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2-Lipschitz Pietsch-q-integral and 2-Lipschitz q-nuclear operators

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In this paper, we extend the concept of q-nuclear to 2-Lipschitz mappings. We provide a factorization theorem and establish that a 2-Lipschitz operator is q-nuclear if and only if its transpose is well-defined and q-nuclear. Additionally, we introduce the concept of 2-Lipschitz Pietsch-q-integral operators, present a factorization theorem, and demonstrate that this class of operators is generated by the composition method. Conclusively, we demonstrate that every 2-Lipschitz q-nuclear operator is a 2-Lipschitz Pietsch-q-integral operator.

Key words and phrases: 2-Lipschitz operator ideal, linear q-nuclear operator, linear Pitsch-q-integral operator, factorization theorem.

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1 Introduction and preliminaries

A. Grothendieck extensively explored the concept of nuclear operators between Banach spaces. In [20], A. Pietsch extended this concept to p-nuclear operators and investigated their properties for $1 < p \leq \infty$. More recently, several authors have delved into various aspects of p-nuclear operators, as evidenced in [11, 14, 15, 17, 20]. In the work [5], D. Chen and B. Zheng introduced the notion of strongly Lipschitz q-nuclear operators between pointed metric and Banach spaces, exploring their properties. They demonstrated that every strongly Lipschitz q-nuclear operator aligns with factorization theory, particularly through the classical Banach spaces ℓ_∞ and ℓ_q.

The concept of Pietsch-q-integral, also known as strictly p-integral operators, for linear operators was introduced by A. Persson and A. Pietsch [18] to establish many of its fundamental properties. In [4], M.G. Cabrera-Padilla et. al. illustrated the role of Pietsch-q-integral operators in characterizing properties such as the factorization theorem. Recently, in [6], R. Cilia and J.M. Gutiérrez characterized the ideal of Pietsch-q-integral polynomials on Banach spaces for $q = 1$, and in [7] for $q > 1$, as a natural polynomial extension of Pietsch-q-integral operators. In [9], E. Dahia and K. Hamidi applied the theory of Pietsch-q-integral operators to the Lipschitz case, investigating this theory and studying its basic properties.

In this paper, following the introduction, Section 2 extends the concept of q-nuclear linear operators in [20] and strongly Lipschitz q-nuclear linear operators in [5] to 2-Lipschitz mappings. We present a factorization theorem that extends a result obtained in [5]. Additionally, we prove that a 2-Lipschitz operator is q-nuclear if and only if its transpose is well-defined and q-nuclear, specifically when E is a reflexive Banach space. We also provide similar results

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for well-known operator ideals. Finally, in Section 3, we introduce the concept of 2-Lipschitz Pietsch-q-integral operators, extending the notion of Pietsch-q-integral in [3] and Lipschitz Pietsch-q-integral operators in [9]. We establish a factorization theorem and demonstrate that this operator ideal is generated by the composition method from the operator ideal of Pietsch-q-nuclear linear operators. Conclusively, we conclude the paper by demonstrating that every 2-Lipschitz q-nuclear operator is also a 2-Lipschitz Pietsch-q-integral operator.

The notation used in this paper is generally standard. Let X and Y be pointed metric spaces with a base point denoted by 0, and the metric denoted by d. Additionally, let E and F be Banach spaces over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) with dual spaces E^* and F^*, respectively. The Banach space E is considered as a pointed metric space with the distinguished point 0 and distance $d(x, y) = \|x - y\|$. The closed unit ball of E is denoted by B_E, and it is endowed with the pointwise topology.

Let $\mathcal{L}(E, F)$ and $Lip_0(X, E)$ denote the Banach spaces of all bounded linear operators and Lipschitz operators, respectively, with the usual operator norm and the Lipschitz norm. We use the notation $Lip_0(X, \mathbb{K}) = X^\#$, where $X^\#$ is the Lipschitz dual of X. It is evident that $\mathcal{L}(E, F)$ is a subspace of $Lip_0(E, F)$, and specifically, E^* is a subspace of $E^\#$. We reserve the symbol $E \otimes F$ for the 2-fold tensor product of E and F, and $E \otimes_\pi F$ for the completed projective tensor product of E and F (see [10, 22]) under the injective tensor norm π defined by $\pi(u) = \inf \sum_{i=1}^{n} \|x_i\| \|y_i\|$, where the infimum is taken over all representations of $u \in E \otimes_\pi F$ in the form $\inf \sum_{i=1}^{n} x_i \otimes y_i$ for all $x_i \in E$ and $y_i \in F$.

A molecule on X is a real-valued function m with a finite support that satisfies the condition $\sum_{x \in X} m(x) = 0$. We denote by $\mathcal{M}(X)$ the real linear space of all molecules on X. For each $m = \sum_{i=1}^{n} \alpha_i m_{x_i x_i'}$, we define the norm $\|m\|_{\mathcal{M}(X)} = \inf \sum_{i=1}^{n} |\alpha_i| \, d(x_i, x_i')$, where $\alpha_i \in \mathbb{R}$ for all $i \in \mathbb{N}$, and the infimum is taken over all representations of the molecule m. The space Æ(X) is the completion of the normed space $(\mathcal{M}(X), \| \cdot \|_{\mathcal{M}(X)})$. It is well known that the space Æ(X) is the predual of $X^\#$, i.e. Æ$(X)^*$ and $X^\#$ are isometrically isomorphic (see [1]). Moreover, for $T \in Lip_0(X, E)$, there exists a unique continuous linear map $T_L :$ Æ$(X) \longrightarrow E$ such that $T = T_L \circ \delta_X$ and $\|T_L\| = Lip(T)$, where $\delta_X : X \longrightarrow$ Æ(X) is an isometric embedding from X into Æ(X) defined by $\delta_X(x) = m_{x0}$ (see [24, Theorem 2.2.4 (b)]).

We use the symbol $BLip_0(X, Y; E)$ for the set of all two-Lipschitz (2-Lipschitz for short) operators T from $X \times Y$ to E with $T(0, y) = T(x, 0) = 0$, such that for all constants $C \geqslant 0$ we have

$$\left\| T(x, y) - T(x, y') - T(x', y) + T(x', y') \right\| \leq C d(x, x') d(y, y').$$

The set $BLip_0(X, Y; E)$ is a Banach space under the 2-Lipschitz operator norm $BLip$ defined by

$$BLip(T) = \sup_{x \neq x',\, y \neq y'} \frac{\left\| T(x, y) - T(x, y') - T(x', y) + T(x', y') \right\|}{d(x, x') d(y, y')}$$

for all $x, x' \in X$, $y, y' \in Y$. For more details, we refer to [12, 13, 23].

The notion of 2-Lipschitz operator ideals can be found in [13, Definition 3.1]. An ideal of 2-Lipschitz mappings, denoted by $\mathcal{I}_{BLip}$, is a subclass of the class $BLip_0$ such that for every pair of pointed metric spaces X and Y, and every Banach space E, the components

$$\mathcal{I}_{BLip}(X, Y; E) := BLip_0(X, Y; E) \cap \mathcal{I}$$

satisfy the following conditions:

(i) $\mathcal{I}_{BLip}(X,Y;E)$ is a vector subspace of $BLip_0(X,Y;E)$,

(ii) for any $h \in X^{\#}$, $g \in Y^{\#}$ and $v \in E$, the map $h \cdot g \cdot v$ belongs to $\mathcal{I}_{BLip}(X,Y;E)$,

(iii) if $h \in Lip_0(W,X)$, $g \in Lip_0(Z,Y)$, $T \in \mathcal{I}_{BLip}(X,Y;E)$ and $U \in \mathcal{L}(E,H)$, then the composition $U \circ T \circ (h,g)$ is in $\mathcal{I}_{BLip}(W,Z;H)$ (*the ideal property*).

A 2-Lipschitz operator ideal $\mathcal{I}_{BLip}$ is a normed (Banach) 2-Lipschitz operator ideal if there is $\|\cdot\|_{\mathcal{I}_{BLip}} : \mathcal{I}_{BLip} \longrightarrow [0,+\infty[$ that satisfies the following conditions:

(i') the pair $\left(\mathcal{I}_{BLip}(X,Y;E), \|\cdot\|_{\mathcal{I}_{BLip}}\right)$ is a normed (Banach) space and $BLip(T) \leq \|T\|_{\mathcal{I}_{BLip}}$ for all $T \in \mathcal{I}_{BLip}(X,Y;E)$ and for every pointed metric spaces X, Y and every Banach space E,

(ii') $\left\|Id_{\mathbb{K}^2}\right\|_{\mathcal{I}_{BLip}} = 1$, where $Id_{\mathbb{K}^2} : \mathbb{K} \times \mathbb{K} \longrightarrow \mathbb{K}$ is defined by $Id_{\mathbb{K}^2}(\alpha,\beta) = \alpha\beta$,

(iii') if $h \in Lip_0(W,X)$, $g \in Lip_0(Z,Y)$, $T \in \mathcal{I}_{BLip}(X,Y;E)$ and $U \in \mathcal{L}(E,H)$, then the inequality $\|U \circ T \circ (h,g)\|_{\mathcal{I}_{BLip}} \leq \|u\| \, \|T\|_{\mathcal{I}_{BLip}} Lip(h) Lip(g)$ holds.

Proposition 1 ([13]). *A 2-Lipschitz mapping $T \in BLip_0(X,Y;E)$ belongs to the composition 2-Lipschitz operator ideal $\mathcal{A} \circ BLip_0$ if its linearization T_L belongs to $\mathcal{A}\left(Æ(X)\widehat{\otimes}_\pi Æ(Y), E\right)$. Furthermore, we have $\|T\|_{\mathcal{A}\circ BLip_0} = \|T_L\|_{\mathcal{A}}$.*

Let $n \in \mathbb{N}^*$ and $1 \leq q < \infty$. We write q^* for the extended real number that satisfies $\frac{1}{q} + \frac{1}{q^*} = 1$. We denote by $\ell_q^n(E)$ the Banach space of all sequences $(v_j)_{j=1}^n$ in the Banach space E with the norm

$$\left\|(v_j)_{j=1}^n\right\|_q = \left(\sum_{j=1}^n \left\|v_j\right\|^q\right)^{\frac{1}{q}},$$

and by $\ell_{q,\omega}^n(E)$ the Banach space of all sequences $(v_j)_{j=1}^n$ in E with the norm

$$\left\|(v_j)_{j=1}^n\right\|_{q,\omega} = \sup_{\varphi \in B_{E^*}} \left(\sum_{j=1}^n \left|\langle v_j, \varphi\rangle\right|^q\right)^{\frac{1}{q}}.$$

Now, we recall our definition for many concept used in this paper. The notion of q-nuclear linear operators was introduced by A. Persson in [17]. Given $1 \leq q \leq \infty$, a linear operator $U : E \longrightarrow F$ is q-nuclear [11, 17] if and only if U can be written in the form $U = \sum_j x_j^* \otimes v_j$, where $(v_j)_j$ in F and $(x_j^*)_j$ in E^* satisfy $\|(x_j^*)_j\|_q < \infty$ and $\|(v_j)_j\|_{w,q^*} < \infty$.

Set

$$\mathcal{N}_q\left((x_j^*)_j, (v_j)_j\right) = \|(x_j^*)_j\|_q \, \|(v_j)_j\|_{w,q^*}.$$

We denote by $\mathcal{N}_q(E,F)$ the collection of all q-nuclear operators endowed with the norm, defined by $\nu_q(U) = \inf \mathcal{N}_q\left((x_j^*)_j, (v_j)_j\right)$, where the infimum is taken over all representations of U. A linear operator $U : E \longrightarrow F$ is q-nuclear with $1 \leq q \leq \infty$, if there are two operators $a \in \mathcal{L}(E, \ell_\infty)$, $b \in \mathcal{L}(\ell_q, F)$ and a sequence $\alpha \in \ell_q$ such that it admits a factorization

$$U : E \xrightarrow{a} \ell_\infty \xrightarrow{D_\alpha} \ell_q \xrightarrow{b} F,$$

where $D_\alpha \in \mathcal{L}\left(\ell_\infty, \ell_q\right)$ is the diagonal operator defined by $D_\alpha(\zeta_n) = (\alpha_n \zeta_n)_n$, $(\zeta_n)_n \in \ell_\infty$. Moreover, $\nu_q(U) = \inf \|a\| \, \|D_\alpha\| \, \|b\|$, where the infimum being extended over all factorizations as above.

For $1 \leq q \leq \infty$, a mapping $T \in Lip_0(X, E)$ is called strongly Lipschitz q-nuclear (see [5]) if it can be written in the form $T = \sum_j f_j \otimes v_j$, where $(v_j)_j$ in E and $(f_j)_j$ in $X^\#$ satisfy $\left\|(Lip(f_j))_j\right\|_q < \infty$ and $\|(v_j)_j\|_{w,q^*} < \infty$.

Set

$$\mathcal{N}_q^L\left((f_j)_j, (v_j)_j\right) = \|(Lip(f_j))_j\|_q \, \|(v_j)_j\|_{w,q^*}.$$

The Banach space of such operators is denoted by $\mathcal{SN}_q^L\,(X, E)$ under the norm, defined by $sv_q^L(T) = \inf \mathcal{N}_q^L\left((f_j)_j, (v_j)_j\right)$, where the infimum is taken over all representations of T.

A Lipschitz operator $T : X \longrightarrow E$ is strongly Lipschitz q-nuclear with $1 \leq q \leq \infty$ (see [5, Theorem 2.2]), if there are a Lipschitz map $A \in Lip_0\,(X, \ell_\infty)$, a linear map $B \in \mathcal{L}\left(\ell_q, E\right)$ and a sequence $\alpha \in \ell_q$ such that the following factorization

$$T : X \xrightarrow{A} \ell_\infty \xrightarrow{D_\alpha} \ell_q \xrightarrow{B} E$$

holds. We set $sv_q^L(T) = \inf \|B\| \, \|D_\alpha\| \, Lip(A)$, where the infimum being extended over all factorizations as above.

A linear operator $U : E \longrightarrow F$ is Pietsch-q-integral with $1 \leq q < \infty$, if there are a regular Borel probability measure space (Ω, Σ, ν), and two linear operators $a \in \mathcal{L}(E, L_\infty(\nu))$, $b \in \mathcal{L}(L_q(\nu), F)$ such that the following factorization

$$U : E \xrightarrow{a} L_\infty(\nu) \xrightarrow{i_q} L_q(\nu) \xrightarrow{b} F$$

holds, where i_q is the canonical mapping. We denoted by $\mathcal{PI}_q\,(E, F)$ the collection of all Pietsch-q-integral operators endowed with the norm $\|U\|_{\mathcal{PI}_q}$, defined by

$$\|U\|_{\mathcal{PI}_q} = \inf \|a\| \, \|b\| \, ,$$

where the infimum is taken over all factorizations of U. If we consider a finite regular Borel measure space (Ω, Σ, ν), then for $U \in \mathcal{PI}_q\,(E, F)$ we infer that

$$\|U\|_{\mathcal{PI}_q} = \inf \|a\| \, \|b\| \, \nu(\Omega)^{\frac{1}{q}}.$$

A Lipschitz operator $T : X \longrightarrow E$ is Pietsch-q-integral with $1 \leq q < \infty$ (see [9, Definition 2.1]), if there are a regular Borel probability measure space (Ω, Σ, ν), a Lipschitz map $A \in Lip_0\,(X, L_\infty(\nu))$ and a linear map $B \in \mathcal{L}(L_q(\nu), E)$ such that the following factorization

$$T : X \xrightarrow{A} L_\infty(\nu) \xrightarrow{i_q} L_q(\nu) \xrightarrow{B} E$$

holds. The Banach space of such operators is denoted by $\mathcal{PI}_q^L\,(X, E)$ under the norm, defined by

$$\|T\|_{\mathcal{PI}_q^L} = \inf Lip(A) \|B\|,$$

where the infimum is taken over all factorizations of T. If we consider a finite regular Borel measure space (Ω, Σ, ν), then for $T \in \mathcal{PI}_q^L\,(X, E)$ we infer that

$$\|T\|_{\mathcal{PI}_q^L} = \inf Lip(A) \, \|B\| \, \nu(\Omega)^{\frac{1}{q}}.$$

2 2-Lipschitz q-nuclear operators

The objective of this section is to introduce the concept of strongly 2-Lipschitz q-nuclear mappings. This introduction is essential for our investigation into the factorization theorem and its applications. We establish a proof demonstrating that the transpose of a strongly Lipschitz q-nuclear operator is itself q-nuclear, and we explore its relationships with other operator ideals.

2.1 Definition and basic properties

Definition 1. *Let $1 \leq q \leq \infty$ and $T : X \times Y \longrightarrow E$ be a 2-Lipschitz operator. We say that T is a 2-Lipschitz q-nuclear operator if it can be written in the form*

$$T(x,y) = \sum_j f_j(x) g_j(y) \otimes v_j \tag{1}$$

for all $x \in X, y \in Y$, where $(f_j)_j \in X^{\#}$, $(g_j)_j \in Y^{\#}$ and $(v_j)_j \in E$ satisfy the following conditions $\|(Lip(f_j)Lip(g_j))_j\|_q < \infty$ and $\|(v_j)_j\|_{w,q^*} < \infty$.

Set

$$\mathcal{N}_q^{BL}\big((f_j)_j, (g_j)_j, (v_j)_j\big) = \|(Lip(f_j)Lip(g_j))_j\|_q \, \|(v_j)_j\|_{w,q^*}.$$

We will denote this set of mappings as $\mathcal{SN}_q^{BL}(X,Y;E)$ and equip it with the norm

$$s\nu_q^{BL}(T) = \inf \mathcal{N}_q^{BL}\big((f_j)_j, (g_j)_j, (v_j)_j\big),$$

where the infimum is taken over all possible representations of T as described in (1).

The next remark collects some elementary facts about 2-Lipschitz q-nuclear operators.

Remark 1. *Every Lipschitz map T from X to E is Lipschitz strongly q-nuclear mapping.*

Remark 2. *As an easy consequence of the Definition 1 we infer that*

$$BLip\,(T) \leq s\nu_q^{BL}(T)$$

for each $T \in \mathcal{SN}_q^{BL}(X,Y;E)$.

Theorem 1. *$\mathcal{SN}_q^{BL}(X,Y;E)$ is a normed space of 2-Lipschitz q-nuclear operators.*

Proof. It is clear that for any operator $T \in BLip_0(X,Y;E)$ and any scalar λ we have $s\nu_q^{BL}(T) \geq 0$ and $s\nu_q^{BL}(\lambda T) = |\lambda| \, s\nu_q^{BL}(T)$.

Let T_1, $T_2 \in \mathcal{SN}_q^{BL}(X,Y;E)$ and $\varepsilon > 0$. We can write $T_i = \sum_j f_{j,i} g_{j,i} \otimes v_{j,i}$ for each $i = 1,2$. By homogeneity, it is possible to choose representations of T_1 and T_2 such that for $i = 1,2$ we have

$$\left\|\big(Lip(f_{j,i})Lip(g_{j,i})\big)_j\right\|_q \leq \big(s\nu_q^{BL}(T_i) + \varepsilon\big)^{\frac{1}{q^*}} \quad \text{and} \quad \|v_{i,j}\|_{w,q^*} \leq \big(s\nu_q^{BL}(T_i) + \varepsilon\big)^{\frac{1}{q^*}}.$$

Consequently,

$$\left\|\big(Lip(f_{j,i})Lip(g_{j,i})\big)_j\right\|_q \leq \big(s\nu_q^{BL}(T_1) + s\nu_q^{BL}(T_2) + \varepsilon\big)^{\frac{1}{q^*}},$$

$$\|v_{i,j}\|_{w,q^*} \leq \big(s\nu_q^{BL}(T_1) + s\nu_q^{BL}(T_2) + \varepsilon\big)^{\frac{1}{q^*}}.$$

Then
$$\left\|\left(Lip(f_{j,i})Lip(g_{j,i})\right)_j\right\|_q \|v_{i,j}\|_{w,q^*} \leq s\nu_q^{BL}(T_1) + s\nu_q^{BL}(T_2) + 2\varepsilon,$$
which means that
$$s\nu_q^{BL}(T_1 + T_2) \leq s\nu_q^{BL}(T_1) + s\nu_q^{BL}(T_2) + 2\varepsilon.$$

□

The next theorem establishes an interesting relationship between a 2-Lipschitz q-nuclear map and its linearization.

Theorem 2. *Let X, Y be pointed metric spaces and E be a Banach space. Let $T \in BLip_0(X, Y; E)$. Then T is 2-Lipschitz q-nuclear if and only if its linearization $T_L : Æ(X)\widehat{\otimes}_\pi Æ(Y) \longrightarrow E$ is q-nuclear.*

Moreover, we have $s\nu_q^{BL}(T) = \nu_q(T_L)$.

Proof. Since $\mathcal{N}_q\left(Æ(X)\widehat{\otimes}_\pi Æ(Y), E\right)$ is a normed operator ideal (see [17]) and $T \in BLip_0(X, Y; E)$, by utilizing [13, Theorem 3.5], we can establish the equivalence mentioned above with the necessary norm. □

As a consequence, we obtain the following corollary, which is a straightforward result of the preceding theorem and [13, Proposition 3.6].

Corollary 1. *$\left(\mathcal{SN}_q^{BL}, s\nu_q^{BL}(\cdot)\right)$ is a 2-Lipschitz operator ideal, generated by the composition method from the Banach operator ideal $\mathcal{N}_q$, i.e.*

$$\mathcal{SN}_q^{BL}(X, Y; E) = \mathcal{N}_q \circ BLip_0(X, Y; E)$$

for all pointed metric spaces X, Y and a Banach space E.

2.2 Factorization theorem and applications

Using some standard techniques from [5, Theorem 2.2.] (see also [2, Theorem 5.1]), we can establish that any 2-Lipschitz q-nuclear operator satisfying the factorization theorem. For the convenience of the reader, we present the proof below.

Theorem 3. *For $1 \leq q \leq \infty$, a 2-Lipschitz mapping $T \in BLip_0(X, Y; E)$ is 2-Lipschitz q-nuclear if there are 2-Lipschitz operator $A \in BLip_0(X, Y; \ell_\infty)$ and a linear operator $B \in \mathcal{L}\left(\ell_q, E\right)$ such that the following diagram*

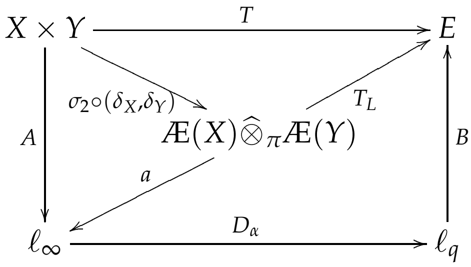
$$\begin{array}{ccc} X \times Y & \xrightarrow{\ T\ } & E \\ {\scriptstyle A}\downarrow & & \uparrow{\scriptstyle B} \\ \ell_\infty & \xrightarrow{\ D_\alpha\ } & \ell_q \end{array} \tag{2}$$

is commutative. Moreover, in this case

$$s\nu_q^{BL}(T) = \inf \|B\| \, \|D_\alpha\| \, BLip(A),$$

where the infimum is taken over all factorizations as in (2).

Proof. Let us first prove the easier sufficient part. Suppose that $T \in \mathcal{SN}_q^{BL}(X,Y;E)$. Then it follows from Theorem 2 that $T_L \in \mathcal{N}_q\big(Æ(X)\widehat{\otimes}_\pi Æ(Y), E\big)$. Consider the canonical bilinear mapping σ_2 from $Æ(X) \times Æ(Y)$ into $Æ(X)\widehat{\otimes}_\pi Æ(Y)$ defined by $\sigma_2\,(m_1, m_2) = m_1 \otimes m_2$, where $m_1 \in Æ(X)$ and $m_2 \in Æ(Y)$. It is clear that $\sigma_2 \in BLip_0\big(Æ(X), Æ(Y); Æ(X)\widehat{\otimes}_\pi Æ(Y)\big)$ with the norm $BLip(\sigma_2) \leq 1$. Moreover, using (1), the following diagram



can be deduced.

Hence, we can see that $T = BD_\alpha A$, where $A = a \circ \sigma_2 \circ (\delta_X, \delta_Y) \in BLip_0(X,Y;\ell_\infty)$ with $BLip(A) \leq \|a\|$ and $\|B\|\,\|D_\alpha\|\,BLip\,(A) \leq \|B\|\,\|D_\alpha\|\,\|a\| \leq \nu_q\,(T_L) + \varepsilon = s\nu_q^{BL}(T) + \varepsilon.$

Let us now prove the more involved necessary part. Assume that $T \in \mathcal{SN}_q^{BL}(X,Y;E)$. Then there exist a 2-Lipschitz operator $A \in BLip_0(X,Y;\ell_\infty)$, and linear operators $B \in \mathcal{L}(\ell_q, E)$, $D_\alpha \in \mathcal{L}(\ell_\infty, \ell_q)$ such that $T = AD_\alpha B$ and $s\nu_q^{BL}(T) \leq \|B\|\,\|D_\alpha\|\,BLip\,(A)$.

Consider the bilinear symmetric mapping $D : \ell_\infty \times \ell_\infty \to \ell_\infty$ defined by

$$D(\gamma_j, \xi_j) = \big(\gamma_j \xi_j\big)_{j=1}^{\infty}, \qquad (\gamma_j) \in \ell_\infty, \ \ (\xi_j) \in \ell_\infty,$$

with the norm equal to one. Define $A : X \times Y \longrightarrow \ell_\infty$ by

$$A(x,y) = D\big((A_1(x))_j, (A_2(y))_j\big) = \big((A_1(x))_j (A_2(y))_j\big)_{j=1}^{\infty},$$

where $A_1 \in Lip_0(X, \ell_\infty)$ and $A_2 \in Lip_0(Y, \ell_\infty)$. Clearly, A is a 2-Lipschitz mapping from $X \times Y$ to ℓ_∞ with $A(x,0) = A(0,y) = 0$ and $BLip\,(T) = Lip\,(A_1)\,Lip\,(A_2)$.

If $D_\alpha = \sum_j \delta_j \beta_j v_j \otimes v_j$ such that $\alpha = \big(\delta_j \beta_j\big)_j \in \ell_p$ with $p < \infty$ and $\alpha = \big(\delta_j \beta_j\big)_j \in c_0$ with $p = \infty$, then for all $x \in X, y \in Y$ we get

$$\begin{aligned} T(x,y) = BD_\alpha A(x,y) &= \sum_j \delta_j \beta_j \left\langle A(x,y), v_j \right\rangle B(v_j) \\ &= \sum_j \delta_j \beta_j \left\langle D(A_1(x), A_2(y)), v_j \right\rangle B(v_j) = \sum_j \delta_j \left\langle A_1(x), v_j \right\rangle \beta_j \left\langle A_2(y), v_j \right\rangle B(v_j). \end{aligned}$$

Let $f_j = \delta_j \left\langle A_1(\cdot), v_j \right\rangle$, $g_j = \beta_j \left\langle A_2(\cdot), v_j \right\rangle$. Then $f_j \in X^\#, g_j \in Y^\#$ and $T = \sum_j f_j g_j \otimes B(v_j)$. For $q = 1$, we have

$$\sum_j Lip(f_j) Lip(g_j) \leq Lip(A_1) Lip(A_2)\,\|D_\alpha\| = BLip(A)\,\|D_\alpha\| \quad \text{and} \quad \sup_j \left\|B(v_j)\right\| \leq \|B\|.$$

For $1 < q < \infty$, we have

$$\left(\sum_j Lip(f_j)^q Lip(g_j)^q\right)^{\frac{1}{q}} \leq Lip\,(A_1)\,Lip\,(A_2)\,\|D_\alpha\| = BLip\,(A)\,\|D_\alpha\|$$

and

$$\sup_{v^*\in B_{E^*}}\left(\sum_j \left|\left\langle v^*, B(v_j)\right\rangle\right|^{q^*}\right)^{\frac{1}{q^*}} = \sup_{v^*\in B_{E^*}}\left(\sum_j \left|\left\langle B^*v^*, v_j\right\rangle\right|^{q^*}\right)^{\frac{1}{q^*}} = \sup_{v^*\in B_{E^*}} \|B^*v^*\| = \|B\|.$$

For $q = \infty$, we have

$$\|D_\alpha\| = \sup_j \left|\delta_j\beta_j\right|, \quad \sup_j Lip(f_j)Lip(g_j) \leq BLip(A)\sup_j \left|\delta_j\beta_j\right|$$

and

$$\lim_j Lip(f_j)Lip(g_j) = 0.$$

Moreover,

$$\sup_{v^*\in B_{E^*}} \sum_j \left|\left\langle v^*, B\left(v_j\right)\right\rangle\right| = \sup_{v^*\in B_{E^*}} \sum_j \left|\left\langle B^*v^*, v_j\right\rangle\right| = \sup_{v^*\in B_{E^*}} \|B^*v^*\| = \|B\|.$$

Thus, for $1 \leq q \leq \infty$, we have

$$\mathcal{N}_q^{BL}\left((f_j)_j, (g_j)_j, (v_j)_j\right) \leq \|B\| \, \|D_\alpha\| \, BLip\left(A\right).$$

This implies that

$$sv_q^{BL}(T) \leq \inf \|B\| \, \|D_\alpha\| \, BLip\left(A\right).$$

□

The following result can be easily derived from Theorem 3.

Proposition 2. *Let $1 < q \leq r \leq \infty$. Then the inclusion $\mathcal{SN}_q^{BL}(X,Y;E) \subset \mathcal{SN}_r^{BL}(X,Y;E)$ holds with $sv_r^{BL}(T) \leq sv_q^{BL}(T)$ for each $T \in \mathcal{SN}_q^{BL}(X,Y;E)$.*

Proof. The proof may be achieved in the same way as in the proof of [2, Proposition 5.1]. □

A definition of the adjoint of an m-linear operator has been proposed in [19,21]. According to [19, Definition 2.5], we define the adjoint of a 2-Lipschitz map as follows. Let X and Y be pointed metric spaces, E be a Banach space. Consider a 2-Lipschitz map $T : X \times Y \longrightarrow E$. We define the adjoint of T by

$$T^t : E^* \longrightarrow BLip_0(X,Y), \quad v^* \longmapsto T^t\left(v^*\right) : X \times Y \longrightarrow \mathbb{K}$$

with $T^t\left(v^*\right)(x,y) = v^*T\left(x,y\right)$. Then again $\left\|T^t\right\| = BLip\left(T\right)$.

We define the operator $K : X \times Y \longrightarrow BLip_0\left(X,Y\right)^*$ by $K\left(x,y\right)\left(f\right) = f\left(x,y\right)$ for all $f \in BLip_0\left(X,Y\right)$. It is clear that K is a 2-Lipschitz operator with the 2-Lipschitz norm ≤ 1. On the other hand, consider the canonical injection $k_E : E \longrightarrow E^{**}$. Then, if $T \in BLip_0\left(X,Y;E\right)$, the following diagram

$$\begin{array}{ccc} X \times Y & \xrightarrow{\quad T \quad} & E \\ {\scriptstyle K}\downarrow & & \downarrow{\scriptstyle k_E} \\ BLip_0\left(X,Y\right)^* & \xrightarrow{\;(T^t)^*\;} & E^{**} \end{array}$$

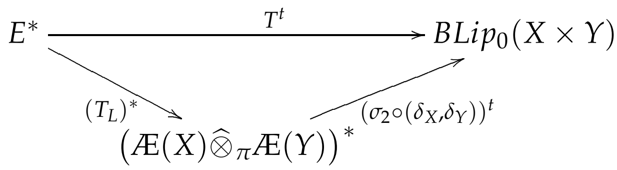
commutes, i.e. $k_E \circ T = \left(T^t\right)^* \circ K$.

The following theorem relates 2-Lipschitz q-nuclear with its adjoint.

Theorem 4. *Let* $1 \leq q \leq \infty$ *and* $T \in BLip_0(X, Y; E)$*. If* E *is a reflexive Banach space, then* T *is* 2*-Lipschitz* q*-nuclear if and only if* T^t *is* q*-nuclear and*

$$sv_q^{BL}(T) = \nu_q\left(T^t\right).$$

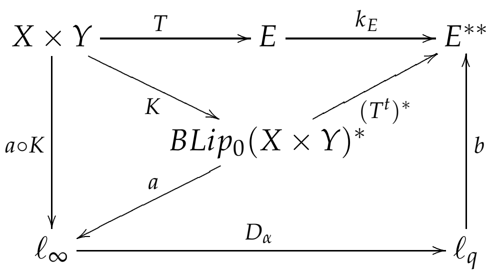
Proof. Assume that $T \in \mathcal{SN}_q^{BL}(X, Y; E)$. Then, from Theorem 2, $T_L \in \mathcal{N}_q\big(Æ(X)\widehat{\otimes}_\pi Æ(Y), E\big)$. Thus, using [16, Proposition 1], we conclude that $(T_L)^* \in \mathcal{N}_q\big(E^*, \big(Æ(X)\widehat{\otimes}_\pi Æ(Y)\big)^*\big)$, where $\nu_q((T_L)^*) = \nu_q(T_L)$. Therefore, we can easily factor T^t. The following diagram



shows this factorization. Thus, we deduce that $T^t \in \mathcal{N}_q\left(E^*, BLip(X, Y)\right)$ and

$$\nu_q\left(T^t\right) = \nu_q\left((\sigma_2 \circ (\delta_X, \delta_Y))^t \circ (T_L)^*\right) \leq \nu_q((T_L)^*) \leq sv_q^{BL}(T). \tag{3}$$

Conversely, assume that $T^t \in \mathcal{N}_q\big(E^*, BLip_0(X, Y)\big)$. Then $\left(T^t\right)^* \in \mathcal{N}_q\left(BLip_0\left(X, Y\right)^*, E^{**}\right)$. Thus, the following diagram



commutes, where $a \in \mathcal{L}(BLip_0(X \times Y)^*, \ell_\infty)$ and $b \in \mathcal{L}(\ell_q, E^{**})$.

Let $A = a \circ K$. Then $A \in BLip_0(X \times Y; \ell_\infty)$ and $BLip(A) \leq \|a\|$. Then from Theorem 3, we deduce that $T \in \mathcal{SN}_q^{BL}(X, Y; E)$ and

$$sv_q^{BL}(T) \leq \|b\| \, \|D_\alpha\| \, BLip\,(A) \leq \|b\| \, \|D_\alpha\| \, \|a\| \leq \nu_q\left(T^t\right). \tag{4}$$

Hence, one finds out from (3) and (4), that $sv_q^{BL}(T) = \nu_q(T^t)$. $\square$

Corollary 2. *Let* $1 < q \leq \infty$ *and* $T \in BLip_0(X, Y; E)$*. Then we have the following statements.*

1) $k_E T \in \mathcal{NS}_q^{BL}(X, Y; E^{**})$ *if and only if* $(T^t)^* \in \mathcal{N}_q\big(BLip_0\left(X, Y\right)^*, E^{**}\big)$.

2) If $T \in \mathcal{NS}_q^{BL}(X, Y; E)$*, then* $(T^t)^* \in \mathcal{N}_q\big(BLip_0\left(X, Y\right)^*, E^{**}\big)$.

Following [13], a 2-Lipschitz map $T \in BLip_0\left(X, Y; E\right)$ is 2-Lipschitz weakly compact if the set

$$\left\{ \frac{T\left(x, y\right) - T\left(x, y'\right) - T\left(x', y\right) + T\left(x', y'\right)}{d(x, x')d(y, y')}, \; x \neq x', \; y \neq y' \right\}$$

is weakly compact in E. We use $BLip_{0,\mathcal{W}}\left(X, Y; E\right)$ to denote the set of 2-Lipschitz weakly compact operators from $X \times Y$ to E.

J.S. Cohen [8, Corollary 2.2.5] has shown that q-nuclear operators are weakly compact. The next corollary follows immediately from Theorem 2 and [13, Theorem 4.3, Remark 4.5].

Corollary 3. *Every 2-Lipschitz q-nuclear operator is 2-Lipschitz weakly compact.*

Proposition 3. *Let $1 \leq q \leq \infty$ and $T \in BLip_0(X,Y;E)$. If $T \in \mathcal{SN}_q^{BL}(X,Y;E)$, then $T^t \in \Pi_q(E^*, BLip_0(X,Y))$ with $\pi_q(T^t) \leq s\nu_q^{BL}(T)$.*

Proof. Let $T \in \mathcal{SN}_q^{BL}(X,Y;E)$. Then it follows from Theorem 4, that $T^t \in \mathcal{N}_q(E^*, BLip_0(X,Y))$. Thus, using [11, Proposition 5.5, Corollary 5.24], we deduce that $T^t \in \Pi_q(E^*, BLip_0(X,Y))$ and $\pi_q(T^t) \leq \nu_q(T^t) \leq s\nu_q^{BL}(T)$. □

K. Hamidi et. al. in [13] introduced the concept of strongly two-Lipschitz q-summing. For $1 < q \leq \infty$, a mapping $T \in BLip_0(X,Y;E)$ is strongly 2-Lipschitz q-summing if there exist a Banach space F and a q^*-summing linear operator $R : E^* \longrightarrow F$ such that for all $x, x' \in X$, $y, y' \in Y$ and $v^* \in E^*$ we have

$$\left|\left\langle T(x,y) - T(x,y') - T(x',y) + T(x',y'), v^*\right\rangle\right| \leq d(x,x')d(y,y')\left\|R(v^*)\right\|. \tag{5}$$

The Banach space of all such mappings is denoted by $\mathcal{D}_q^{BL}(X,Y;E)$ and is endowed with the norm $\|T\|_{\mathcal{D}_q^{BL}} = \inf\{\pi_{q^*}(R)\}$, where infimum is taken over all Banach spaces F and operators R, such that (5) holds.

Proposition 4. *Let $1 \leq q \leq \infty$ and let $T \in BLip_0(X,Y;E)$. If $T \in \mathcal{SN}_q^{BL}(X,Y;E)$, then $T \in \mathcal{D}_q^{BL}(X,Y;E)$ with $\|T\|_{\mathcal{D}_q^{BL}} \leq s\nu_q^{BL}(T)$.*

Proof. Let $T \in \mathcal{SN}_q^{BL}(X,Y;E)$. Then from Theorem 2 we get $T_L \in \mathcal{N}_q(Æ(X)\widehat{\otimes}_\pi Æ(Y), E)$. Thus, using [8, Theorem 2.2.1], we infer that $T_L \in \mathcal{D}_q(Æ(X)\widehat{\otimes}_\pi Æ(Y), E)$ with $\|T_L\|_{\mathcal{D}_q} \leq \nu_q(T_L)$. By [13, Corollary 4.12], we deduce that $T \in \mathcal{D}_q^{BL}(X,Y;E)$ and

$$\|T\|_{\mathcal{D}_q^{BL}} = \|T_L\|_{\mathcal{D}_q} \leq \nu_q(T_L) = s\nu_q^{BL}(T)$$

□

3 2-Lipschitz Pietsch q-integral operators

In this section, we extend the definition of the class of Pietsch-q-integral operators (see [9]) to the 2-Lipschitz case, thereby establishing a new 2-Lipschitz operator ideal.

Definition 2. *Let X, Y be pointed metric spaces, and E be a Banach space. Let $1 \leq q < \infty$. A 2-Lipschitz operator $T \in BLip_0(X,Y;E)$ is called Pietsch-q-integral if there exist regular Borel probability measure space (Ω, Σ, ν), 2-Lipschitz operator $A \in BLip_0(X,Y;L_\infty(\nu))$ and a linear mapping $B \in \mathcal{L}(L_q(\nu), E)$, such that the following diagram*

$$\begin{array}{ccc} X \times Y & \xrightarrow{\quad T \quad} & E \\ {\scriptstyle A}\downarrow & & \uparrow{\scriptstyle B} \\ L_\infty(\nu) & \xrightarrow{\quad i_q \quad} & L_q(\nu) \end{array} \tag{6}$$

commutes. We denote by $\mathcal{PI}_q^{BL}(X,Y;E)$ the collection of all 2-Lipschitz Pietsch-q-integral operators from $X \times Y$ into E, and we set

$$\|T\|_{\mathcal{PI}_q^{BL}} = \inf \|B\| BLip(A),$$

where the infimum is taken over all factorizations (6).

Notice, that the definition remains the same, if we consider a finite regular Borel measure space (Ω, Σ, ν). In this case, for $T \in \mathcal{PI}_q^{BL}(X, Y; E)$, we have

$$\|T\|_{\mathcal{PI}_q^{BL}} = \inf \|B\| \nu(\Omega)^{\frac{1}{q}} BLip(A),$$

where the infimum is taken over all A and B as in (6).

Now, let us study the inclusion theorem. To establish this, it suffices to recall that for $1 \leq q \leq r < \infty$, a canonical mapping i_q can be factored as

$$i_q : L_\infty(\nu) \xrightarrow{i_r} L_r(\nu) \xrightarrow{i_{r,q}} L_q(\nu),$$

where $i_{r,q} : L_r(\nu) \longrightarrow L_q(\nu)$ is the canonical inclusion map. The combination of this fact with Definition 2 immediately yields the inclusion theorem as stated below.

Corollary 4. *Let $1 \leq q \leq r < \infty$. Then we have*

$$\mathcal{PI}_q^{BL}(X, Y; E) \subset \mathcal{PI}_r^{BL}(X, Y; E).$$

Moreover, $\|T\|_{\mathcal{PI}_r^{BL}} \leq \|T\|_{\mathcal{PI}_q^{BL}}$ for all $T \in \mathcal{PI}_q^{BL}(X, Y; E)$.

The following auxiliary lemma is needed for next results.

Lemma 1. *Let $J : \mathcal{M}(X)\widehat{\otimes}_\pi\mathcal{M}(Y) \longrightarrow C(B_{X^\#} \times B_{Y^\#})$ be the operator defined by*

$$J(m^1 \otimes m^2)(f, g) = \sum_{i=1}^{n} \alpha_i \big(f(x_i) - f(x_i')\big) \sum_{j=1}^{s} \beta_j \big(g(y_j) - g(y_j')\big)$$

for all $m^1 = \sum_{i=1}^n \alpha_i m_{x_i x_i'} \in \mathcal{M}(X)$, $m^2 = \sum_{j=1}^s \beta_j m_{y_j y_j'} \in \mathcal{M}(Y)$, $f \in B_{X^\#}$ and $g \in B_{Y^\#}$. Then J is an isometric embedding.

Proof. Since the isometric isomorphism between $Æ(Z)^*$ and $Z^\#$ ($Z = X$ or Y) holds true through linearization, there exist $h \in Z^\#$ and $m^* \in Æ(Z)^*$ such that $h_L = m^*$. Consequently, for all $m^1 = \sum_{i=1}^n \alpha_i m_{x_i x_i'} \in \mathcal{M}(X)$, $m^2 = \sum_{j=1}^s \beta_j m_{y_j y_j'} \in \mathcal{M}(Y)$, $f \in B_{X^\#}$ and $g \in B_{Y^\#}$, we can infer that

$$\begin{aligned}
\|J(m^1 \otimes m^2)\|_{C(B_{X^\#} \times B_{Y^\#})} &= \sup_{f \in B_{X^\#}, g \in B_{Y^\#}} \left|J(m^1 \otimes m^2)(f, g)\right| \\
&= \sup_{\|f_L\| \leq 1} \left|\sum_{i=1}^{n} \alpha_i f_L(m_{x_i x_i'})\right| \sup_{\|g_L\| \leq 1} \left|\sum_{j=1}^{s} \beta_j g_L(m_{y_j y_j'})\right| \\
&= \sup_{\|m_1^*\| \leq 1} \left|\langle m^1, m_1^* \rangle\right| \sup_{\|m_2^*\| \leq 1} \left|\langle m^2, m_2^* \rangle\right| \\
&= \|m^1\|_{Æ(X)} \|m^2\|_{Æ(Y)} = \|m^1\|_{\mathcal{M}(X)} \|m^2\|_{\mathcal{M}(Y)} \\
&= \|m^1 \otimes m^2\|_{\mathcal{M}(X)\widehat{\otimes}_\pi\mathcal{M}(Y)}.
\end{aligned}$$

The proof is finished. □

We now establish one of our main results. In fact, in what follows, we provide a characterization of 2-Lipschitz integral operators through a factorization scheme that emphasizes the significance of the spaces $C(B_{X^\#} \times B_{Y^\#})$ and $L_q(\mu)$.

Theorem 5. *Let $T : X \times Y \longrightarrow E$ be a 2-Lipschitz operator. Then T is a 2-Lipschitz Pietsch-q-integral if and only if there exist a regular Borel probability product measure μ on $B_{X^\#} \times B_{Y^\#}$ and a linear operator $\widetilde{B} \in \mathcal{L}(L_q(\mu), E)$, such that the following diagram*

$$\begin{array}{ccc} X \times Y & \xrightarrow{\quad T \quad} & E \\ {\scriptstyle S}\downarrow & & \uparrow{\scriptstyle \widetilde{B}} \\ C\left(B_{X^\#} \times B_{Y^\#}\right) & \xrightarrow{\quad j_q \quad} & L_q(\mu) \end{array} \tag{7}$$

commutes, where j_q is the natural inclusion and S is a 2-Lipschitz mapping defined by $S(x,y)(f,g) = f(x)g(y)$ with $BLip(S) \le 1$. Moreover,

$$\|T\|_{\mathcal{PI}_q^{BL}} = \inf\left\{\left\|\widetilde{B}\right\| : T = B \circ j_q \circ S\right\}.$$

Proof. We use the symbol Θ for the proposed infimum. Let us assume that T admits a factorization (7). Then there exist regular Borel probability measure μ on $B_{X^\#} \times B_{Y^\#}$ and a linear map $\widetilde{B} \in \mathcal{L}(L_q(\mu), E)$, such that

$$T : X \times Y \xrightarrow{S} C\left(B_{X^\#} \times B_{Y^\#}\right) \xrightarrow{j_q} L_q(\mu) \xrightarrow{\widetilde{B}} E.$$

Note that if j_∞ is the canonical inclusion map from $C\left(B_{X^\#} \times B_{Y^\#}\right)$ to $L_\infty(\mu)$, then we infer the following factorization

$$j_q = i_q \circ j_\infty : C\left(B_{X^\#} \times B_{Y^\#}\right) \xrightarrow{j_\infty} L_\infty(\mu) \xrightarrow{i_q} L_q(\mu).$$

On the other hand, we set $A = j_\infty \circ S$. In addition, it follows that $A \in BLip_0\left(X, Y; L_\infty(\nu)\right)$ and $BLip(A) \le 1$, The factorization $T = \widetilde{B} \circ i_q \circ A$ shows that T is a 2-Lipschitz Pietsch-q-integral operator and

$$\|T\|_{\mathcal{PI}_q^{BL}} \le \|\widetilde{B}\| BLip(A) \le \|\widetilde{B}\|.$$

Passing to the infimum, we get $\|T\|_{\mathcal{PI}_q^{BL}} \le \Theta$.

Next, we prove the sufficient part. If $T \in \mathcal{PI}_q^{BL}\left(X, Y; E\right)$, then the factorization

$$T : X \times Y \xrightarrow{A} L_\infty(\mu) \xrightarrow{i_q} L_q(\mu) \xrightarrow{B} E$$

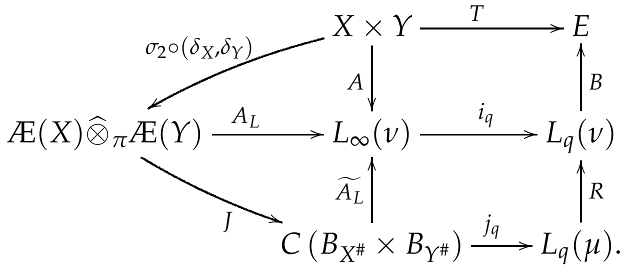
holds for $A \in BLip_0(X, Y; L_\infty(\mu))$, $B \in \mathcal{L}(L_q(\mu), E)$ and $\|B\| BLip(A) \le \|T\|_{\mathcal{PI}_q^{BL}} + \varepsilon$ for all $\varepsilon > 0$.

Since A is a 2-Lipschitz, its linearization $A_L : \text{Æ}(X)\widehat{\otimes}_\pi\text{Æ}(Y) \longrightarrow L_\infty(\mu)$ is a well-defined operator with norm $\|A_L\| = BLip(A)$ (see [13, Theorem 2.6, Remark 2.7]) and J is an isometric embedding defined in Lemma 1. The natural extension to $\text{Æ}(X)\widehat{\otimes}_\pi\text{Æ}(Y)$ we denote also by J. The injectivity of $L_\infty(\mu)$ assures the existence of a linear operator $\widetilde{A_L} \in \mathcal{L}(C(B_{X^\#} \times B_{Y^\#}), L_\infty(\mu))$ that extends A_L with $\left\|\widetilde{A_L}\right\| = \left\|A_L\right\|$, that is $A_L = \widetilde{A_L} \circ J$. By using the fact that i_q is q-summing operator with q-summing norm equal to one, we infer that $i_q \circ \widetilde{A_L}$ is the same with $\pi_q(i_q \circ \widetilde{A_L}) \le \left\|\widetilde{A_L}\right\|$.

Again, from [11, Corollary 2.15], we conclude that there is a regular Borel probability measure μ on $B_{X^\#} \times B_{Y^\#}$. Also, there exists a linear operator $R \in \mathcal{L}(L_q(\mu), L_q(\nu))$, such that

$$i_q \circ \widetilde{A_L} = R \circ j_q : C\left(B_{X^\#} \times B_{Y^\#}\right) \xrightarrow{j_q} L_q(\mu) \xrightarrow{R} L_q(\nu),$$

where $\pi_q(i_q \circ \widetilde{A_L}) = \|R\|$. This implies the following diagram



Therefore,

$$T = B \circ i_q \circ A = B \circ R \circ j_q \circ J \circ \sigma_2 \circ (\delta_X, \delta_Y).$$

Now, take $\widetilde{B} = B \circ R$ and $S = J \circ \sigma_2 \circ (\delta_X, \delta_Y)$. It is clear that $S \in BLip_0\big(X, Y, C(B_{X^\#} \times B_{Y^\#})\big)$ with the norm $BLip\,(S) \leq 1$. As the result, T admits a factorization of the form (7). Then we get

$$\Theta \leq \|\widetilde{B}\| \leq \|B\|\pi_q(i_q \circ \widetilde{A_L}) \leq \|B\|\|\widetilde{A_L}\| = \|B\|BLip(A) \leq \|T\|_{\mathcal{PI}_q^{BL}} + \varepsilon.$$

Letting $\varepsilon \to 0$, we get $\Theta \leq \|T\|_{\mathcal{PI}_q^{BL}}$. □

From Theorem 5 and [11, Corollary 2.15], we easily obtain the following corollary.

Corollary 5. *Let $1 \leq p < +\infty$. A 2-Lipschitz $T \in BLip\,(X, Y; E)$ is Pietsch-p-integral if and only if there is an operator $S \in \mathcal{I}_q\big(C\,(B_{X^\#} \times B_{Y^\#})\,, E\big)$, such that $T = R \circ S$, where S is 2-Lipschitz with the natural inclusion of $X \times Y$ into $C\,(B_{X^\#} \times B_{Y^\#})$. Moreover,*

$$\|T\|_{\mathcal{PI}_q^{BL}} = \inf \|R\|_{\mathcal{I}_q},$$

where the infimum is taken over all possible $S \in \mathcal{I}_q\big(C(B_{X^\#} \times B_{Y^\#}), E\big)$.

The next theorem gives an interesting relationship between the Pietsch integrability of a 2-Lipschitz operator and the Pietsch integrability of its linearization.

Theorem 6. *Let X and Y be pointed metric spaces, E be a Banach space. Let $T \in BLip_0(X, Y; E)$ be such that the following conditions are equivalent:*

(1) T is a 2-Lipschitz Pietsch-q-integral,

(2) $T_L : Æ(X)\widehat{\otimes}_\pi Æ(Y) \longrightarrow E$ is a Pietsch-q-integral.

Then

$$\|T\|_{\mathcal{PI}_q^{BL}} = \|T_L\|_{\mathcal{PI}_q}.$$

Proof. *(2)$\Longrightarrow$(1)* Suppose that $T_L \in \mathcal{PI}_q(Æ(X)\widehat{\otimes}_\pi Æ(Y), E)$. Then, by applying [18, Theorem 18], we already have a typical factorization of T_L, namely

$$T_L = B \circ i_q \circ R : Æ(X)\widehat{\otimes}_\pi Æ(Y) \xrightarrow{R} L_\infty(\mu) \xrightarrow{i_q} L_q(\mu) \xrightarrow{A} E,$$

such that $B \in \mathcal{L}(L_q(\mu), E)$ and $R \in \mathcal{L}(Æ(X)\widehat{\otimes}_\pi Æ(Y), L_\infty(\mu))$ with $\|A\|\,\|R\| \leq \|T_L\|_{\mathcal{PI}_q} + \varepsilon$ for every $\varepsilon > 0$. Consider the mapping $A := R \circ \sigma_2 \circ (\delta_X, \delta_Y) : X \times Y \longrightarrow L_\infty(\mu)$. It is easy to verify that A is a 2-Lipschitz operator and $Lip(A) \leq \|R\|$. And so, by the factorization $T = T_L \circ \sigma_2 \circ (\delta_X, \delta_Y) = B \circ i_q \circ A$, we conclude that $T \in \mathcal{PI}_q^{BL}(X, Y; E)$ and

$$\|T\|_{\mathcal{PI}_q^{BL}} \leq \|B\| \, BLip\,(A) \leq \|T_L\|_{\mathcal{PI}_q} + \varepsilon.$$

(1)$\Longrightarrow$(2) Assume that $T \in \mathcal{PI}_q^{BL}(X, Y; E)$. Then for $\varepsilon > 0$ choose the following factorization of T

$$T = B \circ i_q \circ A : X \times Y \xrightarrow{A} L_\infty(\mu) \xrightarrow{i_q} L_q(\mu) \xrightarrow{B} E,$$

such that $B \in \mathcal{L}\left(L_q(\mu), E\right)$ and $A \in BLip_0\big(X, Y; L_\infty(\mu)\big)$ with $\|B\| \, BLip(A) \leq \|T\|_{\mathcal{PI}_q^{BL}} + \varepsilon$. Then we can apply the uniqueness of liearization to infer that

$$T_L = \big(B \circ i_q \circ A\big)_L = B \circ i_q \circ A_L.$$

This shows that $T_L \in \mathcal{PI}_q\big(Æ(X)\widehat{\otimes}_\pi Æ, E\big)$ and

$$\|T_L\|_{\mathcal{PI}_q} \leq \|B\| \, \|A_L\| = \|B\| \, BLip(A) \leq \|T\|_{\mathcal{PI}_q^{BL}} + \varepsilon.$$

This completes the proof. □

The following result is a direct consequence of Theorem 6 and Proposition 1.

Corollary 6. *The 2-Lipschitz operator ideal $\mathcal{PI}_q^{BL}(X, Y; E)$ is generated by the composition method from the operator ideal $\mathcal{PI}_q$, i.e.*

$$\mathcal{PI}_q^{BL}(X, Y; E) = \mathcal{PI}_q \circ BLip_0(X, Y; E)$$

for all pointed metric spaces X, Y and Banach space E.

We now give a slightly different factorization for Pietsch-p-integral 2-Lipschitz operators. The proof of the following is inspired by [7, Theorem 2.11].

Theorem 7. *Let $1 \leq q < \infty$. Let $T : X \times Y \longrightarrow E$ be a 2-Lipschitz mapping. Then the following assertions are equivalent:*

(1) $T \in \mathcal{PI}_q^{BL}(X, Y; E)$,

(2) there exist a finite nonnegative regular countably additive Borel measures ν_1 on K_1, ν_2 on K_2 and ν on $K_1 \times K_2$, and a linear map $B : L_q(\nu) \longrightarrow E$, such that the following diagram

$$\begin{array}{ccccc} X \times Y & & \xrightarrow{\quad T \quad} & & E \\ \Big\downarrow{\scriptstyle \iota_X \times \iota_Y} & & & & \Big\uparrow{\scriptstyle B} \\ C(K_1) \times C(K_2) & \xrightarrow{S} & C(K_1 \times K_2) & \xrightarrow{j_q} & L_q(\nu) \end{array}$$

commutes, where S is the 2-Lipschitz given by

$$S(f, g) = fg$$

for all $(f, g) \in C(K_1) \times C(K_2)$, *(3) there are a finite measure spaces* $(\Omega_1, \Sigma_1, \nu_1)$, $(\Omega_2, \Sigma_2, \nu_2)$ *and* (Ω, Σ, ν), *a linear mapping* $B : L_q(\mu) \longrightarrow E$, *and two Lipschitz mappings* $A_1 : X \longrightarrow L_\infty(\nu_1)$ *and* $A_2 : X \longrightarrow L_\infty(\nu_2)$, *such that the following diagram*

$$\begin{array}{ccccc} X \times Y & & \xrightarrow{\quad T \quad} & & E \\ \Big\downarrow{\scriptstyle A_1 \times A_2} & & & & \Big\uparrow{\scriptstyle B} \\ L_\infty(\nu_1) \times L_\infty(\nu_2) & \xrightarrow{A} & L_\infty(\nu) & \xrightarrow{i_q} & L_q(\nu) \end{array}$$

commutes, where A *is the* 2*-Lipschitz, given by*

$$A(f, g) = fg$$

for all $(f, g) \in L_\infty(\nu_1) \times L_\infty(\nu_2)$.

In addition, we can choose B *in (3) such that*

$$\|T\|_{\mathcal{PI}_q^{BL}} = \inf Lip\,(A_1)\, Lip\,(A_2)\, \nu(\Omega)^{\frac{1}{q}},$$

where the infimum is taken over all factorizations as in (3) and

$$\|T\|_{\mathcal{PI}_q^{BL}} = \inf \nu\,(K_1 \times K_2)^{\frac{1}{q}},$$

where the infimum is taken over all factorizations as in (2).

Proof. *(1)*$\Longrightarrow$*(2)* By Theorem 5, there exist a regular Borel probability product measure μ on $B_{X^\#} \times B_{Y^\#}$ and a linear operator $\widetilde{B} \in \mathcal{L}\left(L_q(\mu), E\right)$ such that

$$T : X \times Y \xrightarrow{S} C\left(B_{X^\#} \times B_{Y^\#}\right) \xrightarrow{j_q} L_q(\mu) \xrightarrow{\widetilde{B}} E,$$

where j_q is the natural inclusion. Consider the 2-Lipschitz S of the statement. Obviously, $S(x, y) = R \circ (\iota_X(x), \iota_Y(y))$ for all $x \in X, y \in Y$ and we obtain a factorization as in *(2)*.

(2)$\Longrightarrow$*(3)* Let T factors as in *(2)*. Since $j_q = i_q \circ j_\infty$, where $j_\infty : C(K_1 \times K_2) \longrightarrow L_\infty(\nu)$ and $i_q : L_\infty(\nu) \longrightarrow L_q(\nu)$ are the natural inclusions. Letting A be as in the statement, and setting $\Omega_1 := K_1, \Omega_2 := K_2$, $A_1 := j^1_\infty \circ \iota_X$, and $A_2 := j^2_\infty \circ \iota_Y$, we obtain the factorization required in *(3)*.

(3)$\Longrightarrow$*(1)* Since A can obviously be seen as the natural 2-Lipschitz associated with the identity on $L_\infty(\nu)$, the 2-Lipschitz $B \circ i_q \circ A$ satisfies Definition 2, so it is 2-Lipschitz Pietsch-q-integral. By the ideal property, T is 2-Lipschitz Pietsch-q-integral too. □

Proposition 5. *Let* X, Y, W *and* Z *be Banach spaces with* $X \subset W$ *and* $Y \subset Z$. *Each* 2*-Lipschitz Pietsch-*q*-integral operator admits a* 2*-Lipschitz Pietsch* q*-integral extension* $\widetilde{T} : W \times Z \longrightarrow E$ *with* $\|T\|_{\mathcal{PI}_q^{BL}} = \|\widetilde{T}\|_{\mathcal{PI}_q^{BL}}$.

One can easily check the relation between 2-Lipschitz Pietsch-q-integral operators and another Banach 2-Lipschitz operator ideal.

As an application of Theorem 6 and [13, Remark 4.5], we infer the following corollary.

Corollary 7. *Every 2-Lipschitz Pietsch-q-integral operator is 2-Lipschitz weakly compact.*

Proof. Let $T \in \mathcal{PI}_q^{BL}(X, Y; E)$. Then we get from Theorem 6, that T_L is Pietsch-q-integral. Since $\mathcal{PI}_q(Æ(X)\widehat{\otimes}_\pi Æ(Y), E) \subset \mathcal{W}(Æ(X)\widehat{\otimes}_\pi Æ(Y), E)$ (see [3]), then by virtue of [13, Theorem 4.4 and Remark 4.5], we conclude that the T is weakly compact. □

Proposition 6. *Let $1 \leq q < \infty$. Every 2-Lipschitz q-nuclear is Pietsch-q-integral and*

$$\|T\|_{\mathcal{PI}_q^{BL}} \leq \|T\|_{\mathcal{SN}_q^{BL}}.$$

Proof. Let $T \in \mathcal{SN}_q^{BL}(X, Y; E)$ with the factorization as in Theorem 3, such that

$$\|B\| \, \|D_\alpha\| \leq \|T\|_{\mathcal{PI}_q^L} + \varepsilon,$$

where $BLip(A) \leq 1$. In this case, ℓ_∞ and ℓ_q are the spaces $L_\infty(\mu)$ and $L_q(\mu)$ with μ being the counting measure on $\mathbb{N}$, respectively, and $D_\alpha : L_\infty(\mu) \longrightarrow L_q(\mu)$ is the multiplication operator induced by $\alpha \in L_q(\mu)$, i.e. $D_\alpha(f) = \alpha \cdot f$. We use [11, Page 111] to see that D_α is a p-integral linear operator and $\|D_\alpha\|_{\mathcal{I}_q} \leq \|D_\alpha\|$ and by the ideal property of $B \circ D_\alpha \circ j_\infty \in \mathcal{I}_q(L_\infty(\mu), E)$, where j_∞ is the canonical inclusion map from $C\left(B_{X^\#} \times B_{Y^\#}\right)$ to $L_\infty(\mu)$.

Thus, using Corollary 5, we conclude that $T \in \mathcal{PI}_q^L(X, Y; E)$ and

$$\|T\|_{\mathcal{PI}_q^{BL}} \leq \|B \circ D_\alpha \circ j_\infty\|_{\mathcal{I}_q} \leq \|B\| \, \|D_\alpha\| \leq \|T\|_{\mathcal{SN}_q^{BL}} + \varepsilon.$$

If $\varepsilon \to 0$, we are done. □

Open problem. *Under which condition every 2-Lipschitz Pietsch-q-integral is q-nuclear and*

$$\|T\|_{\mathcal{SN}_q^{BL}} \leq \|T\|_{\mathcal{PI}_q^{BL}}.$$

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Хаміді Х. *2-ліпшицеві Пітч-q-інтегральні та 2-ліпшицеві q-ядерні оператори* // Карпатські матем. публ. — 2025. — Т.17, №2. — С. 416–432.

У цій статті ми розширюємо поняття q-ядерності на 2-ліпшицеві відображення. Ми наводимо теорему факторизації та встановлюємо, що 2-ліпшицевий оператор є q-ядерним тоді і тільки тоді, коли його транспонований оператор існує та є q-ядерним. Додатково ми вводимо поняття 2-ліпшицевих Пітч-q-інтегральних операторів, подаємо теорему факторизації та показуємо, що цей клас операторів породжується методом композиції. У підсумку доводиться, що кожен 2-ліпшицевий q-ядерний оператор є 2-ліпшицевим Пітч-q-інтегральним оператором.

Ключові слова і фрази: 2-ліпшицевий операторний ідеал, лінійний q-ядерний оператор, лінійний Пітч-q-інтегральний оператор, теорема факторизації.