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Normal curves using the Bishop frame in Minkowski 3-space

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The geometric characterization of curves with zero curvature points presents inherent limitations when employing the classical Frenet frame. This research investigates the mathematical properties of normal curves in E_1^3 (Minkowski 3-space) through the application of the Bishop frame. The study systematically analyzes the geometric and topological properties of both spacelike and timelike normal curves, establishing rigorous mathematical conditions necessary and sufficient for their classification within the Bishop frame formalism. We derive fundamental theorems characterizing these curves and examine their differential geometric invariants. The investigation extends to the analysis of curvature relationships, parallel transport properties, and the behavior of normal curves under the Bishop frame parametrization. Our findings contribute to the theoretical framework of curve theory in pseudo-Riemannian geometry, particularly in spaces with indefinite metrics, demonstrating the robustness of the Bishop frame approach for characterizing curves where traditional Frenet analysis becomes singular.

Key words and phrases: Lorentz-Minkowski space, parallel transport frame, Bishop equation, Bishop curvature, normal curve, differential geometry.

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Introduction

The geometric study of curves in Minkowski space has garnered significant attention due to its fundamental role in differential geometry and mathematical physics, particularly in special relativity theory. The classical approach to studying curves typically employs the Frenet frame, which provides a moving reference frame along a curve. However, this framework encounters significant limitations when analyzing curves at points where the second derivative approaches zero or becomes undefined.

The Bishop frame, alternatively known as the parallel transport frame, was first introduced by R.L. Bishop [1] as an alternative approach to curve framing. This framework addresses the limitations of the Frenet frame by maintaining well-defined behavior even at points where traditional curvature becomes undefined. The fundamental principle behind the Bishop frame lies in its parallel transport of frame vectors along the curve, providing a more robust geometric description.

A.J. Hanson and H. Ma [14] further developed this concept by conducting a comparative analysis between the Bishop and Frenet frames in Euclidean 3-space, establishing the advantages of the Bishop frame in various geometric contexts. The extension of these ideas to higher dimensions was subsequently explored by F. Gökçelik et al. [12] in their investigation of the

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Bishop frame in Euclidean 4-space.

Recent years have witnessed substantial advances in the application of the Bishop frame to Minkowski space. Notable contributions include the works of B. Bükcü and M.K. Karacan [2, 3, 17], who extensively investigated non-null curves using the Bishop frame in Minkowski 3-space. Their research has revealed intricate relationships between the geometric properties of curves and their Bishop frame representations.

Additionally, numerous mathematicians have studied related curves in different spaces and according to various frames such as [4–6, 9].

Let us outline the structure of our investigation. First, we present a comprehensive analysis of the prerequisites for spacelike curves to be normal in Minkowski three-space. Specifically, we identify the necessary and sufficient conditions for this to occur and proceed to examine various geometric characteristics of this space as it pertains to E_1^3. Subsequently, we turn our attention to studying timelike normal curves in Minkowski three-space with the Bishop frame. Our investigation delves into the intricacies of these curves and aims to provide an insightful analysis of their behavior.

1 Preliminaries

The mathematical entity referred to as Lorentz-Minkowski space E_1^3 comprises a pair $(\mathbb{R}^3, g)$, in which $\mathbb{R}^3$ denotes the Euclidean 3-dimensional space, while g denotes the associated metric

$$g = -dv_1^2 + dv_2^2 + dv_3^2,$$

where (v_1, v_2, v_3) is a coordinate system of E_1^3. Assuming that y is a vector belonging to the three-dimensional Euclidean space E_1^3, it is considered spacelike if $g(y, y)$ is greater than zero or if y is the zero vector. On the other hand, if $g(y, y)$ is less than zero, then y is referred to as a timelike vector. A vector that satisfies $g(y, y) = 0$ and is not the zero vector is classified as a null or lightlike vector. A vector in E_1^3 is considered non-null if it is either spacelike or timelike. The value $\|y\| = \sqrt{|g(y, y)|}$ determines the norm of the non-null vector y. If $\|y\| = 1$, the vector y is called a unit. If $g(y, z) = 0$, we say that the vectors y and z are orthogonal. The Lorentzian vector product of the vectors y and z is defined in [10, 13, 15, 16] by

$$y \wedge z = (y_3 z_2 - y_2 z_3,\ y_3 z_1 - y_1 z_3,\ y_1 z_2 - y_2 z_1),$$

where $y = (y_1, y_2, y_3), z = (z_1, z_2, z_3)$.

A curve in E_1^3 can be mathematically represented as a mapping from an open interval I to E_1^3. The velocity vectors of the curve are calculated using the first derivative of each curve component. The classification of a curve as either spacelike, timelike, or null (lightlike) is determined by the nature of its velocity vectors: a spacelike curve has spacelike velocity vectors, a timelike curve has timelike velocity vectors, and a null curve has null velocity vectors.

To fully describe the motion of a curve in E_1^3, we use the Frenet frame, which is a set of orthogonal vectors that move along the curve. This frame is described by three vectors: the tangent vector T, the principal normal vector N, and the binormal vector B. The tangent vector T points in the direction of the curve's motion, the principal normal vector N points in the direction of the curve's curvature, and the binormal vector B completes the orthogonal set. Consequently, concerning a specific point on a curve $\alpha(s)$ situated in E_1^3, the vectors T, N, and B correspondingly represent its tangential, principal normal, and binormal vectors, where

s is the parameter used to trace the curve. These advancements in the use of the Frenet frame have been widely discussed and developed in [7,8,11,22].

Additionally, in conformity with the discoveries from the publication referenced as [20], the planes established by the vectors $\{T,N\}$, $\{T,B\}$, and $\{N,B\}$ are known as the osculating plane, the rectifying plane, and the normal plane, correspondingly. The characteristic term "normal curve" is commonly employed to indicate a unit speed curve present in Minkowski 3-space E_1^3, whose position vector perpetually lies in the plane that it defines as normal. This definition of the normal curve allows us to express $\alpha(s)$ as

$$\alpha(s) = E(s)N(s) + D(s)B(s),$$

where E and D are differentiable functions of $s \in I \subset R$.

Various descriptions of normal curves that are spacelike, timelike, or null are outlined in [15, 16]. These curves lie completely in the Minkowski three-dimensional space. The following Frenet formulae are introduced in [13, 15, 19, 23] for a curve $\alpha(s)$ in the Minkowski 3-space.

If the curve α is spacelike and its principal normal N is either spacelike or timelike, the Frenet formulas can be expressed as

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\epsilon\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where $g(T,T) = 1, g(N,N) = \epsilon = \pm 1, g(B,B) = -\epsilon$ and $g(T,N) = g(T,B) = g(N,B) = 0$.

Assuming α represents a timelike curve in E_1^3, the Frenet formulae can be given as

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where $g(T,T) = -1,\ g(N,N) = 1,\ g(B,B) = 1$ and $g(T,N) = g(T,B) = g(N,B) = 0$.

Assuming the curve α possesses a unitary velocity, then

$$\kappa = \left\| \alpha''(s) \right\|, \qquad \tau = \left\| b'(s) \right\|.$$

The following are the definitions of three pseudo-Riemannian objects of interest in this paper. The pseudo-Riemannian sphere denoted as $S_1^2(x,r)$, the pseudo-Riemannian hyperbolic space denoted as $H_0^2(x,r)$ and the pseudo-Riemannian lightlike cone denoted as $C(x)$ are defined, respectively, by

$$S_1^2(x,r) = \{y \in E_1^3 : g(y-x, y-x) = r^2\},$$

$$H_0^2(x,r) = \{y \in E_1^3 : g(y-x, y-x) = -r^2\},$$

$$C(x) = \{y \in E_1^3 : g(y-x, y-x) = 0\},$$

where x denotes a point inside E_1^3 and r indicates a positive constant value [15,16].

The "Bishop frame", or "parallel transport frame", provides an alternative method for constructing a moving frame that remains well-defined even when the curve has a vanishing second derivative. This is achieved by transporting each component of an orthonormal frame in parallel along the curve to keep the frame aligned with the curve. Although each curve model has a unique $T(s)$ tangent vector, any arbitrary orthonormal basis $(U(s), V(s))$ can be used for the rest of the frame as long as it lies in the normal plane perpendicular to $T(s)$ at each point and the derivatives of $(U(s), V(s))$ depend only on $T(s)$ and not each other. This ensures that $U(s)$ and $V(s)$ move smoothly along the path, independent of the curve's curvature. Assuming that the curve α is parameterized by arc length, we can derive that $V = T \wedge U$ along α by using C^1 unit vector fields U and T, where U and V are orthogonal vectors.

As we move along the curve, the frames T, U, and V gradually transform into right-handed orthonormal frames. However, to impose the additional requirement that the field is parallel to the curve α, it is necessary for $g(U', V)$ to hold. This indicates that the unit's first normal vector field, U, moves only in the direction of T, without any other movement. It should be noted that the Frenet frame would be suitable for a curve C^3 with $\kappa \neq 0$ up to this point.

Despite the presence of regions where $\kappa = 0$, it is possible to establish a Bishop frame that comprises the first B-normal U, followed by the second B-normal V. The complementary frame equations are developed as a result [14, 18, 21].

When considering a spacelike curve with a B-normal vector field U that is either spacelike or timelike with ϵ values of 1 or -1 respectively, the Bishop formulas can be mathematically expressed as

$$\begin{bmatrix} T' \\ U' \\ V' \end{bmatrix} = \begin{bmatrix} 0 & K_1 & -K_2 \\ -\epsilon K_1 & 0 & 0 \\ -\epsilon K_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ U \\ V \end{bmatrix}, \tag{1}$$

where $g(T, T) = 1, g(U, U) = \epsilon = \pm 1, g(V, V) = -\epsilon, \kappa(s) = \sqrt{|K_1^2 - K_2^2|}, \theta(s) = \arg\tanh\left(\frac{K_2}{K_1}\right)$, $\tau(s) = -\epsilon \frac{d\theta}{ds}$ (see [18]).

In the case of α is a timelike curve, the Bishop formulas are

$$\begin{bmatrix} T' \\ U' \\ V' \end{bmatrix} = \begin{bmatrix} 0 & K_1 & K_2 \\ K_1 & 0 & 0 \\ K_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ U \\ V \end{bmatrix}, \tag{2}$$

where $g(T, T) = -1, g(U, U) = 1, g(V, V) = 1$ and $\kappa(s) = \sqrt{K_1^2 + K_2^2}$, $\theta(s) = \arctan\left(\frac{K_2}{K_1}\right)$, $\tau(s) = \frac{d\theta}{ds}$ (see [18]).

Timelike and null normal curves in Minkowski 3-space according to the Frenet frame were introduced in [16]. Also, K. İlarslan studied spacelike normal curves in Minkowski space E_1^3 according to the Frenet frame in [15].

2 Spacelike Bishop normal curves in E_1^3

In this section, we present the definition and fundamental properties of spacelike Bishop normal curves in the three-dimensional Minkowski space E_1^3. Specifically, we establish several theorems concerning the geometric characteristics of these curves, providing a deeper understanding of their behavior and structure.

Definition 1. *A spacelike curve $\alpha(s)$ can be considered a Bishop normal curve in Minkowski 3-space if and only if its position vector remains within the normal plane. This normal plane is defined as the one that is spanned by vectors U and V. In the other words, $\alpha(s)$ can be written as*

$$\alpha(s) = E(s)U(s) + D(s)V(s)$$

for some differentiable functions $E(s)$ and $D(s)$.

Theorem 1. *If $\alpha(s)$ is a spacelike Bishop normal curve that has either a spacelike or a timelike B-normal vector U in E_1^3, then the following assertions are valid.*

i) The functions $K_1(s)$ and $K_2(s)$ satisfy

$$AK_1(s) + CK_2(s) = -\epsilon,$$

where A and C are non-zero constants.

ii) The components of the first and the second B-normal components with the position vector are provided, respectively, by

$$g(\alpha, U) = A\epsilon,$$

$$g(\alpha, V) = -C\epsilon.$$

iii) If the curve's position vector is null, then α lies on $C(x)$, and the Bishop curvatures satisfy

$$K_1 \pm K_2 = \frac{-\epsilon}{A}.$$

If $\alpha(s)$ is a spacelike curve, it can be deduced that α is a Bishop normal curve or has equivalency to a Bishop normal curve if any of the conditions (i), (ii), or (iii) hold.

Proof. Assume that $\alpha(s)$ is a spacelike Bishop normal curve, then we can write

$$\alpha(s) = E(s)U(s) + D(s)V(s). \tag{3}$$

Differentiating with respect to s and using formulas (1), we deduce

$$E' = 0, \quad D' = 0, \quad -\epsilon EK_1 - \epsilon DK_2 = 1. \tag{4}$$

Therefore, we get

$$E(s) = A, \quad D(s) = C, \tag{5}$$

where A and C are constants.

Thus from the third relation of equation (4), we obtain

$$AK_1 + CK_2 = -\epsilon. \tag{6}$$

By substituting (5) into (3), we get

$$\alpha(s) = AU(s) + CV(s).$$

Therefore, we obtain

$$g(\alpha,\alpha) = \epsilon(A^2 - C^2), \tag{7}$$
$$g(\alpha,U) = A\epsilon,$$
$$g(\alpha,V) = -C\epsilon.$$

Let us continue and assume that α is a spacelike Bishop normal curve that has a null position vector. Therefore, we get that $g(\alpha,\alpha)$ is equal to zero. Therefore, using equation (7), we have

$$A = \pm C.$$

Hence, from (6), we obtain

$$AK_1 \pm AK_2 = -\epsilon.$$

Thus,

$$K_1 \pm K_2 = \frac{-\epsilon}{A}.$$

Now, let x be a vector written as

$$x = \alpha(s) - AU - CV(s).$$

After doing the differentiation with regard to s and using the related Bishop equation (1), we obtain that $x' = 0$, and therefore $x = \text{const}$.

Thus, we get

$$g(\alpha - x, \alpha - x) = A^2\epsilon - C^2\epsilon = 0.$$

That is to say that α is located on $C(x)$.

In the other direction, let us assume that statement *(i)* is correct. Then the curvatures satisfy

$$AK_1 + CK_2 = -\epsilon.$$

Using Bishop formulas (4), we get

$$\frac{d}{ds}[\alpha(\sigma) - AU - CV] = 0.$$

As a direct consequence, α is congruent to a Bishop normal curve.

Secondly, let *(ii)* be correct. Then

$$g(\alpha,U) = \epsilon A, \quad g(\alpha,V) = -\epsilon C.$$

By differentiating these equations with regard to s and using the Bishop formulas (1), we obtain $g(\alpha,T) = 0$, which implies that α is a Bishop normal curve.

Finally, let us suppose that *(iii)* is true. Therefore, α is located on $C(x)$ with vertex at m, m is constant and curvatures $K_1(s)$ and $K_2(s)$ satisfy

$$K_1(s) \pm AK_2(s) = \frac{-\epsilon}{A}.$$

As a result, we deduce

$$g(\alpha - x, \alpha - x) = 0.$$

After differentiating three times with respect to s and using the Bishop formulas (1), we arrive to the following

$$\alpha - x = 0T + \frac{\epsilon\kappa_2'}{\kappa_1' K_2 - K_1\kappa_2'}U - \frac{\epsilon\kappa_1'}{\kappa_1' K_2 - K_1\kappa_2'}V.$$

This indicates that, up to a translation for the vector x, the curve α is congruent to a Bishop normal curve. Let us assume that $x = 0$, and then by using the formula $K_1(s) \pm AK_2(s) = \frac{-\epsilon}{A}$, we can readily determine that $g(\alpha, \alpha) = 0$, which concludes the proof. □

Theorem 2. *If a unit-speed spacelike Bishop normal curve denoted as $\alpha = \alpha(s)$ in E_1^3 and accompanied by a non-null B-normal vector field U and non-null position vector, then*

(i) the position vector can be regarded as spacelike if and only if the curve α is positioned on $S_1^2(x, r)$ and satisfies

$$CK_2 \pm \sqrt{\frac{r^2}{\epsilon} + C^2} = -\epsilon;$$

(ii) the position vector can be regarded as timelike if and only if the curve α is positioned on $H_0^2(x, r)$ and satisfies

$$CK_2 \pm \sqrt{\frac{r^2}{\epsilon} - C^2} = -\epsilon.$$

Proof. Let the position vector α be spacelike. This means that $g(\alpha, \alpha) = r^2$, where $r \in R^+$. In equation (7), we get by substituting that

$$A = \pm\sqrt{\frac{r^2}{\epsilon} + C^2}.$$

By substituting in equation (6), we have

$$CK_2 \pm \sqrt{\frac{r^2}{\epsilon} + C^2} = -\epsilon. \tag{8}$$

Next, assume that the vector x is given by

$$x = \alpha(s) - AU(s) - CV(s).$$

We have obtained this result by applying formulas (1) and differentiating this with regard to s, we can find that $x' = 0$, and therefore $x = \text{const}$.

Then

$$g(\alpha - x, \alpha - x) = \epsilon A^2 - \epsilon C^2 = r^2.$$

This indicates that α is located on $S_1^2(x, r)$ with center x and of radius $r \in R^+$.

On the other hand, let us assume that equation (8) is true and that the value of α can be found on $S_1^2(x, r)$. If this is the case, $g(\alpha - x, \alpha - x) = r^2$, where r is contained inside R^+. We get this result by doing three separate differentiation with regard to s, i.e.

$$\alpha - x = 0T + \frac{\epsilon\kappa_2'}{\kappa_1' K_2 - K_1\kappa_2'}U - \frac{\epsilon\kappa_1'}{\kappa_1' K_2 - K_1\kappa_2'}V.$$

This means that α is congruent to a normal curve up to a translation for a vector m. For convenience, let $x = 0$. As a result, we conclude that $g(\alpha, \alpha) = r^2$, which provides indisputable evidence for statement *(i)*.

A similar argument may be made for *(ii)*, as seen in the proof of statement *(i)*. □

3 Timelike Bishop normal curves E_1^3

In this section, we present the concept of a timelike Bishop normal curve in E_1^3 from the perspective of the Bishop frame and provide some characterization theorems for timelike normal curves in E_1^3.

Definition 2. *The timelike curve $\alpha(s)$ is a Bishop normal curve in E_1^3 if and only if its position vector is always contained inside the normal plane spanned by U, V, i.e. $g(\alpha, T) = 0$.*

Theorem 3. *Let $\alpha = \alpha(s)$ be a timelike curve in E_1^3. Then α is a Bishop normal curve if and only if the position vector α of U and V are given respectively by*

$$g(\alpha, U) = C_1,$$

$$g(\alpha, V) = C_2,$$

where C_1 and C_2 are constants.

Proof. If we take it for granted that $\alpha(s)$ is a timelike Bishop normal curve, then we may write $\alpha(s)$ as

$$\alpha(s) = G(s)U(s) + F(s)V(s). \tag{9}$$

When we differentiate with regard to s, we get the following

$$G' = 0, \quad F' = 0, \quad EK_1 + FK_2 = 1. \tag{10}$$

Then, we obtain

$$G = C_1, \quad F = C_2. \tag{11}$$

Thus from the third relation of equation (10) we obtain

$$C_1K_1 + C_2K_2 = 1. \tag{12}$$

From (11) and (9) we obtain

$$\alpha(s) = C_1U(s) + C_2V(s).$$

Thus, we deduce

$$g(\alpha, \alpha) = C_1^2 - C_2^2,$$

$$g(\alpha, U) = C_1, \tag{13}$$

$$g(\alpha, V) = C_2. \tag{14}$$

Conversely, suppose that equations (13), (14) are true. By differentiating equations (13), (14) with respect to s and using (2), we get $g(\alpha, T) = 0$, which implies that α is a normal curve. □

Theorem 4. *Let $\alpha = \alpha(s)$ be a timelike curve in E_1^3. Then α is congruent to a normal Bishop curve if and only if*

$$C_1K_1 + C_2K_2 = 1.$$

Proof. First, assume that α is congruent to a normal Bishop curve, then equation (12) holds. Conversely, suppose that $C_1K_1 + C_2K_2 = 1$. By applying (2), we can easily obtain

$$\frac{d}{ds}[\alpha(\sigma) - AU - CV] = 0.$$

In this way, α is congruent to a Bishop normal curve. □

Theorem 5. *Let $\alpha = \alpha(s)$ be a timelike curve in E_1^3 according to Bishop frame. Then α is located on $S_1^2(x,r)$ if and only if α is a Bishop normal curve.*

Proof. If we assume that α is located on $S_1^2(x,r)$, we may deduce that $g(\alpha - x, \alpha - x) = r^2$ and that $r \in R^+$. We discover this by doing three separate differentiations with regard to s and using (2), so we get

$$\alpha - x = 0T + \frac{\kappa_2'}{\kappa_1' K_2 - K_1 \kappa_2'} U + \frac{\kappa_1'}{\kappa_1' K_2 - K_1 \kappa_2'} V.$$

This indicates that a vector named x may be translated in such a way that it would provide the same result as a Bishop normal curve, and that vector is α. Put $x = 0$ for convenience.

On the other hand, suppose that α is a Bishop normal curve, and let us put the vector x in the form

$$x = \alpha(s) - C_1 U(s) - C_2 V(s).$$

By differentiating this with respect to s and then applying equations (2), we get $x' = 0$ and therefore $x = \text{const}$. Then $g(\alpha - x, \alpha - x) = C_1^2 + C_2^2 = r^2$. This demonstrates that α is located on $S_1^2(x,r)$, which has x as its center and $r \in R^+$ as its radius. □

4 Conclusion

This investigation has established several fundamental theorems characterizing the behavior of normal curves in Minkowski 3-space E_1^3 utilizing the Bishop frame formalism. Our analysis yields the following several significant results.

1. For spacelike Bishop normal curves with non-null B-normal vector fields, we have demonstrated that the Bishop curvatures $K_1(s)$ and $K_2(s)$ satisfy the relationship

   $$AK_1(s) + CK_2(s) = -\epsilon,$$

   where A and C are non-zero constants. This relationship provides a complete characterization of such curves and establishes necessary and sufficient conditions for their existence.

2. The study reveals that spacelike Bishop normal curves with non-null position vectors exhibit distinct geometric properties depending on their causal character:

   - for spacelike position vectors, the curves lie on $S_1^2(x,r)$ with the Bishop curvatures satisfying $CK_2 \pm \sqrt{r^2\epsilon + C^2} = -\epsilon$;
   - for timelike position vectors, the curves are confined to $H_0^2(x,r)$ with the Bishop curvatures satisfying $CK_2 \pm \sqrt{r^2\epsilon - C^2} = -\epsilon$.

3. For timelike Bishop normal curves, we have established that their characterization is completely determined by the inner products $g(\alpha, U) = C_1$ and $g(\alpha, V) = C_2$, where C_1 and C_2 are constants. This result provides a powerful geometric criterion for identifying timelike normal curves in E_1^3.

4. The investigation demonstrates that timelike curves in E_1^3 are congruent to Bishop normal curves if and only if their Bishop curvatures satisfy $C_1K_1 + C_2K_2 = 1$, establishing a fundamental relationship between the curve's geometry and its Bishop frame parameters.

These results significantly extend the existing theory of normal curves in Minkowski space by providing a comprehensive framework using the Bishop frame approach. The advantages of this approach become particularly evident when dealing with curves having vanishing curvature points, where the traditional Frenet frame analysis becomes singular. Our findings contribute to the broader understanding of curve theory in pseudo-Riemannian manifolds and provide new tools for analyzing geometric properties of curves in Minkowski space.

Future research directions may include:

- extension of these results to higher-dimensional Minkowski spaces;
- investigation of Bishop normal curves with null position vectors;
- analysis of special classes of Bishop normal curves with prescribed curvature functions;
- applications to physical problems in special relativity, where timelike and spacelike curves play significant roles.

The theoretical framework, developed in this study, provides a foundation for further investigations in differential geometry and its applications to mathematical physics, particularly in contexts, where the causal character of curves plays a crucial role.

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Геометрична характеристика кривих із точками нульової кривини має притаманні обмеження при використанні класичного репера Френе. У цій роботі досліджуються математичні властивості нормальних кривих у просторі E^3_1 (тривимірному просторі Мінковського) із застосуванням репера Бішопа. Систематично проаналізовано геометричні та топологічні властивості як просторовоподібних, так і часоподібних нормальних кривих, а також встановлено строгі необхідні та достатні умови їх класифікації в межах формалізму репера Бішопа. Отримано основні теореми, що характеризують ці криві, та досліджено їх диференціально-геометричні інваріанти. Крім того, проведено аналіз співвідношень між кривими, властивостей паралельного перенесення та поведінки нормальних кривих за параметризації, пов'язаної з репером Бішопа. Отримані результати розширюють теоретичні засади теорії кривих у псевдорімановій геометрії, зокрема в просторах із невизначеною метрикою, і демонструють ефективність підходу, заснованого на репері Бішопа, для дослідження кривих у випадках, коли традиційний аналіз за допомогою репера Френе стає сингулярним.

Ключові слова і фрази: простір Лоренца-Мінковського, репер паралельного перенесення, рівняння Бішопа, кривина Бішопа, нормальна крива, диференціальна геометрія.