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Analyzing the normalized Laplacian and Randić spectrum of cozero-divisor graph of the ring $\mathbb{Z}_n$

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In this article, we investigate the normalized Laplacian and Randić spectrum of the cozero-divisor graph of a finite commutative ring $\mathfrak{R}$ with identity $1 \neq 0$. Let $Z'(\mathfrak{R})$ be the set of non-unit and non-zero elements of ring $\mathfrak{R}$. The cozero-divisor graph of $\mathfrak{R}$, denoted by $\Gamma'(\mathfrak{R})$, is a simple undirected graph having vertex set $Z'(\mathfrak{R})$ and two distinct vertices u and v are joined by an edge if and only if $u \notin v\mathfrak{R}$ and $v \notin u\mathfrak{R}$, where $\alpha\mathfrak{R}$ is the ideal generated by the element α in $\mathfrak{R}$. Specifically, we describe the normalized Laplacian spectrum and Randić spectrum of the graph $\Gamma'(\mathbb{Z}_n)$ for various values of n.

Key words and phrases: cozero-divisor graph, normalized Laplacian spectrum, Randić spectrum, ring of integer modulo n.

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1 Introduction

Consider $\mathfrak{R}$ as a commutative ring with a non-zero identity element denoted by $1 \neq 0$. We define $\alpha\mathfrak{R}$ as the ideal generated by an element α within $\mathfrak{R}$ as follows $\alpha\mathfrak{R} = \{\alpha\beta : \beta \in \mathfrak{R}\}$. We denote the set of non-unit and non-zero elements in the ring $\mathfrak{R}$ as $Z'(\mathfrak{R})$. When dealing with a positive integer n, we employ $\mathbb{Z}_n$ to symbolize the ring of integers under modulo n.

We use V to denote the vertex set and E to represent the edge set of the graph $\mathcal{G}$, which can be denoted as $\mathcal{G} = (V, E)$. The *neighbourhood* of a vertex v is defined as the set of vertices in $\mathcal{G}$ that are connected to v, and we symbolize it as $N_{\mathcal{G}}(v)$. The number of edges that connect to a vertex $v \in V$ is denoted as $deg(v)$, which is referred to as the *degree* of the vertex v. A graph $\mathcal{G}$ is considered *s-regular* if the degree of every vertex $v \in V$ is equal to s. We represent the spectrum of any graph $\mathcal{G}$, including its eigenvalues and multiplicities, using the notation $\sigma(\mathcal{G})$. If two vertices u and v are connected in $\mathcal{G}$, we express it as $u \sim v$. The complete graph with n vertices is denoted as K_n, and the complete bipartite graph with order $u + v$ is denoted as $K_{u,v}$. Also, it is important to mention that the references [11, 17] might have more symbols and definitions that we have not defined here.

The *adjacency matrix* of a graph $\mathcal{G}$, denoted by $A(\mathcal{G})$ is an $n \times n$ matrices, defined as follows

$$A(\mathcal{G}) = (a_{ij}) = \begin{cases} 1, & v_i v_j \in E(\mathcal{G}), \\ 0, & \text{otherwise.} \end{cases}$$

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The matrices $L(\mathcal{G}) = Deg(\mathcal{G}) - A(\mathcal{G})$ and $SL(\mathcal{G}) = Deg(\mathcal{G}) + A(\mathcal{G})$ are called the *Laplacian* and *signless Laplacian matrices* of $\mathcal{G}$, respectively. Note that $Deg(\mathcal{G})$ is the diagonal matrix of vertex degrees given by $Deg(\mathcal{G}) = diag(deg(v_1), deg(v_2), \ldots, deg(v_n))$. The *signless Laplacian spectral radius* of a graph $\mathcal{G}$ is the largest eigenvalue of $SL(\mathcal{G})$. The spectrum of the Laplacian matrix and signless Laplacian matrix are known as the *Laplacian spectrum* and *signless Laplacian spectrum* $\mathcal{G}$, respectively. For more on spectrum of graphs we refer the readers to [5, 9, 13, 14, 16, 18, 19, 21].

The normalized Laplacian matrix introduced by F.R.K. Chung [10] in 1997 to study random walks, is denoted by $NL(\mathcal{G})$ and is defined as

$$NL(\mathcal{G}) = D(\mathcal{G})^{-\frac{1}{2}} L(\mathcal{G}) D(\mathcal{G})^{-\frac{1}{2}},$$

where $D(\mathcal{G})^{-\frac{1}{2}}$ is the diagonal matrix whose ith diagonal entry is $\frac{1}{\sqrt{d_i}}$. Note that $NL(\mathcal{G})$ is real symmetric positive semidefinite matrix. We order the normalized Laplacian eigenvalues of $NL(\mathcal{G})$ as $0 = \lambda_1(NL) \geq \lambda_2(NL) \geq \cdots \geq \lambda_n(NL) = 2$. In certain situations normalized Laplacian matrix is a natural tool that works better than adjacency and Laplacian matrices. More literature about $NL(\mathcal{G})$ can be found in [8, 12] and references therein.

The Randić matrix of a graph $\mathcal{G}$ is defined as

$$\mathbf{R}_{\mathcal{G}} = D(\mathcal{G})^{-\frac{1}{2}} A(\mathcal{G}) D(\mathcal{G})^{-\frac{1}{2}}.$$

This matrix is real symmetric and all its eigenvalues are real. So, we can order its eigenvalues as $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. It is easy to verify that

$$NL(\mathcal{G}) = I - \mathbf{R}_{\mathcal{G}}.$$

More about the Randić matrix can be found in [4, 6].

This research article explores a unique concept known as the *cozero-divisor graph* of the commutative ring $\mathfrak{R}$. M. Afkhami and K. Khashyarmanesh [1] introduced the concept of cozero-divisor graph. The cozero-divisor graph of the ring $\mathfrak{R}$ is denoted by $\Gamma'(\mathfrak{R})$ and is defined as the graph with vertex set $Z'(\mathfrak{R})$ and $u \notin v\mathfrak{R}$ and $v \notin u\mathfrak{R}$ if and only if two distinct vertices u and v are adjacent. For a more comprehensive exploration of the cozero-divisor graph, please consult [1–3] and the references therein. The main focus of this paper is to provide a detailed analysis of two spectral properties of the cozero-divisor graph, namely the *normalized Laplacian spectrum* and the *Randić spectrum*. These spectral characteristics offer insights into the structural and algebraic properties of the graph $\Gamma'(\mathbb{Z}_n)$, for different values of n.

P. Mathil et. al. [15] analysed the Laplacian spectrum of the cozero-divisor graph $\Gamma'(\mathbb{Z}_n)$, for $n = p^{n_1}q^{n_2}$, where $p < q$ are primes and $n_1, n_2 \in \mathbb{N}$. In this research article, we extend their study by exploring both the normalized Laplacian and Randić spectra of the cozero-divisor graph, but with a different parametrization. Specifically, we consider the case when the number n is expressed as $n = \phi_1^M \phi_2^N$, with ϕ_1 and ϕ_2 as prime numbers satisfying $\phi_1 < \phi_2$, and M, N are positive integers. It is noteworthy that the authors employed `matrixcalc.org` for the computational tasks related to characteristic polynomials and approximate eigenvalues of various matrices.

2 Preliminaries

We start with the important definitions that will be used to support the major findings.

Definition 1. *Let $\mathcal{G}$ be a graph with vertex set $\{v_1, \ldots, v_n\}$ and let $\mathcal{G}_1, \ldots, \mathcal{G}_n$ be disjoint graphs of order $n_1, \ldots, n_n$, respectively. The $\mathcal{G}$-join of graphs $\mathcal{G}_1, \ldots, \mathcal{G}_n$ is denoted by $\mathcal{G}[\mathcal{G}_1, \ldots, \mathcal{G}_\mathrm{n}]$ and is a graph obtained by replacing the vertex v_k of $\mathcal{G}$ by the graph $\mathcal{G}_k$ and joining each vertex of $\mathcal{G}_i$ to every vertex of $\mathcal{G}_j$ whenever the vertices v_i and v_j are adjacent in $\mathcal{G}$.*

This operation $\mathcal{G}[\mathcal{G}_1, \ldots, \mathcal{G}_\mathrm{m}]$ is also called generalized join graph operation [7] and $\mathcal{G}$-join operation. If $\mathcal{G} = K_2$, the K_2-join is the usual join operation, namely $\mathcal{G}_1 \nabla \mathcal{G}_2$. Herein we follow later name with notation $\mathcal{G}[\mathcal{G}_1, \ldots, \mathcal{G}_\mathrm{m}]$ and call it $\mathcal{G}$-join.

Let $\tau(n)$ denotes the number of positive divisors of a positive integer n. The number of positive integers less than or equal to n that are relatively prime to n is denoted by $\phi(n)$, which stands for *Euler's function*. We refer to n as being in *prime decomposition* if $n = {q_1}^{m_1} \ldots {q_k}^{m_k}$, where $m_1, \ldots, m_k$ are positive integers and $q_1, \ldots, q_k$ are distinct primes.

An integer d is said to be a proper divisor of n if and only if $d|n$ and $1 < d < n$. Let δ'_n be the simple graph with vertex set $\{d_1, \ldots, d_k\}$ of n, where $d_1, \ldots, d_k$ are the distinct proper divisors of n. Two vertices of graph δ'_n are adjacent if and only if $d_i \nmid d_j$ and $d_j \nmid d_i$. If $n = {p_1}^{n_1} \ldots {p_r}^{n_r}$ is a prime decomposition of n, then the order of the graph δ'_n is given by

$$|V(\delta'_n)| = \prod_{i=1}^{r} (n_i + 1) - 2.$$

For $1 \leq r \leq k$, consider the sets

$$A_{d_r} = \{x \in \mathbb{Z}_n : (x, n) = d_r\},$$

where (x, n) denotes the greatest common divisor of x and n. Also, we see that $A_{d_r} \cap A_{d_x} = \phi$, when $r \neq x$. Further, note that the sets $A_{d_1}, \ldots, A_{d_k}$ are pairwise disjoint and partitions the vertex set of $\Gamma'(\mathbb{Z}_n)$ as

$$V(\Gamma'(\mathbb{Z}_n)) = A_{d_1} \cup \cdots \cup A_{d_k}.$$

3 Normalized Laplacian spectrum of $\Gamma'(\mathbb{Z}_n)$

In this section, the normalized Laplacian spectrum of $\Gamma'(\mathbb{Z}_n)$ for different n is examined with proper divisors of n as $d_1, \ldots, d_k$. For $1 \leq r \leq k$, we give the weight $|A_{d_r}| = \phi(\frac{n}{d_r})$ to the vertex d_r of the graph δ'_n. Let $\mathbb{L}(\delta'_n)$ denotes the k^{th} order weighted normalized Laplacian matrix of δ'_n, which is defined in [20, Theorem 3.1] and is given by

$$\mathbb{L}(\delta'_n) = \begin{bmatrix} 1 & t_{1,2} & \ldots & t_{1,k} \\ t_{2,1} & 1 & \ldots & t_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ t_{k,1} & t_{k,2} & \ldots & 1 \end{bmatrix}, \tag{1}$$

where

$$t_{i,j} = \begin{cases} -\sqrt{\frac{\phi(\frac{n}{d_i})\phi(\frac{n}{d_j})}{M_{d_i} M_{d_j}}}, & d_i \sim d_j \in \delta'_n, \\ 0, & \text{otherwise}, \end{cases} \qquad 1 \leq i \neq j \leq k, \qquad M_{d_j} = \sum_{d_i \in N_{\delta'_n}(d_j)} \phi\Big(\frac{n}{d_i}\Big).$$

The vertex weighted normalized Laplacian matrix of δ'_n is $\mathbb{L}(\delta'_n)$. It can be seen that the matrices $Y(G)$ and $\mathbb{L}(\delta'_n)$ are similar and hence $\sigma(Y(G)) = \sigma(\mathbb{L}(\delta'_n))$.

Theorem 1. *The normalized Laplacian spectrum of $\Gamma'(\mathbb{Z}_n)$ is given by*

$$\sigma_{\mathscr{N}\mathscr{L}}(\Gamma'(\mathbb{Z}_n)) = \left(\bigcup_{r=1}^{t} \left(1 + 0\left(\sigma_{\mathscr{N}\mathscr{L}}(\Gamma'(A_{d_i})) \setminus \{0\}\right)\right)\right) \bigcup \sigma_{\mathscr{N}\mathscr{L}}(\mathbb{L}(\delta'_n)),$$

where $1 + 0(\sigma_{\mathscr{N}\mathscr{L}}(\Gamma'(A_{d_i})) \setminus \{0\})$ indicates that each element of the multiset

$$0(\sigma_{\mathscr{N}\mathscr{L}}(\Gamma'(A_{d_i})) \setminus \{0\})$$

has 1 *added to it.*

Proof. By using [15, Lemma 3.6], we have $\Gamma'(\mathbb{Z}_n) = \delta'_n[\Gamma'(A_{d_1}), \dots, \Gamma'(A_{d_k})]$. Thus, by using the relation $\sigma(Y(G)) = \sigma(\mathbb{L}(\delta'_n))$ and consequence of [20, Theorem 3.1], the result holds. □

By [15, Corollary 3.4], $\Gamma'(A_{d_r})$ is isomorphic to $\overline{K}_{\phi(\frac{n}{d_r})}$ for $r \in \{1, \dots, t\}$. Thus, by Theorem 1, $n - \phi(n) - 1$ normalized Laplacian eigenvalues of $\Gamma'(\mathbb{Z}_n)$ exists. Out of which $n - \phi(n) - 1 - t$ are known. The remaining t normalized Laplacian eigenvalues of $\Gamma'(\mathbb{Z}_n)$ are the roots of the characteristic polynomial of the matrix $\mathbb{L}(\delta'_n)$ given in equation (1).

Lemma 1. *Let $n = \phi_1\phi_2$ with distinct primes ϕ_1 and ϕ_2. Then the normalized Laplacian spectrum of $\Gamma'(\mathbb{Z}_n)$ is given by*

$$\begin{Bmatrix} 0 & 1 & 1 & 2 \\ 1 & \phi_2 - 2 & \phi_1 - 2 & 1 \end{Bmatrix}.$$

Proof. Let $n = \phi_1\phi_2$, where distinct primes ϕ_1 and ϕ_2 are the proper divisors of n. So, by [15, Lemma 3.6], we get $\Gamma'(\mathbb{Z}_{\phi_1\phi_2}) = \delta'_{\phi_1\phi_2}[\Gamma'(A_{\phi_1}), \Gamma'(A_{\phi_2})]$, where $\delta_{\phi_1\phi_2} = K_2$, $\Gamma'(A_{\phi_1}) = \overline{K}_{\phi(\phi_2)}$ and $\Gamma'(A_{\phi_2}) = \overline{K}_{\phi(\phi_1)}$. Thus, by using [21, Proposition 2.1] and [15, Corollary 3.4], we get

$$\Gamma'(\mathbb{Z}_{\phi_1\phi_2}) = \delta'_{\phi_1\phi_2}[\overline{K}_{\phi(\phi_2)}, \overline{K}_{\phi(\phi_1)}].$$

Also, we have $M_{\phi_1} = \phi_1 - 1$ and $M_{\phi_2} = \phi_2 - 1$. So, by the consequence of Theorem 1, the normalized Laplacian spectrum of $\Gamma'(\mathbb{Z}_{\phi_1\phi_2})$ is

$$\begin{aligned} \sigma_{\mathscr{N}\mathscr{L}}(\Gamma'(\mathbb{Z}_{\phi_1\phi_2})) = & \left((1 + 0(\sigma_{\mathscr{N}\mathscr{L}}(\Gamma'(A_{\phi_1})) \setminus \{0\})\right) \\ & \bigcup \left((1 + 0(\sigma_{\mathscr{N}\mathscr{L}}(\Gamma'(A_{\phi_2})) \setminus \{0\})\right) \bigcup \sigma_{\mathscr{N}\mathscr{L}}(\mathbb{L}(\delta'_{\phi_1\phi_2})) \\ = & \begin{Bmatrix} 1 & 1 \\ \phi_2 - 2 & \phi_1 - 2 \end{Bmatrix} \bigcup \sigma_{\mathscr{N}\mathscr{L}}(\mathbb{L}(\delta'_{\phi_1\phi_2})). \end{aligned}$$

Now, by using equation (1), the matrix $\mathbb{L}(\delta'_{\phi_1\phi_2})$ is given by

$$\mathbb{L}(\delta'_{\phi_1\phi_2}) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

which has the characteristic polynomial $x^2 - 2x$ and the eigenvalues are 0 and 2. Hence, we have the result. □

Lemma 2. *Let $n = \phi_1^2\phi_2$ with ϕ_1 and ϕ_2 as distinct primes. Then the normalized Laplacian spectrum of $\Gamma'(\mathbb{Z}_n)$ is given by*

$$\begin{Bmatrix} 1 & 1 & 1 & 1 \\ \phi(\phi_1\phi_2)-1 & \phi(\phi_1^2)-1 & \phi(\phi_2)-1 & \phi(\phi_1)-1 \end{Bmatrix}$$

and the remaining eigenvalues are the roots of the characteristic polynomial of matrix (2).

Proof. Let $n = \phi_1^2\phi_2$, where ϕ_1 and ϕ_2 are distinct primes and we see that ϕ_1, ϕ_2, ϕ_1^2 and $\phi_1\phi_2$ are the proper divisors of n. So, $\delta'_{\phi_1^2\phi_2} : \phi_1 \sim \phi_2 \sim \phi_1^2 \sim \phi_1\phi_2$. By [15, Lemma 3.6], we have

$$\Gamma'(\mathbb{Z}_{\phi_1^2\phi_2}) = \delta'_{\phi_1^2\phi_2}[\Gamma'(A_{\phi_1}), \Gamma'(A_{\phi_2}), \Gamma'(A_{\phi_1^2}), \Gamma'(A_{\phi_1\phi_2})],$$

where $\Gamma'(A_{\phi_1}) = \overline{K}_{\phi(\phi_1\phi_2)}$, $\Gamma'(A_{\phi_2}) = \overline{K}_{\phi(\phi_1^2)}$, $\Gamma'(A_{\phi_1^2}) = \overline{K}_{\phi(\phi_2)}$ and $\Gamma'(A_{\phi_1\phi_2}) = \overline{K}_{\phi(\phi_1)}$. Thus, by using [21, Proposition 2.1] and [15, Corollary 3.4], we get

$$\Gamma'(\mathbb{Z}_{\phi_1^2\phi_2}) = \delta'_{\phi_1^2\phi_2}[\overline{K}_{\phi(\phi_1\phi_2)}, \overline{K}_{\phi(\phi_1^2)}, \overline{K}_{\phi(\phi_2)}, \overline{K}_{\phi(\phi_1)}].$$

Also, the values of M_{s_r} are as follows

$$M_{\phi_1} = \phi_1^2 - \phi_1, \quad M_{\phi_2} = \phi_1(\phi_2 - 1), \quad M_{\phi_1^2} = \phi_1^2 - 1 \quad \text{and} \quad M_{\phi_1\phi_2} = \phi_2 - 1.$$

Therefore, by Theorem 1, the normalized Laplacian spectrum of $\Gamma'(\mathbb{Z}_{\phi_1^2\phi_2})$ is given by

$$\begin{aligned}\sigma_{\mathcal{NL}}(\Gamma'(\mathbb{Z}_{\phi_1^2\phi_2})) =& \Big((1 + 0(\sigma_{\mathcal{NL}}(\Gamma'(A_{\phi_1})) \setminus \{0\})\Big) \\ &\bigcup \Big((1 + 0(\sigma_{\mathcal{NL}}(\Gamma'(A_{\phi_2})) \setminus \{0\})\Big) \bigcup \Big((1 + 0(\sigma_{\mathcal{NL}}(\Gamma'(A_{\phi_1^2})) \setminus \{0\})\Big) \\ &\bigcup \Big((1 + 0(\sigma_{\mathcal{NL}}(\Gamma'(A_{\phi_1\phi_2})) \setminus \{0\})\Big) \bigcup \sigma_{\mathcal{NL}}(\mathbb{L}(\delta'_{\phi_1\phi_2})) \\ =& \begin{Bmatrix} 1 & 1 & 1 & 1 \\ \phi(\phi_1\phi_2)-1 & \phi(\phi_1^2)-1 & \phi(\phi_2)-1 & \phi(\phi_1)-1 \end{Bmatrix} \bigcup \sigma_{\mathcal{NL}}(\delta'_{\phi_1^2\phi_2}).\end{aligned}$$

Now, by using equation (1), the matrix $\mathbb{L}(\delta'_{\phi_1^2\phi_2})$ is given by

$$\mathbb{L}(\delta'_{\phi_1^2\phi_2}) = \begin{bmatrix} 1 & -\sqrt{\frac{\phi_1-1}{\phi_1}} & 0 & 0 \\ -\sqrt{\frac{\phi_1-1}{\phi_1}} & 1 & -\sqrt{\frac{1}{\phi_1+1}} & 0 \\ 0 & -\sqrt{\frac{1}{\phi_1+1}} & 1 & -\sqrt{\frac{1}{\phi_1+1}} \\ 0 & 0 & -\sqrt{\frac{1}{\phi_1+1}} & 1 \end{bmatrix}. \tag{2}$$

□

Next, we calculate the normalized Laplacian eigenvalues of $\Gamma'(\mathbb{Z}_{\phi_1^m\phi_2})$, which is the second main result of this section.

Theorem 2. *Let $n = \phi_1^m\phi_2$, where m is a positive integer and ϕ_1, ϕ_2 are distinct primes. Then the normalized Laplacian spectrum of $\Gamma'(\mathbb{Z}_{\phi_1^m\phi_2})$ consists of the eigenvalues*

$$\begin{Bmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ \phi(\phi_1^{m-1}\phi_2)-1 & \phi({\phi^{m-2}}_1\phi_2)-1 & \cdots & \phi(\phi_2)-1 & \phi(\phi_1^m)-1 & \phi(\phi_1^{m-1})-1 & \cdots & \phi(\phi_1)-1 \end{Bmatrix}$$

and the remaining eigenvalues are the eigenvalues of matrix (3).

Proof. Let $n = \phi_1^m\phi_2$, where m is a positive integer and ϕ_1, ϕ_2 are distinct primes. The proper divisors of n are $\{\phi_1, \phi_1^2, \ldots, \phi_1^m, \phi_2, \phi_1\phi_2, \phi_1^2\phi_2, \ldots, \phi_1^{m-1}\phi_2\}$. Now, by definition of δ'_n, we have the following adjacency relations:

$$\begin{aligned}
&\phi_2 \sim \phi_1^\alpha, && 1 \le \alpha \le m,\\
&\phi_1^\alpha \sim \phi_1^{x-1}\phi_2, && 1 \le \alpha \le m,\ 1 \le x \le \alpha,\\
&\phi_1^\alpha\phi_2 \sim \phi_1^{\alpha+x}, && 1 \le \alpha \le m-1,\ 1 \le x \le m-\alpha.
\end{aligned}$$

By [15, Lemma 3.6], we get

$$\Gamma'(\mathbb{Z}_{\phi_1^m\phi_2}) = \delta'_{\phi_1^m\phi_2}[\Gamma'(A_{\phi_1}), \Gamma'(A_{\phi_1^2}), \ldots, \Gamma'(A_{\phi_1^m}), \Gamma'(A_{\phi_2}), \Gamma'(A_{\phi_1\phi_2}), \ldots, \Gamma'(A_{\phi_1^{m-1}\phi_2})].$$

By using [15, Corollary 3.4], we have

$$\Gamma'(\mathbb{Z}_{\phi_1^m\phi_2}) = \delta'_{\phi_1^m\phi_2}[\overline{K}_{\phi(\phi_1^{m-1}\phi_2)}, \overline{K}_{\phi(\phi_1^{m-2}\phi_2)}, \ldots, \overline{K}_{\phi(\phi_2)}, \overline{K}_{\phi(\phi_1^m)}, \overline{K}_{\phi(\phi_1^{m-1})}, \ldots, \overline{K}_{\phi(\phi_1)}].$$

It also follows that

$$\begin{aligned}
M_{\phi_1} &= \phi(\phi_1^m) = \phi_1^m - \phi_1^{m-1},\\
M_{\phi_1^2} &= \phi(\phi_1^m) + \phi(\phi_1^{m-1}) = \phi_1^m - \phi_1^{m-2},\\
M_{\phi_1^3} &= \phi(\phi_1^m) + \phi(\phi_1^{m-1}) + \phi(\phi_1^{m-2}) = \phi_1^m - \phi_1^{m-3},\\
&\vdots\\
M_{\phi_1^m} &= \phi(\phi_1^m) + \phi(\phi_1^{m-1}) + \cdots + \phi(\phi_1^2) + \phi(\phi_1) = \phi_1^{m-1},\\
M_{\phi_2} &= \phi(\phi_1^{m-1}\phi_2) + \phi(\phi_1^{m-2}\phi_2) + \cdots + \phi(\phi_1\phi_2) + \phi(\phi_2) = \phi_1^{m-1} \cdot \phi(\phi_2),\\
M_{\phi_1\phi_2} &= \phi(\phi_1^{m-2}\phi_2) + \phi(\phi_1^{m-3}\phi_2) + \cdots + \phi(\phi_1\phi_2) + \phi(\phi_2) = \phi_1^{m-2} \cdot \phi(\phi_2),\\
M_{\phi_1^2\phi_2} &= \phi(\phi_1^{m-3}\phi_2) + \phi(\phi_1^{m-4}\phi_2) + \cdots + \phi(\phi_1\phi_2) + \phi(\phi_2) = \phi_1^{m-3} \cdot \phi(\phi_2),\\
&\vdots\\
M_{\phi_1^{m-1}\phi_2} &= \phi(\phi_2).
\end{aligned}$$

By the consequence of Theorem 1, the normalized Laplacian spectrum of $\Gamma'(\mathbb{Z}_{\phi_1^m\phi_2})$ is given by

$$\begin{aligned}
\sigma_{\mathscr{N}\mathscr{L}}(\Gamma'(\mathbb{Z}_{\phi_1^m\phi_2})) =& \Big((1+0(\sigma_{\mathscr{N}\mathscr{L}}(\Gamma'(A_{\phi_1})) \setminus \{0\})\Big)\\
&\bigcup \Big((1+0(\sigma_{\mathscr{N}\mathscr{L}}(\Gamma'(A_{\phi_1^2})) \setminus \{0\})\Big) \bigcup \cdots \bigcup \Big((1+0(\sigma_{\mathscr{N}\mathscr{L}}(\Gamma'(A_{\phi_1^m})) \setminus \{0\})\Big)\\
&\bigcup \Big((1+0(\sigma_{\mathscr{N}\mathscr{L}}(\Gamma'(A_{\phi_2})) \setminus \{0\})\Big) \bigcup \Big((1+0(\sigma_{\mathscr{N}\mathscr{L}}(\Gamma'(A_{\phi_1\phi_2})) \setminus \{0\})\Big)\\
&\bigcup \cdots \bigcup \Big((1+0(\sigma_{\mathscr{N}\mathscr{L}}(\Gamma'(A_{\phi_1^{m-1}\phi_2})) \setminus \{0\})\Big) \bigcup \sigma_{\mathscr{N}\mathscr{L}}(\mathbb{L}(\delta'_{\phi_1\phi_2}))\\
=& \left\{\begin{matrix} 1 & 1 & \cdots & 1 & 1\\ \phi(\phi_1^{m-1}\phi_2)-1 & \phi(\phi_1^{m-2}\phi_2)-1 & \cdots & \phi(\phi_2)-1 & \phi(\phi_1^m)-1 \end{matrix}\right.\\
&\left.\begin{matrix} 1 & \cdots & 1\\ \phi(\phi_1^{m-1})-1 & \cdots & \phi(\phi_1)-1 \end{matrix}\right\} \bigcup \sigma_{\mathscr{N}\mathscr{L}}(\delta'_{\phi_1^2\phi_2}).
\end{aligned}$$

Thus, the rest $2m$ normalized Laplacian eigenvalues are the latent roots of the matrix

$$\mathbb{L}(\delta'(_{\phi_1^m\phi_2})) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & B_1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & B_2 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & B_3 & A_3 & C_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & B_m & A_m & C_m & \cdots & D_m \\ B_1 & B_2 & B_3 & \cdots & B_m & 1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & A_3 & \cdots & A_m & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & C_1 & \cdots & C_m & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & D_m & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \tag{3}$$

where

$$A_2 = -\sqrt{\frac{\phi(\phi_1^{m-2})\phi(\phi_1^{m-1})}{\phi_1^{m-2}(\phi_1^m - \phi_1^{m-2})}},\quad A_3 = -\sqrt{\frac{\phi(\phi_1^{m-3})\phi(\phi_1^{m-1})}{\phi_1^{m-2}(\phi_1^m - \phi_1^{m-3})}},\quad A_m = -\sqrt{\frac{\phi(\phi_1^{m-1})}{\phi_1^{m-2}(\phi_1^m - 1)}},$$

$$B_1 = -\sqrt{\frac{\phi(\phi_1^{m-1})\phi(\phi_1^m)}{\phi_1^{m-1}(\phi_1^m - \phi_1^{m-1})}},\quad B_2 = -\sqrt{\frac{\phi(\phi_1^{m-2})\phi(\phi_1^m)}{\phi_1^{m-1}(\phi_1^m - \phi_1^{m-2})}},\quad B_3 = -\sqrt{\frac{\phi(\phi_1^{m-3})\phi(\phi_1^m)}{\phi_1^{m-1}(\phi_1^m - \phi_1^{m-3})}},$$

$$B_m = -\sqrt{\frac{\phi(\phi_1^m)}{\phi_1^{m-1}(\phi_1^m - 1)}},\quad C_1 = -\sqrt{\frac{\phi(\phi_1^{m-3})\phi(\phi_1^{m-2})}{\phi_1^{m-3}(\phi_1^m - \phi_1^{m-3})}},\quad C_m = -\sqrt{\frac{\phi(\phi_1^{m-2})}{\phi_1^{m-3}(\phi_1^m - 1)}},$$

$$D_m = -\sqrt{\frac{\phi(\phi_1)}{\phi_1^m - 1}}.$$

□

For distinct primes ϕ_1 and ϕ_2, our next result gives the normalized Laplacian spectrum of $\Gamma'(\mathbb{Z}_n)$, where $n = \phi_1^M\phi_2^N$.

Theorem 3. *Let $n = \phi_1^M\phi_2^N$, where M, N are positive integers and ϕ_1, ϕ_2 distinct primes. Then the normalized Laplacian spectrum of $\Gamma'(\mathbb{Z}_n)$ contains the eigenvalues*

$$\begin{Bmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ \phi(\phi_1^{M-1}\phi_2^N) - 1 & \phi(\phi_1^{M-2}\phi_2^N) - 1 & \cdots & \phi(\phi_2^N) - 1 & \phi(\phi_1^M\phi_2^{N-1}) - 1 & \phi(\phi_1^M\phi_2^{N-2}) - 1 \\ \cdots & 1 & 1 & 1 & \cdots & 1 \\ \cdots & \phi(\phi_1^M) - 1 & \phi(\phi_1^{M-1}\phi_2^{N-1}) - 1 & \phi(\phi_1^{M-2}\phi_2^{N-1}) - 1 & \cdots & \phi(\phi_2^{N-1}) - 1 \\ 1 & \cdots & 1 & \cdots & 1 & \\ \phi(\phi_1^{M-1}\phi_2^{N-2}) - 1 & \cdots & \phi(\phi_1^{M-1}) - 1 & \cdots & \phi(\phi_1) - 1 & \end{Bmatrix}$$

and the remaining $(M+1)(N+1) - 2$ eigenvalues are the eigenvalues of the matrix given in equation (1).

Proof. Let $n = \phi_1^M\phi_2^N$, where M, N are positive integers and ϕ_1, ϕ_2 distinct primes. The proper divisors of n are

$$\{\phi_1, \phi_1^2, \ldots, \phi_1^M, \phi_2, \phi_2^2, \ldots, \phi_2^N, \phi_1\phi_2, \phi_1^2\phi_2, \ldots, \phi_1^M\phi_2, \phi_1\phi_2^2, \\ \phi_1^2\phi_2^2, \ldots, \phi_1^M\phi_2^2, \ldots, \phi_1\phi_2^N, \phi_1^2\phi_2^N, \ldots, \phi_1^{M-1}\phi_2^N\}.$$

By using the definition of δ'_n, we have the following adjacency relations:

$$\begin{aligned} \phi_1^u &\sim \phi_2^v, \quad \text{for all } u, v,\\ \phi_1^u &\sim \phi_1^w\phi_2^x, \quad \text{for } u > w \text{ and } x > 0,\\ \phi_2^v &\sim \phi_1^u\phi_2^x, \quad \text{for } v > x \text{ and } u > 0,\\ \phi_1^w\phi_2^x &\sim \phi_1^y\phi_2^z, \quad \text{if either } w > y, x < z \text{ or } x > z, w < y. \end{aligned}$$

By using [15, Lemma 3.6], we get

$$\Gamma'(\mathbb{Z}_{\phi_1^M\phi_2^N}) = \delta'_{\phi_1^M\phi_2^N}[\Gamma'(A_{\phi_1}), \Gamma'(A_{\phi_1^2}), \ldots, \Gamma'(A_{\phi_1^m}), \Gamma'(A_{\phi_2}), \Gamma'(A_{\phi_1\phi_2}), \ldots, \Gamma'(A_{\phi_1^{m-1}\phi_2})].$$

By using [15, Corollary 3.4] and [21, Proposition 2.1], we can write

$$\begin{aligned} \Gamma'(A_{\phi_1^u}) &= \overline{K}_{\phi_1^{M-u}\phi_2^N}, \text{ where } 1 \le u \le M,\\ \Gamma'(A_{\phi_2^v}) &= \overline{K}_{\phi_1^M\phi_2^{N-v}}, \text{ where } 1 \le v \le M,\\ \Gamma'(A_{\phi_1^u\phi_2^v}) &= \overline{K}_{\phi_1^{M-u}\phi_2^{N-v}}. \end{aligned}$$

Therefore, by Theorem 1, the normalized Laplacian spectrum of $\Gamma'(\mathbb{Z}_{\phi_1^M\phi_2^N})$ is

$$\begin{aligned} \sigma_{\mathcal{NL}}(\Gamma'(\mathbb{Z}_{\phi_1^M\phi_2^N})) =& \Big((1+0(\sigma_{\mathcal{NL}}(\Gamma'(A_{\phi_1}))\setminus\{0\})\Big)\bigcup\Big((1+0(\sigma_{\mathcal{NL}}(\Gamma'(A_{\phi_1^2}))\setminus\{0\})\Big)\\ &\bigcup\cdots\bigcup\Big((1+0(\sigma_{\mathcal{NL}}(\Gamma'(A_{\phi_1^m}))\setminus\{0\})\Big)\bigcup\Big((1+0(\sigma_{\mathcal{NL}}(\Gamma'(A_{\phi_2}))\setminus\{0\})\Big)\\ &\bigcup\Big((1+0(\sigma_{\mathcal{NL}}(\Gamma'(A_{\phi_2^2}))\setminus\{0\})\Big)\bigcup\cdots\bigcup\Big((1+0(\sigma_{\mathcal{NL}}(\Gamma'(A_{\phi_2^N}))\setminus\{0\})\Big)\\ &\bigcup\Big((1+0(\sigma_{\mathcal{NL}}(\Gamma'(A_{\phi_1\phi_2}))\setminus\{0\})\Big)\bigcup\cdots\bigcup\Big((1+0(\sigma_{\mathcal{NL}}(\Gamma'(A_{\phi_1^M\phi_2}))\setminus\{0\})\Big)\\ &\bigcup\cdots\bigcup\Big((1+0(\sigma_{\mathcal{NL}}(\Gamma'(A_{\phi_1\phi_2^N}))\setminus\{0\})\Big)\\ &\bigcup\cdots\bigcup\Big((1+0(\sigma_{\mathcal{NL}}(\Gamma'(A_{\phi_1^{M-1}\phi_2^N}))\setminus\{0\})\Big)\bigcup\sigma_{\mathcal{NL}}(\mathbb{L}(\delta'_{\phi_1^M\phi_2^N}))\\ =&\left\{\begin{matrix} 1 & 1 & \cdots & 1 & 1\\ \phi(\phi_1^{M-1}\phi_2^N)-1 & \phi(\phi_1^{M-2}\phi_2^N)-1 & \cdots & \phi(\phi_2^N)-1 & \phi(\phi_1^M\phi_2^{N-1})-1 \end{matrix}\right.\\ &\begin{matrix} 1 & \cdots & 1 & 1 & 1\\ \phi(\phi_1^M\phi_2^{N-2})-1 & \cdots & \phi(\phi_1^M)-1 & \phi(\phi_1^{M-1}\phi_2^{N-1})-1 & \phi(\phi_1^{M-2}\phi_2^{N-1})-1 \end{matrix}\\ &\begin{matrix} \cdots & 1 & 1 & \cdots & 1\\ \cdots & \phi(\phi_2^{N-1})-1 & \phi(\phi_1^{M-1}\phi_2^{N-2})-1 & \cdots & \phi(\phi_1^{M-1})-1 \end{matrix}\\ &\left.\begin{matrix} \cdots & 1\\ \cdots & \phi(\phi_1)-1 \end{matrix}\right\}\bigcup\sigma_{\mathcal{NL}}(\delta'_{\phi_1^2\phi_2}). \end{aligned}$$

The remaining $(M+1)(N+1)-2$ eigenvalues are the eigenvalues of the matrix $\mathbb{L}(\delta'_{\phi_1^M\phi_2^N})$ given in equation (1). □

Example 1. *The normalized Laplacian spectrum of cozero-divisor graph of $\mathbb{Z}_{30}$ is*

$$\left\{\begin{matrix} 1 & -0.017 & 0.577 & 0.976 & 1.066 & 1.617 & 1.780\\ 15 & 1 & 1 & 1 & 1 & 1 & 1 \end{matrix}\right\}.$$

Proof. Let $n = 30$. Then the proper divisors of n are 2, 3, 5, 6, 10 and 15. So, we obtain $\delta'_{30}: 3 \sim 5 \sim 2 \sim 3 \sim 10 \sim 6 \sim 15 \sim 2, 5 \sim 6, 10 \sim 15$. Now, increasing the divisor sequence to order the vertices and using [15, Lemma 3.6], we have

$$\Gamma'(\mathbb{Z}_{30}) = \delta'_{30}[\Gamma'(A_2), \Gamma'(A_3), \Gamma'(A_5), \Gamma'(A_6), \Gamma'(A_{10}), \Gamma'(A_{15})].$$

By [15, Corollary 3.4], we have

$$\Gamma'(\mathbb{Z}_{30}) = \delta'_{30}[\overline{K}_8, \overline{K}_4, \overline{K}_2, \overline{K}_4, \overline{K}_2, \overline{K}_1].$$

The values of M_{s_r} are given by

$$M_2 = 7,\ M_3 = 12,\ M_5 = 16,\ M_6 = 5,\ M_{10} = 9,\ M_{15} = 14.$$

Thus, by Theorem 1, the normalized Laplacian spectrum of $\Gamma'(\mathbb{Z}_{30})$ consists of the eigenvalue 1 with multiplicity 15 together with the eigenvalues of the matrix $\mathbb{L}(\delta'_{30})$ given below

$$\mathbb{L}(\delta'_{30}) = \begin{bmatrix} 1 & -\sqrt{\frac{8}{21}} & -\sqrt{\frac{1}{7}} & 0 & 0 & -\frac{2}{7} \\ -\sqrt{\frac{8}{21}} & 1 & -\sqrt{\frac{1}{24}} & 0 & -\sqrt{\frac{2}{27}} & 0 \\ -\sqrt{\frac{1}{7}} & -\sqrt{\frac{1}{24}} & 1 & -\sqrt{\frac{1}{10}} & 0 & 0 \\ 0 & 0 & -\sqrt{\frac{1}{10}} & 1 & -\frac{2}{3}\sqrt{\frac{2}{5}} & -\sqrt{\frac{2}{35}} \\ 0 & -\sqrt{\frac{2}{27}} & 0 & -\frac{2}{3}\sqrt{\frac{2}{5}} & 1 & -\frac{1}{3}\sqrt{\frac{1}{7}} \\ -\frac{2}{7} & 0 & 0 & -\sqrt{\frac{2}{35}} & -\frac{1}{3}\sqrt{\frac{1}{7}} & 1 \end{bmatrix}.$$

The approximated eigenvalues of the above matrix $\mathbb{L}(\delta'_{30})$ are

$$\{-0.017, 0.577, 0.976, 1.066, 1.617, 1.780\}.$$

□

4 Randiĉ spectrum of $\Gamma'(\mathbb{Z}_n)$

In this section, the Randiĉ spectrum of $\Gamma'(\mathbb{Z}_n)$ for different n is examined with proper divisors of n as $d_1, \ldots, d_k$. For $1 \le r \le k$, we give the weight $|A_{d_r}| = \phi(\frac{n}{d_r})$ to the vertex d_r of the graph δ'_n. Let $\mathbb{L}(\delta'_n)$ denotes the k^{th} order weighted Randiĉ matrix of δ'_n, which is defined in [4, Theorem 2] and is given by

$$\mathbb{L}(\delta'_n) = \begin{bmatrix} 0 & l_{1,2} & \cdots & l_{1,k} \\ l_{2,1} & 0 & \cdots & l_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ l_{k,1} & l_{k,2} & \cdots & 0 \end{bmatrix}, \tag{4}$$

where

$$l_{i,j} = \begin{cases} \sqrt{\frac{\phi(\frac{n}{d_i})\phi(\frac{n}{d_j})}{M_{d_i} M_{d_j}}}, & d_i \sim d_j \in \delta'_n, \\ 0, & \text{otherwise}, \end{cases} \qquad 1 \le i \ne j \le k, \qquad M_{d_j} = \sum_{d_i \in N_{\delta'_n}(d_j)} \phi\left(\frac{n}{d_i}\right).$$

Remark 1. *The normalized Laplacian matrix and the Randić matrix are connected by the relation*

$$NL(\mathcal{G}) = I - \mathbf{R}_{\mathcal{G}}, \tag{5}$$

where $NL(\mathcal{G})$ is the normalized Laplacian matrix, $\mathbf{R}_{\mathcal{G}}$ is the Randić matrix and I is the identity matrix of order n. It is clear from the relation (5) *that the spectrum of $\mathbf{R}_{\mathcal{G}}$ can be read from the spectrum of $NL(\mathcal{G})$ and vice-versa.*

The proofs for the subsequent results are straightforwardly derived from relation (5). Consequently, these results hold trivially based on the provided Remark 1.

Lemma 3. *Let $n = \phi_1\phi_2$ with distinct primes ϕ_1 and ϕ_2. Then the Randić spectrum of $\Gamma'(\mathbb{Z}_n)$ is given by*

$$\left\{\begin{matrix} 0 & 1 & -1 \\ \phi_1+\phi_2-4 & 1 & 1 \end{matrix}\right\}.$$

Lemma 4. *Let $n = \phi_1^2\phi_2$ with ϕ_1 and ϕ_2 as distinct primes. Then the Randić spectrum of $\Gamma'(\mathbb{Z}_n)$ is given by*

$$\left\{\begin{matrix} 0 \\ \phi(\phi_1\phi_2)+\phi(\phi_1^2)+\phi(\phi_2)+\phi(\phi_1)-4 \end{matrix}\right\}$$

and the remaining eigenvalues are the roots of the characteristic polynomial of matrix (4).

Theorem 4. *Let $n = \phi_1^m\phi_2$, where m is a positive integer and ϕ_1, ϕ_2 are distinct primes. Then the Randić spectrum of $\Gamma'(\mathbb{Z}_{\phi_1^m\phi_2})$ consists of the eigenvalues*

$$\left\{\begin{matrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \phi(\phi_1^{m-1}\phi_2)-1 & \phi(\phi^{m-2}{}_1\phi_2)-1 & \cdots & \phi(\phi_2)-1 & \phi(\phi_1^m)-1 & \phi(\phi_1^{m-1})-1 & \cdots & \phi(\phi_1)-1 \end{matrix}\right\}$$

and the remaining eigenvalues are the eigenvalues of matrix (4).

Theorem 5. *If $n = \phi_1^M\phi_2^N$, where M, N are positive integers and ϕ_1, ϕ_2 distinct primes. Then the Randić spectrum of $\Gamma'(\mathbb{Z}_n)$ contains the eigenvalues*

$$\left\{\begin{matrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ \phi(\phi_1^{M-1}\phi_2^N)-1 & \phi(\phi_1^{M-2}\phi_2^N)-1 & \cdots & \phi(\phi_2^N)-1 & \phi(\phi_1^M\phi_2^{N-1})-1 & \phi(\phi_1^M\phi_2^{N-2})-1 \end{matrix}\right.$$

$$\begin{matrix} \cdots & 0 & 0 & 0 & \cdots & 1 \\ \cdots & \phi(\phi_1^M)-1 & \phi(\phi_1^{M-1}\phi_2^{N-1})-1 & \phi(\phi_1^{M-2}\phi_2^{N-1})-1 & \cdots & \phi(\phi_2^{N-1})-1 \end{matrix}$$

$$\left.\begin{matrix} 0 & \cdots & 0 & \cdots & 0 \\ \phi(\phi_1^{M-1}\phi_2^{N-2})-1 & \cdots & \phi(\phi_1^{M-1})-1 & \cdots & \phi(\phi_1)-1 \end{matrix}\right\}$$

and the remaining $(M+1)(N+1)-2$ eigenvalues are the eigenvalues of the matrix given in equation (4).

Example 2. *The Randić spectrum of cozero-divisor graph of $\mathbb{Z}_{30}$ is*

$$\left\{\begin{matrix} 0 & -0.780 & -0.617 & -0.066 & 0.024 & 0.423 & 1.017 \\ 15 & 1 & 1 & 1 & 1 & 1 & 1 \end{matrix}\right\}.$$

5 Conclusion

In this article, we extended the study of P. Mathil et. al. [15] on the Laplacian spectrum of the cozero-divisor graph $\Gamma'(\mathbb{Z}_n)$, where $n = p^{n_1}q^{n_2}$ with $p < q$ being primes and $n_1, n_2 \in \mathbb{N}$. Our work focused on exploring both the normalized Laplacian and Randić spectrum of the cozero-divisor graph with a different parametrization. Specifically, we considered n expressed as $n = \phi_1^M \phi_2^N$, with ϕ_1 and ϕ_2 being prime numbers such that $\phi_1 < \phi_2$, and M, N as positive integers. This new parametrization allowed us to derive significant insights into the spectral properties of the cozero-divisor graph, broadening the understanding and applicability of spectral graph theory in algebraic structures. Our findings contribute to the ongoing research in this field and open up new avenues for further investigation.

One may generalize these results to extend the study of cozero-divisor graphs to find the normalized Laplacian and Randić spectrum for the ring $\mathbb{Z}_n$, where n is a product of more than two distinct primes, i.e. $n = \phi_1^M \phi_2 \phi_3$, $n = \phi_1^M \phi_2^N \phi_3$, $n = \phi_1^M \phi_2^N \phi_3^O$ for positive integers M, N, O and ϕ_1, ϕ_2, ϕ_3 are distinct primes.

Future research could focus on further exploring the spectral properties of the cozero-divisor graph. Specifically, the following questions may be of interest.

Question 1. *What are the bounds for the smallest and largest eigenvalues of the normalized Laplacian and Randić matrices of the cozero-divisor graph for* $n = \phi_1^M \phi_2^N$*?*

Question 2. *How does the spectral radius of the cozero-divisor graph vary with different parametrizations of* $n = \phi_1^M \phi_2^N$*?*

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Рехман Н., Назім, Мір Ш.А., Назім М. *Аналіз нормалізованого лапласіана та спектра Рандича конуль-дільникового графа кільця $\mathbb{Z}_n$* // Карпатські матем. публ. — 2025. — Т.17, №2. — С. 766–777.

У цій статті досліджується нормалізований лапласіан та спектр Рандича конуль-дільникового графа скінченного комутативного кільця $\mathfrak{R}$ з одиницею $1 \neq 0$. Нехай $Z'(\mathfrak{R})$ позначає множину неунітарних і ненульових елементів кільця $\mathfrak{R}$. Конуль-дільниковий граф кільця $\mathfrak{R}$, який позначається через $\Gamma'(\mathfrak{R})$, є простим неорієнтованим графом із множиною вершин $Z'(\mathfrak{R})$, у якому дві різні вершини u та v з'єднані ребром тоді і лише тоді, коли $u \notin v\mathfrak{R}$ та $v \notin u\mathfrak{R}$, де $\alpha\mathfrak{R}$ – ідеал, породжений елементом α в кільці $\mathfrak{R}$. Зокрема, описано нормалізований лапласіан-спектр та спектр Рандича графа $\Gamma'(\mathbb{Z}_n)$ для різних значень n.

Ключові слова і фрази: конуль-дільниковий граф, нормалізований лапласіан-спектр, спектр Рандича, кільце цілих чисел за модулем n.