f_{\lambda_2+1} & \cdots & f_{\lambda_2+n-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ f_{\lambda_n-(n-1)} & f_{\lambda_n-(n-2)} & f_{\lambda_n-(n-3)} & \cdots & f_{\lambda_n} \end{vmatrix} = \prod_{i=1}^{n} F(\boldsymbol{y}, x_i)$$

holds, where

$$F(\boldsymbol{y}, z) = \sum_{i=0}^{\infty} f_i z^i.$$

We prove that this identity implies the Cauchy identities.

Also, for the set of indeterminates $t_1, t_2, \ldots, t_n$ the following identity

$$\sum_{\lambda\in\mathcal{P}_n} s_\lambda \det\left(t_j^{\lambda_i+j-i}\right) = \frac{\det\left(\dfrac{x_j^{n-i}}{1-x_jt_i}\right)}{\det\left(x_j^{n-i}\right)}$$

holds. Here we take $f_i = 0$ and $t_i = 0$ for all $i < 0$.

In [7], D.E. Littlewood presented the following identity

$$\sum_{\lambda\in\mathcal{P}_n} s_\lambda(x) = \prod_{i=1}^{n}\frac{1}{1-x_i}\prod_{1\le i<j\le n}\frac{1}{1-x_ix_j}.$$

I.G. Macdonald [2, p.84] proved the following bounded analogue of the Littlewood identities [7]

$$\sum_{\lambda\in\mathcal{P}_n} s_\lambda(\boldsymbol{x}) = \frac{\det\left(x_i^{a+2n-j} - x_i^{j-1}\right)}{\prod_{i=1}^{n}(x_i-1)\prod_{1\le i<j\le n}(x_i-x_j)(x_ix_j-1)}$$

for the case $\lambda \le (a, a, \ldots, a)$.

Very recent work on bounded Littlewood identities can be found in [8].

Let

$$\sum_{i=0}^{a} f_i(\boldsymbol{y})z^i = F(\boldsymbol{y}, z, a), \quad a \in \mathbb{N}.$$

In this paper, we improve the Macdonald's result and prove the folowing two bounded identities:

$$\sum_{\substack{\lambda\in\mathcal{P}_n\\ \lambda\leq(a_1,a_2,\ldots,a_n)}} \boldsymbol{s}_\lambda\left(\boldsymbol{x}\right)\det\left(f_{\lambda_i-i+j}\right)=\frac{\det\left(x_j^{n-i}F\left(\boldsymbol{y},x_i,a_i\right)\right)}{\det\left(x_j^{n-i}\right)},\quad a_i\in\mathbb{N},$$

and

$$\sum_{\substack{\lambda\in\mathcal{P}_n\\ \lambda\leq(a_1,a_2,\ldots,a_n)}} \boldsymbol{s}_\lambda\det\left(t_j^{\lambda_i+j-i}\right)=\frac{\det\left(\dfrac{x_j^{n-i}\left(1-\left(x_jt_i\right)^{a_i+1}\right)}{1-x_it_j}\right)}{\det\left(x_j^{n-i}\right)}.$$

If $\lambda_i+j-i<0$ or $\lambda_i+j-i>a_i$, then corresponding entries of the matrices $\left(f_{\lambda_i+j-i}\right)$ and $\left(t_j^{\lambda_i+j-i}\right)$ are equal to 0.

We will also consider some specifications of these identities.

1 Unbounded identities

The symmetric group S_n acts on $\mathbb{N}^n$ by permuting coordinates. Let us define a binary relation $\sim$ on the set $\mathbb{N}^n$ as follows: for $\mu,\nu\in\mathbb{N}^n$ we have $\mu\sim\nu$, if there exists a permutation $\pi_\mu\in S_n$ such that $\mu+\delta_n=\pi_\mu\left(\nu+\delta_n\right)$. Here $\delta_n=(n-1,n-2,\ldots,1,0)$. It is easy to show that the relation $\sim$ is an equivalence relation on $\mathbb{N}^n$. Then $\mathbb{N}^n$ is a union of mutually disjoint equivalence classes $[\mu]_\sim$, namely

$$\mathbb{N}^n=\bigcup_{\mu\in\mathbb{N}^n/\sim}[\mu]_\sim,$$

where μ runs the set of representatives of the equivalence relation $\sim$.

Let $f_i=f_i(\boldsymbol{y})$, $i=0,1,\ldots$, be a polynomial family and for any $\mu\in\mathbb{N}^n$ we denote $f_\mu=f_{\mu_1}f_{\mu_2}\cdots f_{\mu_n}$.

The following lemma plays important role in the proof of our main results.

Lemma 1. *Suppose that for $\mu,\lambda\in\mathbb{N}^n$ we have $\mu\sim\lambda$ and λ is a partition. Then*

$$\sum_{\mu\in[\lambda]_\sim}\operatorname{sgn}\left(\pi_\mu\right)f_\mu=\det\left(f_{\lambda_i-i+j}\right).$$

Proof. Since $\mu=\pi_\mu(\lambda+\delta_n)-\delta_n$, we get $\mu_j=\lambda_{\pi_\mu(j)}+(n-\pi_\mu(j))-(n-j)=\lambda_{\pi_\mu(j)}-\pi_\mu(j)+j$. Then

$$\sum_{\mu\in[\lambda]_\sim}\operatorname{sgn}\left(\pi_\mu\right)f_\mu=\sum_{\pi\in S_n}\operatorname{sgn}(\pi)f_{\pi(\lambda+\delta_n)-\delta_n}=\sum_{\pi\in S_n}\operatorname{sgn}(\pi)\prod_{j=1}^n f_{\lambda_{\pi(j)}-\pi(j)+j}=\det\left(f_{\lambda_i-i+j}\right).$$

For the case $\lambda_i-i+j<0$ the corresponding entries of the determinant should be equal to zero. □

To clarify Lemma 1, we present an example.

Example 1. *For $n = 3$ and for a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ we have the following table.*

$\mu = \pi_\mu(\lambda + \delta_n) - \delta_n$	π_μ	$\mathrm{sgn}(\pi_\mu)$
$(\lambda_1, \lambda_2, \lambda_3)$	e	$+1$
$(\lambda_1, \lambda_3 - 1, \lambda_2 + 1)$	(23)	-1
$(\lambda_2 - 1, \lambda_1 + 1, \lambda_3)$	(12)	-1
$(\lambda_2 - 1, \lambda_3 - 1, \lambda_1 + 2)$	(123)	$+1$
$(\lambda_3 - 2, \lambda_1 + 1, \lambda_2 + 1)$	(132)	$+1$
$(\lambda_3 - 2, \lambda_2, \lambda_1 + 2)$	(13)	-1

Then

$$\sum_{\pi_\mu \in S_n} \mathrm{sgn}(\pi_\mu) f_{\pi_\mu(\lambda+\delta_n)-\delta_n} = f_{\lambda_1} f_{\lambda_2} f_{\lambda_3} - f_{\lambda_1} f_{\lambda_3-1} f_{\lambda_2+1} - f_{\lambda_3-2} f_{\lambda_2} f_{\lambda_1+2} - f_{\lambda_2-1} f_{\lambda_1+1} f_{\lambda_3}$$

$$+ f_{\lambda_3-2} f_{\lambda_1+1} f_{\lambda_2+1} + f_{\lambda_2-1} f_{\lambda_3-1} f_{\lambda_1+2} = \begin{vmatrix} f_{\lambda_1} & f_{\lambda_1+1} & f_{\lambda_1+2} \\ f_{\lambda_2-1} & f_{\lambda_2} & f_{\lambda_2+1} \\ f_{\lambda_3-2} & f_{\lambda_3-1} & f_{\lambda_3} \end{vmatrix} = \det\left(f_{\lambda_i-i+j}\right).$$

Note that the Lemma 1 remains true for any finite sequence of polynomials $f_0, f_1, \dots, f_a$ for $a \in \mathbb{N}$. Then for the case $\lambda_i - i + j > a$ the corresponding entries of the determinant $\det\left(f_{\lambda_i-i+j}\right)$ should be equal to zero.

Now, for any $\mu \in \mathbb{N}^n$ define the polynomial

$$\boldsymbol{S}_\mu(\boldsymbol{x}) = \frac{\det\left(x_j^{\mu_i+n-i}\right)}{\det\left(x_j^{n-i}\right)}.$$

The following lemma states that $\boldsymbol{S}_\mu(\boldsymbol{x})$ either equals zero or up to sign is equal to the Schur polynomial $\boldsymbol{s}_\lambda(\boldsymbol{x})$, where the partition λ is a representative of equivalence class to which μ belongs.

Lemma 2. *If $\mu \in \mathbb{N}^n$ is equivalent to $\lambda \in \mathcal{P}_n$ and π_μ is the corresponding permutation, then $\boldsymbol{S}_\mu(\boldsymbol{x}) = \mathrm{sgn}(\pi_\mu)\, \boldsymbol{s}_\lambda(\boldsymbol{x})$. If μ is not equivalent to any partition then $\boldsymbol{S}_\mu(\boldsymbol{x}) = 0$.*

Proof. For a partition λ the sum $\lambda + \delta_n$ consists of a strong decreasing sequence of integers $\lambda_1 + n - 1 > \lambda_2 + n - 2 > \cdots > \lambda_n$.

Since $\mu + \delta_n$ is obtained from $\lambda + \delta_n$ by permutation π_μ, the corresponding determinants are equal up to a sign to $\det(\mu + \delta_n) = \mathrm{sgn}(\pi_\mu) \det(\lambda + \delta_n) \neq 0$. Dividing the both sides by $\det\left(x_j^{n-i}\right)$, we get $\boldsymbol{S}_\mu(\boldsymbol{x}) = \mathrm{sgn}(\pi_\mu)\, \boldsymbol{s}_\lambda(\boldsymbol{x})$, as required.

Suppose now that μ does not equivalent to any partition λ. Then $\mu + \delta_n$ has two equal parts and it implies that $\det(\mu + \delta_n) = 0$. $\square$

The following theorem is the main result of the section.

Theorem 1. (i) *Let $f_i = f_i(\boldsymbol{y})$ be a polynomial family and*

$$\sum_{i=0}^{\infty} f_i z^i = F(\boldsymbol{y}, z).$$

Then the following identity holds

$$\sum_{\lambda\in\mathcal{P}_n} \boldsymbol{s}_\lambda(\boldsymbol{x}) \begin{vmatrix} f_{\lambda_1} & f_{\lambda_1+1} & f_{\lambda_1+2} & \cdots & f_{\lambda_1+n-1} \\ f_{\lambda_2-1} & f_{\lambda_2} & f_{\lambda_2+1} & \cdots & f_{\lambda_2+n-2} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ f_{\lambda_n-(n-1)} & f_{\lambda_n-(n-2)} & f_{\lambda_n-(n-3)} & \cdots & f_{\lambda_n} \end{vmatrix} = \prod_{i=1}^{n} F\left(\boldsymbol{y}, x_i\right).$$

(ii) *Let $t_1, t_2, \ldots, t_n$ be some set of variables. Then*

$$\sum_{\lambda\in\mathcal{P}_n} \boldsymbol{s}_\lambda \det\left(t_j^{\lambda_i+j-i}\right) = \frac{\det\left(\dfrac{x_j^{n-i}}{1-x_j t_i}\right)}{\det\left(x_j^{n-i}\right)}.$$

Here we take $f_i = 0$ and $t_i = 0$ for all $i < 0$.

Proof. (i) Lemma 1 and Lemma 2 imply

$$\sum_{\mu\in[\lambda]} \boldsymbol{S}_\mu(\boldsymbol{x}) f_\mu = \boldsymbol{s}\lambda(\boldsymbol{x}) \begin{vmatrix} f_{\lambda_1} & f_{\lambda_1+1} & f_{\lambda_1+2} & \cdots & f_{\lambda_1+n-1} \\ f_{\lambda_2-1} & f_{\lambda_2} & f_{\lambda_2+1} & \cdots & f_{\lambda_2+n-2} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ f_{\lambda_n-(n-1)} & f_{\lambda_n-(n-2)} & f_{\lambda_n-(n-3)} & \cdots & f_{\lambda_n} \end{vmatrix}.$$

On the one hand, we have

$$\sum_{\mu\in\mathbb{N}^n} \boldsymbol{S}_\mu(\boldsymbol{x}) f_\mu = \sum_{\lambda\in\mathcal{P}_n} \boldsymbol{s}_\lambda(\boldsymbol{x}) \begin{vmatrix} f_{\lambda_1} & f_{\lambda_1+1} & f_{\lambda_1+2} & \cdots & f_{\lambda_1+n-1} \\ f_{\lambda_2-1} & f_{\lambda_2} & f_{\lambda_2+1} & \cdots & f_{\lambda_2+n-2} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ f_{\lambda_n-(n-1)} & f_{\lambda_n-(n-2)} & f_{\lambda_n-(n-3)} & \cdots & f_{\lambda_n} \end{vmatrix}.$$

We can find this sum explicitly, namely

$$\begin{aligned} \det\left(x_i^{n-j}\right) \sum_{\mu\in\mathbb{N}^n} \boldsymbol{S}_\mu f_\mu &= \sum_{\mu_1,\cdots,\mu_n=0}^{\infty} \begin{vmatrix} x_1^{\mu_1+n-1} & x_2^{\mu_1+n-1} & \dots & x_n^{\mu_1+n-1} \\ x_1^{\mu_2+n-2} & x_2^{\mu_2+n-2} & \dots & x_n^{\mu_2+n-2} \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{\mu_n} & x_2^{\mu_n} & \dots & x_n^{\mu_n} \end{vmatrix} f_{\mu_1} f_{\mu_2} \cdots f_{\mu_n} \\ &= \begin{vmatrix} \sum\limits_{\mu_1=0}^{\infty} x_1^{\mu_1+n-1} f_{\mu_1} & \sum\limits_{\mu_1=0}^{\infty} x_2^{\mu_1+n-1} f_{\mu_1} & \dots & \sum\limits_{\mu_1=0}^{\infty} x_n^{\mu_2+n-1} f_{\mu_1} \\ \sum\limits_{\mu_2=0}^{\infty} x_1^{\mu_2+n-2} f_{\mu_2} & \sum\limits_{\mu_2=0}^{\infty} x_2^{\mu_2+n-2} f_{\mu_2} & \dots & \sum\limits_{\mu_2=0}^{\infty} x_n^{\mu_2+n-2} f_{\mu_2} \\ \vdots & \vdots & \dots & \vdots \\ \sum\limits_{\mu_n=0}^{\infty} x_1^{\mu_n} f_{\mu_n} & \sum\limits_{\mu_n=0}^{\infty} x_2^{\mu_n} f_{\mu_n} & \dots & \sum\limits_{\mu_n=0}^{\infty} x_n^{\mu_n} f_{\mu_n} \end{vmatrix} \\ &= \begin{vmatrix} x_1^{n-1} F\left(\boldsymbol{y}, x_1\right) & x_2^{n-1} F\left(\boldsymbol{y}, x_2\right) & \dots & x_n^{n-1} F\left(\boldsymbol{y}, x_n\right) \\ x_1^{n-2} F\left(\boldsymbol{y}, x_1\right) & x_2^{n-2} F\left(\boldsymbol{y}, x_2\right) & \dots & x_n^{n-2} F\left(\boldsymbol{y}, x_n\right) \\ \vdots & \vdots & \vdots & \dots \\ F\left(\boldsymbol{y}, x_1\right) & F\left(\boldsymbol{y}, x_2\right) & \dots & F\left(\boldsymbol{y}, x_n\right) \end{vmatrix} \\ &= \det\left(x_i^{n-j}\right) \prod_{j=1}^{n} F\left(\boldsymbol{y}, x_j\right). \end{aligned}$$

Therefore

$$\sum_{\lambda\in\mathcal{P}_n} \boldsymbol{s}_\lambda(\boldsymbol{x}) \begin{vmatrix} f_{\lambda_1} & f_{\lambda_1+1} & f_{\lambda_1+2} & \cdots & f_{\lambda_1+n-1} \\ f_{\lambda_2-1} & f_{\lambda_2} & f_{\lambda_2+1} & \cdots & f_{\lambda_2+n-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ f_{\lambda_n-(n-1)} & f_{\lambda_n-(n-2)} & f_{\lambda_n-(n-3)} & \cdots & f_{\lambda_n} \end{vmatrix} = \prod_{j=1}^{n} F\left(\boldsymbol{y}, x_j\right).$$

Let us prove the part (ii) of the theorem. By a similar argument we have

$$\sum_{\mu\in\mathbb{N}^n} \boldsymbol{S}_\mu \boldsymbol{t}^\mu = \sum_{\lambda\in\mathcal{P}_n} \boldsymbol{s}_\lambda \begin{vmatrix} t_1^{\lambda_1} & t_2^{\lambda_1+1} & t_3^{\lambda_1+2} & \cdots & t_n^{\lambda_1+n-1} \\ t_1^{\lambda_2-1} & t_2^{\lambda_2} & t_3^{\lambda_2+1} & \cdots & t_n^{\lambda_2+n-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ t_1^{\lambda_n-(n-1)} & t_2^{\lambda_n-(n-2)} & t_3^{\lambda_n-(n-3)} & \cdots & t_n^{\lambda_n} \end{vmatrix}.$$

On the other hand

$$\det\left(x_j^{n-i}\right) \sum_{\mu\in\mathbb{N}^n} \boldsymbol{S}_\mu \boldsymbol{t}^\mu = \sum_{\lambda_1,\cdots,\lambda_n=0}^{\infty} \begin{vmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \cdots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \cdots & x_n^{\lambda_2+n-2} \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \cdots & x_n^{\lambda_n} \end{vmatrix} t_1^{\lambda_1} t_2^{\lambda_2} \cdots t_n^{\lambda_n}$$

$$= \begin{vmatrix} \sum\limits_{\lambda_1=0}^{\infty} x_1^{\lambda_1+n-1} t_1^{\lambda_1} & \sum\limits_{\lambda_1=0}^{\infty} x_2^{\lambda_1+n-1} t_1^{\lambda_1} & \cdots & \sum\limits_{\lambda_1=0}^{\infty} x_n^{\lambda_1+n-1} t_1^{\lambda_1} \\ \sum\limits_{\lambda_2=0}^{\infty} x_1^{\lambda_2+n-2} t_2^{\lambda_2} & \sum\limits_{\lambda_2=0}^{\infty} x_2^{\lambda_2+n-2} t_2^{\lambda_2} & \cdots & \sum\limits_{\lambda_2=0}^{\infty} x_n^{\lambda_2+n-2} t_2^{\lambda_2} \\ \vdots & \vdots & \cdots & \vdots \\ \sum\limits_{\lambda_n=0}^{\infty} x_1^{\lambda_n} t_n^{\lambda_n} & \sum\limits_{\lambda_n=0}^{\infty} x_2^{\lambda_n} t_n^{\lambda_n} & \cdots & \sum\limits_{\lambda_n=0}^{\infty} x_n^{\lambda_n} t_n^{\lambda_n} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{x_1^{n-1}}{1-x_1t_1} & \dfrac{x_2^{n-1}}{1-x_2t_1} & \cdots & \dfrac{x_n^{n-1}}{1-x_nt_1} \\ \dfrac{x_1^{n-2}}{1-x_1t_2} & \dfrac{x_2^{n-2}}{1-x_2t_2} & \cdots & \dfrac{x_n^{n-2}}{1-x_nt_2} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{1}{1-x_1t_n} & \dfrac{1}{1-x_2t_n} & \cdots & \dfrac{1}{1-x_nt_n} \end{vmatrix} = \det\left(\frac{x_j^{n-i}}{1-x_jt_i}\right).$$

Thus

$$\sum_{\lambda\in\mathcal{P}_n} \boldsymbol{s}_\lambda(\boldsymbol{x}) \det\left(t_j^{\lambda_i-i+j}\right) = \frac{\det\left(\dfrac{x_j^{n-i}}{1-x_jt_i}\right)}{\det\left(x_j^{n-i}\right)}.$$

□

The theorem implies the two classical Cauchy identities. In fact, put $f_i = h_i$, where h_i is the complette symmetrical polynomial in the m variables $\boldsymbol{y} = (y_1, y_2, \ldots, y_m)$. Taking into account the identity

$$\sum_{i=0}^{\infty} h_i z^i = \prod_{i=0}^{n} \frac{1}{1-y_i z},$$

the Theorem 1 implies

$$\sum_{\lambda} s_\lambda(\boldsymbol{x}) \begin{vmatrix} h_{\lambda_1} & h_{\lambda_1+1} & h_{\lambda_1+2} & \dots & h_{\lambda_1+n-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & h_{\lambda_2+1} & \dots & h_{\lambda_2+n-2} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ h_{\lambda_n-(n-1)} & h_{\lambda_n-(n-2)} & h_{\lambda_n-(n-3)} & \dots & h_{\lambda_n} \end{vmatrix} = \prod_{i=1}^{n}\prod_{j=1}^{m} \frac{1}{1-x_i y_j}.$$

By using the Jacobi-Trudi identity

$$s_\lambda(\boldsymbol{y}) = \det\left(h_{\lambda_i-i+j}\right),$$

we get the Cauchy identity

$$\sum_{\lambda\in\mathcal{P}_n} s_\lambda(\boldsymbol{x}) s_\lambda(\boldsymbol{y}) = \prod_{i=1}^{n}\prod_{j=1}^{m} \frac{1}{1-x_i y_j}.$$

Similarly, by using the dual Jacobi-Trudi identity

$$s_{\lambda'}(\boldsymbol{y}) = \det\left(e_{\lambda_i-i+j}\right),$$

we obtain the dual Cauchy identity

$$\sum_{\lambda\in\mathcal{P}_n} s_\lambda(\boldsymbol{x}) s_{\lambda'}(\boldsymbol{y}) = \prod_{i=1}^{n}\prod_{j=1}^{m} \left(1+x_i y_j\right).$$

Let us consider some specialization of the formulas.

Example 2. *Put $f_i = p_i$, where p_i is the power sum polynomial. Then, taking into account the equality*

$$\sum_{i=0}^{\infty} p_i z^i = \sum_{i=0}^{n} \frac{1}{1-x_i z},$$

we obtain

$$\sum_{\lambda\in\mathcal{P}_n} s_\lambda(\boldsymbol{x}) \det\left(p_{\lambda_i-i+j}\right) = \prod_{j=1}^{m}\sum_{i=1}^{i} \frac{1}{1-x_i y_j}.$$

Example 3. *Put $t_1 = t_2 = \dots = t_n = z$. It is easy to see that for $\lambda_2 \neq 0$ the first two rows of the determinant $\det\left(z^{\lambda_i-i+j}\right)$ are proportional. Then for all partitions except $\lambda = (n)$ it equals to zero. For $\lambda = (n)$ we get upper-triangular matrices with the main diagonal $(z^n, 1, 1, \dots, 1)$. Thus*

$$\det\left(z^{\lambda_i-i+j}\right) = \det\left(\operatorname{diag}\left(z^n, 1, 1, \dots, 1\right)\right) = z^n.$$

Since $s_{(n)}\left(\boldsymbol{x}\right) = h_i$, then

$$\sum_{i=0}^{\infty} h_i z^i = \prod_{i=0}^{n} \frac{1}{1-x_i z}.$$

2 Bounded identities

For bounded sums the following theorem holds.

Theorem 2. *Let* $\sum_{i=0}^{a} f_i z^i = F(\boldsymbol{y}, z, a)$, $a \in \mathbb{N}$. *Then*

$$\sum_{\substack{\lambda\in\mathcal{P}_n\\ \lambda\le(a_1,a_2,\dots,a_n)}} s_\lambda\left(\boldsymbol{x}\right)\det\left(f_{\lambda_i-i+j}\right) = \frac{\det\left(x_j^{n-i}F\left(\boldsymbol{y},x_i,a_i\right)\right)}{\det\left(x_j^{n-i}\right)},$$

$$\sum_{\substack{\lambda\in\mathcal{P}_n\\ \lambda\le(a_1,a_2,\dots,a_n)}} s_\lambda\det\left(t_j^{\lambda_i+j-i}\right) = \frac{\det\left(\dfrac{x_j^{n-i}\left(1-\left(x_jt_i\right)^{a_i+1}\right)}{1-x_it_j}\right)}{\det\left(x_j^{n-i}\right)}.$$

Here f_{λ_i+j-i} *and* $t_j^{\lambda_i+j-i}$ *is equal to* 0*, if* $\lambda_i+j-i<0$ *or* $\lambda_i+j-i>a_i$*.*

Proof. We have

$$\sum_{\substack{\mu\in\mathbb{N}^n\\ \mu\le(a_1,a_2,\dots,a_n)}} S_\mu(\boldsymbol{x})f_\mu = \sum_{\lambda\in\mathcal{P}_n} s_\lambda(\boldsymbol{x})\det\left(f_{\lambda_i-i+j}\right).$$

On the other hand, we can find this sum explicitly

$$\det\left(x_j^{n-i}\right)\sum_{\substack{\mu\in\mathbb{N}^n\\ \mu\le(a_1,a_2,\dots,a_n)}} S_\mu f_\mu = \sum_{\mu_1=0}^{a_1}\sum_{\mu_2=0}^{a_2}\cdots\sum_{\mu_n=0}^{a_n}\det\left(x_j^{\mu_i+n-i}\right)f_{\mu_1}f_{\mu_2}\cdots f_{\mu_n}$$

$$=\sum_{\mu_1=0}^{a_1}\sum_{\mu_2=0}^{a_2}\cdots\sum_{\mu_n=0}^{a_n}\det\left(x_j^{\mu_i+n-i}f_{\mu_i}\right) = \det\left(\sum_{\mu_i=0}^{a_i}x_j^{\mu_i+n-i}f_{\mu_i}\right) = \det\left(x_j^{n-i}F\left(\boldsymbol{y},x_i,a_i\right)\right).$$

Thus

$$\sum_{\lambda\le(a_1,a_2,\dots,a_n)} s_\lambda(\boldsymbol{x})\det\left(f_{\lambda_i-i+j}\right) = \frac{\det\left(x_j^{n-i}F\left(\boldsymbol{y},x_i,a_i\right)\right)}{\det\left(x_j^{n-i}\right)},$$

as required. In the same way

$$\det\left(x_j^{n-i}\right)\sum_{\substack{\lambda\in\mathcal{P}_n\\ \lambda_i\le a_i}} S_\lambda t_1^{\lambda_1}\cdots t_n^{\lambda_n} = \sum_{\substack{\lambda\mathcal{P}_n\\ \lambda_i\le a_i}}\det\left(x_j^{\lambda_i+n-i}\right)t_1^{\lambda_1}t_2^{\lambda_2}\cdots t_n^{\lambda_n}$$

$$=\sum_{\substack{\lambda\\ \lambda_i\le a_i}}\det\left(x_j^{\lambda_i+n-i}t_i^{\lambda_i}\right) = \det\left(\sum_{\lambda_i=0}^{a_i}x_j^{\lambda_i+n-i}t_i^{\lambda_i}\right) = \det\left(\frac{x_j^{n-i}\left(1-\left(x_jt_i\right)^{a_i+1}\right)}{1-x_jt_i}\right)$$

$$=\begin{vmatrix} \dfrac{x_1^{n-1}\left(1-(x_1t_1)^{a_1+1}\right)}{1-x_1t_1} & \dfrac{x_2^{n-1}\left(1-(x_2t_1)^{a_1+1}\right)}{1-x_2t_1} & \dots & \dfrac{x_n^{n-1}\left(1-(x_nt_1)^{a_1+1}\right)}{1-x_nt_1}\\ \dfrac{x_1^{n-2}\left(1-(t_2x_1)^{a_2+1}\right)}{1-x_1t_2} & \dfrac{x_2^{n-2}\left(1-(t_2x_2)^{a_2+1}\right)}{1-x_2t_2} & \dots & \dfrac{x_1^{n-2}\left(1-(x_nt_2)^{a_2+1}\right)}{1-x_nt_2}\\ \dots & \dots & \dots & \dots\\ \dfrac{1-(x_1t_n)^{a_n+1}}{1-x_1t_n} & \dfrac{1-(x_2t_n)^{a_n+1}}{1-x_2t_n} & \dots & \dfrac{1-(x_nt_n)^{a_n+1}}{1-x_nt_n}\end{vmatrix}.$$

□

For the case $a_1 = a_2 = \cdots = a_n = a$ we have

$$\frac{\det\left(x_j^{n-i} F\left(\boldsymbol{y}, x_i, a\right)\right)}{\det\left(x_j^{n-i}\right)} = \frac{\det\left(x_j^{n-i}\right) \prod\limits_{i=1}^{n} F\left(\boldsymbol{y}, x_i, a\right)}{\det\left(x_j^{n-i}\right)} = \prod_{i=1}^{n} F\left(\boldsymbol{y}, x_i, a\right)$$

and

$$\sum_{\lambda \leq (a,a,\ldots,a)} \boldsymbol{s}_\lambda(\boldsymbol{x}) \det\left(f_{\lambda_i - i + j}\right) = \prod_{i=1}^{n} F\left(\boldsymbol{y}, x_i, a\right).$$

Example 4. *Put $f_i = e_i$, where e_i is the elementary symmetrical polynomial in $y_1, y_2, \ldots, y_m$. Since*

$$F(z) = \sum_{i=0}^{m} e_i z^i = \prod_{i=1}^{m} (1 + y_i z),$$

we have

$$\sum_{\lambda \leq (m,m,\ldots,m)} \boldsymbol{s}_\lambda(\boldsymbol{x}) \det\left(e_{\lambda_i - i + j}\right) = \prod_{i=1}^{n} F\left(x_i\right) = \prod_{i=1}^{n} \prod_{j=1}^{m} \left(1 + x_i y_j\right).$$

Note that here $\det\left(e_{\lambda_i - i + j}\right)$ does not equal to $\boldsymbol{s}_{\lambda'}(\boldsymbol{y})$ because the corresponding entry of the matrix $\left(e_{\lambda_i - i + j}\right)$ is zero, if $\lambda_i - i + j > m$.

For the case $a_1 = a_2 = \cdots = a_n = a$ and $t_1 = t_2 = \cdots = t_n = t$ we have

$$\frac{\det\left(\dfrac{x_j^{n-i}\left(1 - (x_j t)^{a+1}\right)}{1 - x_j t}\right)}{\det\left(x_j^{n-i}\right)} = \frac{\det\left(x_j^{n-i}\right) \prod\limits_{j=1}^{m} \frac{1 - \left(x_j t\right)^{a+1}}{1 - x_j t}}{\det\left(x_j^{n-i}\right)} = \prod_{i=1}^{n} \frac{1 - \left(x_j t\right)^{a+1}}{1 - x_j t}.$$

Thus

$$\sum_{\lambda \leq (a,a,\ldots,a)} \boldsymbol{s}_\lambda\left(\boldsymbol{x}\right) \det\left(t^{\lambda_i + j - i}\right) = \prod_{j=1}^{n} \frac{1 - \left(x_j t\right)^{a+1}}{1 - x_j t}.$$

Example 5. *Let $n = 2$ and $a = 2$. Then there exists 6 partitions λ of length 2 such that $\lambda \leq (2,2)$, namely*

$$(0,0), (1,0), (1,1), (2,0), (2,1), (2,2).$$

We have the following table.

λ	$\det\left(t^{\lambda_i + j - i}\right)$	$\boldsymbol{s}_\lambda$
$(0,0)$	$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$	*1*
$(1,0)$	$\begin{vmatrix} t & t^2 \\ 0 & 1 \end{vmatrix} = t$	$x_1 + x_2$
$(1,1)$	$\begin{vmatrix} t & t^2 \\ 1 & t \end{vmatrix} = 0$	$x_2 x_1$
$(2,0)$	$\begin{vmatrix} t^2 & 0 \\ 0 & 1 \end{vmatrix} = t^2$	${x_1}^2 + x_2 x_1 + {x_2}^2$
$(2,1)$	$\begin{vmatrix} t^2 & 0 \\ 1 & t \end{vmatrix} = t^3$	$\left(x_1 + x_2\right) x_2 x_1$
$(2,2)$	$\begin{vmatrix} t^2 & 0 \\ t & t^2 \end{vmatrix} = t^4$	${x_1}^2 {x_2}^2$

Note that for $\lambda_i + j - i < 0$ or $\lambda_i + j - i > 2$ the corresponding entries of the matrix $\left(t^{\lambda_i+j-i}\right)$ are equal to zero. Then

$$\sum_{\lambda \leq (2,2)} s_\lambda(\boldsymbol{x}) \det\left(t^{\lambda_i+j-i}\right) = 1 + (x_1+x_2)\, t + \left(x_1{}^2+x_2x_1+x_2{}^2\right) t^2 + (x_1+x_2)\, x_2x_1t^3 + x_1{}^2x_2{}^2t^4$$

$$= \left(1 + tx_1 + t^2x_1{}^2\right)\left(1 + tx_2 + t^2x_2{}^2\right) = \frac{1-(x_1t)^3}{1-x_1t} \cdot \frac{1-(x_2t)^3}{1-x_2t}.$$

For the specialization $a_1 = a_2 = \cdots = a_n = a$ and $t_1 = t_2 = \cdots = t_n = 1$ let us consider the matrix $\left(1^{\lambda_i+j-i}\right)_{i,j} = c(\lambda, a)$, where

$$c_{i,j}(\lambda, a) = \begin{cases} 0, & \text{if } \lambda_i + j - i < 0 \ \text{ or } \ \lambda_i + j - i > a, \\ 1, & \text{otherwise.} \end{cases}$$

Then

$$\sum_{\lambda \leq (a,a,\ldots,a)} s_\lambda(\boldsymbol{x}) \det\left(c(\lambda, a)\right) = \prod_{i=1}^{n} \frac{1 - x_i^{a+1}}{1 - x_i}.$$

Example 6. *Put $a = 1$. Then all partitions which are less or equal to $(1, 1, \ldots, 1)$ have the form*

$$\varepsilon^{(k)} = \Big(\underbrace{1, 1, \ldots, 1}_{n-k \text{ times}}, \underbrace{0, 0, \ldots, 0}_{k \text{ times}}\Big), \quad k \leq n.$$

The matrices $c\left(\varepsilon^{(k)}, 1\right)$ are lower-triangular ones with the main diagonal $(1, 1, 1, \ldots, 1)$. Since for all k $\det\left(c\left(\varepsilon^{(k)}, 1\right)\right) = 1$, we obtain

$$\sum_{\lambda \leq (1,1,\ldots,1)} s_\lambda(\boldsymbol{x}) = \prod_{i=1}^{n} \frac{1 - x_i^2}{1 - x_i} = \prod_{i=1}^{n} (1 + x_i).$$

The bounded Macdonald formula gives the same result.

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Бедратюк Л.П. *Нові тотожності для многочленів Шура* // Карпатські матем. публ. — 2025. — Т.17, №2. — С. 706–716.

Многочлени Шура відіграють центральну роль у комбінаториці, теорії зображень і симетричних функціях. Класичні тотожності, такі як формули Коші та Літтлвуда, встановлюють фундаментальні взаємозв'язки між цими многочленами. У цій статті ми пропонуємо нові узагальнення цих тотожностей для обмежених і необмежених випадків. Зокрема, доведено, що сімейство розкладів многочленів Шура, параметризоване довільними послідовностями многочленів, задовольняє властивість факторизації, засновану на визначниках. Цей результат розширює класичні тотожності Коші та надає уніфіковану структуру для обмежених аналогів. Крім того, отримано нові обмежені тотожності для многочленів Шура, які уточнюють результати Макдональда та враховують обмеження на розбиття. Ці результати включають представлення у вигляді визначників, які охоплюють як класичні, так і обмежені випадки, створюючи основу для подальших узагальнень. Також обговорюються застосування цих результатів до комбінаторних структур і теорії симетричних функцій.

Ключові слова і фрази: многочлен Шура, тотожність Коші, тотожність Літлвуда, обмежена тотожність Макдональдса.