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Coincidence points of pairwise eventually H-restrictive multi-valued maps with applications in fuzzy fixed point results

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In this article, we define a generalized concept of pairwise eventually H-restrictive multi-valued mappings which is based on the restrictive conditions described in [Symmetry 2020, **12** (1), 127]. We also utilize the concept of $D(\epsilon)$-restrictive mappings to explore the coincidence points of the pair of a continuous single-valued map and an H-continuous multi-valued map. Our main results are further applied to deduce their corresponding fuzzy fixed point theorems. The latter technique is in appreciation of the fact that many results from the existing literature on contractive or nonexpansive multifunctions have their fuzzy counterparts. Finally, for the validity of our results, some useful examples are also added.

Key words and phrases: H-restrictive mapping, $D(\epsilon)$-restrictive mapping, coincidence point.

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1 Preliminaries

For the space $CB\left(\Omega\right)$ of closed and bounded subsets of the metric space (Ω, d), the distance function $H : CB\left(\Omega\right) \times CB\left(\Omega\right) \to [0, \infty)$ is defined by

$$H\left(U, V\right) = \max\left\{ \sup D\left(\zeta, U\right), \sup D\left(\eta, V\right) \right\}, \quad U, V \in CB\left(\Omega\right),$$

where $D\left(\eta, V\right) = \inf_{\zeta \in V} d\left(\eta, \zeta\right)$. Then the pair $(CB(\Omega), H)$ is called Hausdorff metric space.

Definition 1 ([17]). *Let $\Phi : \Omega \to CB\left(\Omega\right)$ be a multi-valued map. Then $v \in \Omega$ is termed as a fixed point of Φ if $v \in \Phi v$.*

Remark 1. *In the metric space $(CB(\Omega), H)$, $v \in \Omega$ is a fixed point of Φ if and only if $D(v, \Phi v) = 0$.*

Definition 2. *The function D is called continuous if $v_n \to v$ implies $D\left(v_n, U\right) \to D\left(v, U\right)$ as $n \to \infty$ for any $U \subseteq \Omega$.*

Some properties regarding d, D and H are discussed in [8, 9]. For further study of fixed points for multi-valued mappings we refer to [2, 14, 16]. The notions of upper and lower semi continuity are used to define the continuity of a multi-valued map in a metric space [15]. For the Hausdorff distance H, the concept of H-continuity is defined in the following way.

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Definition 3. *Let (Ω, d) be a metric space. A multi-valued map $\Phi : \Omega \to CB(\Omega)$ is said to be H-continuous at a point v_0, if for each sequence $\{v_n\} \subset \Omega$ we have $\Phi v_n \to \Phi v_0$ whenever $v_n \to v_0$ as $n \to \infty$.*

Definition 4 ([17])**.** *A multi-valued map $\Phi : \Omega \to CB(\Omega)$ is a multi-valued contraction if*

$$H(\Phi v, \Phi \nu) \leq \lambda d(v, \nu)$$

for all $v, \nu \in \Omega$, where $\lambda \in [0, 1)$.

Definition 5 ([7,10])**.** *Let (Ω, d) be a metric space and $\Phi : \Omega \to CB(\Omega)$ be a multi-valued map. An Φ-Orbital (or in a simple way Orbital) sequence at a point $v \in \Omega$ is a collection* $\mathrm{Or}(v, \Phi)$ *of points in Ω defined by* $\mathrm{Or}(v, \Phi) = \{v_0 = v, v_{n+1} \in \Phi v_n, n = 0, 1, 2, \ldots\}$.

M. Edelstein [12,13] proposed various modifications of the well-known Banach contraction principle by improving the set of assumptions. In this continuation, the notion of ε-contractive mappings was initiated [12] and some fixed and periodic point results were proved.

In this article, inspired by work of P. Debnath and M. de La Sen [11], we extend the concepts of eventually D-restrictive and $D(\varepsilon)$-restrictive multi-valued maps to eventually H-restrictive to evaluate coincidence point results for the pair of multi-valued maps. In this work, Edelstein type restrictive conditions [3,6] are utilized for the pairs of multi-valued maps.

2 Main results

In this section, we define a generalized concept of pairwise eventually H-restrictive multi-valued mappings, which is based on the restrictive conditions described in [11], and evaluate the coincidence points of the pair of mappings in Theorem 1. We also utilize the concept of $D(\epsilon)$-restrictive mappings to explore the coincidence points of the pair of a continuous single-valued map and an H-continuous multi-valued map. For the validity of our results some useful examples are also added.

The notation *coin* (Φ, Ψ) refers to the set of all coincidence points of the multi-valued mappings $\Phi, \Psi : \Omega \to 2^{\Omega}$.

Definition 6. *Let (Ω, d) be a metric space. A multi-valued map $\Phi : \Omega \to CB(\Omega)$ is called an eventually D-restrictive if there exists $k \in \mathbb{N}$ such that $D(v_n, \Phi v_n) < D(v, \Phi v)$ for all $n \geq k$, whenever $D(v, \Phi v) > 0$ for some $v \in \Omega$, where $v_n \in \mathrm{Or}(v, \Phi)$.*

Definition 7. *Let (Ω, d) be a metric space. A multi-valued map $\Phi : \Omega \to CB(\Omega)$ is termed as p-Orbitally continuous if whenever a subsequence $\{v_{n_i}\}$ of a sequence $\{v_n\}$ converges to some $v \in \Omega$, i.e. if $v_{n_i} \to v$, then for any fixed positive integer p, $v_{n_i+p} \to v_p$ and $\Phi v_{n_i+p} \to \Phi v_p$, where v_p is the pth term of the Φ-Orbital sequence* $\mathrm{Or}(v, \Phi)$.

Definition 8. *Let (Ω, d) be a metric space and $\Phi, \Psi : \Omega \to CB(\Omega)$ be any two multi-valued maps. An (Φ, Ψ)-Orbital (or in a simple way Orbital) sequence of Φ and Ψ at a point $v \in \Omega$ is a set* $\mathrm{Or}(v, \Phi, \Psi)$ *of points in Ω defined by*

$$\mathrm{Or}(v, \Phi, \Psi) = \{v_0 = v, v_{2n+1} \in \Phi v_{2n}, v_{2n+2} \in \Psi v_{2n+1}, n = 0, 1, 2, \ldots\}.$$

Definition 9. *Let (Ω, d) be a metric space. Two multi-valued maps $\Phi, \Psi : \Omega \to 2^{\Omega}$ are called pairwise eventually H-restrictive if there exists a positive integer k such that $H(\Phi v_{2n}, \Psi v_{2n+1}) < H(\Phi v, \Psi v)$ for all $n \geq k$, whenever $H(\Phi v, \Psi v) > 0$ for some $v \in \Omega$, where $v_n \in \mathrm{Or}(v, \Phi, \Psi)$.*

Theorem 1. *Let (Ω, d) be a metric space and $\Phi, \Psi : \Omega \to 2^{\Omega}$ be any two pairwise eventually H-restrictive p-Orbitally continuous multi-valued mappings. If for some $v_0 \in \Omega$, the Orbital sequence $\mathrm{Or}(v, \Phi, \Psi)$ of the pair (Φ, Ψ) has a subsequence $\{v_{n_i}\}$ converging to v, then $v \in coin\,(\Phi, \Psi)$.*

Proof. Firstly, we assume that $v_{n+1} \neq v_n$ and also $v_n \notin coin\,(\Phi, \Psi)$ for all $n \in \mathbb{N}$, otherwise we obviously get a coincidence point.

As $\lim\limits_{i\to\infty} v_{n_i} = v$ and the pair (Φ, Ψ) is p-Orbitally continuous, for a fixed element $p \in \mathbb{N}$, we get

$$v_{n_i+p} \to v_p, \qquad \Phi v_{2(n_i+p)} \to \Phi v_{2p}, \qquad \Psi v_{2(n_i+p)+1} \to \Psi v_{2p+1}.$$

Let $\Phi v \cap \Psi v = \phi$ and $H(\Phi v, \Psi v) > 0$. Since (Φ, Ψ) is pairwise eventually H-restrictive, there exists $r \in \mathbb{N}$ such that $H(\Phi v_{2n}, \Psi v_{2n+1}) < H(\Phi v, \Psi v)$ for all $n \geq r$. Also

$$H(\Phi v_{2r}, \Psi v_{2r+1}) < H(\Phi v, \Psi v).$$

Let

$$\varepsilon' = \frac{1}{2}\big[H(\Phi v, \Psi v) - H(\Phi v_{2r}, \Psi v_{2r+1})\big] > 0.$$

By using the continuity of H and p-Orbital continuity of the pair (Φ, Ψ), we get

$$\lim_{i\to\infty} H\big(\Phi v_{2(n_i+r)}, \Psi v_{2(n_i+r)+1}\big) = H\big(\Phi v_{2r}, \Psi v_{2r+1}\big).$$

For an arbitrary positive real number $\varepsilon > 0$, there exists an adequately large $n_i = q$ such that

$$H\big(\Phi v_{2(q+r)}, \Psi v_{2(q+r)+1}\big) < H\big(\Phi v_{2r}, \Psi v_{2r+1}\big) + \varepsilon.$$

For $\varepsilon = \varepsilon'$, we have

$$\begin{aligned} H\big(\Phi v_{2(q+r)}, \Psi v_{2(q+r)+1}\big) &< H\big(\Phi v_{2r}, \Psi v_{2r+1}\big) + \frac{1}{2}H\big(\Phi v, \Psi v\big) - \frac{1}{2}H\big(\Phi v_{2r}, \Psi v_{2r+1}\big) \\ &< \frac{1}{2}\big[H(\Phi v_{2r}, \Psi v_{2r+1}) + H(\Phi v, \Psi v)\big]. \end{aligned}$$

Since $H(\Phi v_{2n}, \Psi v_{2n+1}) > 0$ (for $v_{2n} \neq v_{2n+1}$ and $v_n \notin coin\,(\Phi, \Psi)$) and (Φ, Ψ) is pairwise eventually H-restrictive, there exists a non-negative integer k such that

$$H\big(\Phi v_{2(n+q+r)}, \Psi v_{2(n+q+r)+1}\big) < H\big(\Phi v_{2(q+r)}, \Psi v_{2(q+r)+1}\big)$$

for all $n \geq k$, where $v_{n+q+r} \in \mathrm{Or}(v_{q+r}, \Phi, \Psi)$. Similarly

$$H\big(\Phi v_{2n}, \Psi v_{2n+1}\big) < H\big(\Phi v_{2(q+r)}, \Psi v_{2(q+r)+1}\big)$$

for all $n \geq k + q + r$, where $v_n \in \mathrm{Or}(v_{q+r}, S, \Phi)$. So, we obtain

$$H\big(\Phi v_{2n}, \Psi v_{2n+1}\big) < \frac{1}{2}\big[H(\Phi v, \Psi v) + H(\Phi v_{2r}, \Psi v_{2r+1})\big]$$

for all $n > k + q + r$. Thus,

$$H\left(\Phi v_{2n_i}, \Psi v_{2n_i+1}\right) < \frac{1}{2}\left[H(\Phi v, \Psi v) + H(\Phi v_{2r}, \Psi v_{2r+1})\right].$$

Consequently, applying limit $i \to \infty$ and using the facts, we obtain

$$H(\Phi v, \Psi v) = \lim_{i\to\infty} H\left(\Phi v_{2n_i}, \Psi v_{2n_i+1}\right) \leq \frac{1}{2}\left[H(\Phi v, \Psi v) + H(\Phi v_{2r}, \Psi v_{2r+1})\right].$$

Hence

$$H(\Phi v, \Psi v) \leq H(\Phi v_{2r}, \Psi v_{2r+1}),$$

which is a contradiction. Therefore $\Phi v \cap \Psi v \neq \phi$. $\square$

Example 1. *Let* $\Omega = [0,1] \cup \{\xi, \eta\}$, *where* $\xi, \eta \in \mathbb{R} \setminus [0,1]$, $d(x,y) = |x-y|$. *Let us define* $\Phi, \Psi : \Omega \to CB(\Omega)$ *by*

$$\Phi(x) = \begin{cases} \left[1 - \frac{x}{3}, 1\right], & x \in [0,1], \\ \{\xi, \eta\}, & x = \xi, \\ \{\eta\}, & x = \eta, \end{cases} \qquad \Psi(x) = \begin{cases} \left[1 - \frac{x}{2}, 1\right], & x \in [0,1], \\ \{\xi\}, & x = \xi, \\ \{\xi, \eta\}, & x = \eta. \end{cases}$$

Now it is easy to verify that $\Phi(x) \cap \Psi(x) = \left[1 - \frac{x}{3}, 1\right] \cup \{\xi, \eta\}$. *An* (Φ, Ψ)*-orbital sequence* $\{\lambda_n\}_{n=0}^{\infty}$ *such that* $\lambda_{n+1} = 1 - \frac{\lambda_n}{3}$ *can be constructed at a point* $\lambda = \lambda_0 = 1$ *as follows:*

$$\lambda_1 = \frac{2}{3}, \quad \lambda_2 = \frac{7}{9}, \quad \lambda_3 = \frac{20}{27}, \quad \lambda_4 = \frac{61}{81}, \dots, \text{ and}$$
$$\lambda_1 \in \Phi\lambda_0 = \left[\frac{2}{3}, 1\right], \quad \lambda_2 \in \Psi\lambda_1 = \left[\frac{2}{3}, 1\right],$$
$$\lambda_3 \in \Phi\lambda_2 = \left[\frac{20}{27}, 1\right], \quad \lambda_4 \in \Psi\lambda_3 = \left[\frac{17}{27}, 1\right], \dots .$$

Clearly, we have

$$\lambda_{2n+1} \in \Phi\lambda_{2n}, \quad \lambda_{2n+2} \in \Psi\lambda_{2n+1}, \quad n = 0, 1, 2, \dots,$$

and the pair (Φ, Ψ) *is eventually* H*-restrictive. Also, there exists a monotone increasing subsequence* $\{\lambda_{3n} : n = 1, 2, 3, \dots\}$ *converging to* 1. *Thus all the conditions of the above theorem are satisfied and hence* $1 \in coin(\Phi, \Psi)$.

Theorem 2. *Let* (Ω, d) *be a compact metric space,* $\Phi : \Omega \to CB(\Omega)$ *be a* H*-continuous set valued map and* $f : \Omega \to X$ *be a continuous map such that for each* $v \in \Omega$, *there exists* $n \in \mathbb{N}$ *for which* $D(fv, \Phi v) > 0$ *implies* $D(fv_n, \Phi v_n) < D(fv, \Phi v)$, *where* fv_n *is the* n*th term of the Orbital sequence*

$$\mathrm{Or}(v, \Phi, f) = \{v_0 = v, \ fv_{n+1} \in \Phi v_n, \ n = 0, 1, 2, \dots\}.$$

Then Φ *and* f *have a coincidence point.*

Proof. Consider a function $\psi : \Omega \to [0, \infty)$ defined as $\psi(v) = D(fv, \Phi v)$. Since d and D both are continuous, so ψ is continuous as well. Thus ψ achieves its least value at some point, say $\theta \in \Omega$. Now if $D(f\theta, \Phi\theta) > 0$, then by hypothesis there exists $n \in \mathbb{N}$ such that

$$D(f\theta_n, \Phi\theta_n) < D(f\theta, \Phi\theta).$$

This implies $\psi(\theta_n) < \psi(\theta)$, which is a contradiction to the choice of θ.

Hence $D(f\theta, \Phi\theta) = 0$, i.e. $f\theta \in \Phi\theta$. □

Definition 10. *Let* (Ω, d) *be a metric space. A multi-valued map* $\Phi : \Omega \to CB(\Omega)$ *and a mapping* $f : \Omega \to X$ *are said to be pairwise eventually* $D(\varepsilon)$*-restrictive if there exists* $\varepsilon > 0$ *such that* $D(fv, \Phi v) < D(f\gamma, \Phi\gamma)$ *for all* $fv \in \Phi\gamma$, *whenever* $0 < D(f\gamma, \Phi\gamma) < \varepsilon$ *for some* $\gamma \in \Omega$.

Theorem 3. *Let* (Ω, d) *be a metric space. Let* $\Phi : \Omega \to CB(\Omega)$ *be a* H*-continuous multi-valued map and* $f : \Omega \to \Omega$ *be a continuous map. If* f *and* Φ *are pairwise eventually* $D(\varepsilon)$*-restrictive and the Orbital sequence*

$$\mathrm{Or}(v_0, \Phi, f) = \{v_0 \in \Omega, \ fv_{n+1} \in \Phi v_n\},$$

contains a convergent subsequence v_{n_i}, *which converges to* θ, *then* $f\theta \in \Phi\theta$, *i.e.* θ *is a coincidence point of* f *and* Φ.

Proof. Assume that $v_{n+1} \neq v_n$ and $v_n \notin coin\,(\Phi, f)$. Since v_{n_i} is a convergent subsequence of v_n, for $\varepsilon > 0$ there exists integers n_p and n_{p+1} such that $d\big(v_{n_p}, v_{n_p+1}\big) < \varepsilon$, since f is continuous so $D\big(fv_{n_p}, fv_{n_p+1}\big) < \varepsilon$. Thus we have

$$D\big(fv_{n_p}, \Phi v_{n_p}\big) < \varepsilon,$$

since $fv_{n_p+1} \in \Phi v_{n_p}$. Now $fv_{n_p} \notin \Phi v_{n_p}$, so

$$D\big(fv_{n_{p+1}}, \Phi v_{n_p+1}\big) < D\big(fv_{n_p}, \Phi v_{n_p}\big)$$

by definition of $D(\varepsilon)$-restrictive.

Thus the sequence $\big\{D\,(fv_m, \Phi v_m)\,\big\}_{m=n_p}^{\infty}$ is strictly decreasing sequence of real numbers and hence is convergent. We have $fv_{n_p+1} \in \Phi v_{n_p}$. This implies

$$\begin{aligned} D\big(fv_{n_{p+1}}, \Phi v_{n_p}\big) = 0 &\Rightarrow \lim_{p\to\infty} D\big(fv_{n_{p+1}}, \Phi v_{n_p}\big) = 0 \\ &\Rightarrow D\big(fw, \Phi\theta\big) = 0 \quad \big(\text{for some } w \in \Omega, \text{ since } \Phi \text{ is } H\text{-continuous}\big) \\ &\Rightarrow fv_{n_{p+1}} \to fw \quad \text{and} \quad fw \in \Phi\theta. \end{aligned}$$

Now $\lim\limits_{n\to\infty} D\big(fv_{n_p}, \Phi v_{n_p}\big) = D\big(f\theta, \Phi\theta\big) < \varepsilon$ and $\lim\limits_{p\to\infty} D\big(fv_{n_{p+1}}, \Phi v_{n_p+1}\big) = D\big(fw, \Phi w\big)$. The sequence $\{D\,(fv_l, \Phi v_l)\}_{l=n_p+1}^{\infty}$ is a subsequence of convergent sequence $\{D\,(fv_m, \Phi v_m)\}_{m=n_p}^{\infty}$ and hence converges to same limit. It follows that

$$D\,(f\theta, \Phi\theta) = D\,(fw, \Phi w)\,.$$

Now if $f\theta \notin \Phi\theta$, since $fw \in \Phi\theta$ and $D\,(f\theta, \Phi\theta) < \varepsilon$, then we have

$$D\,(fw, \Phi w) < D\,(f\theta, \Phi\theta)\,,$$

which is a contradiction. Hence $f\theta \in \Phi\theta$. □

Example 2. *Let* $\Omega = [0,5]$, $d\,(x,y) = |x-y|$, $\Phi : \Omega \longrightarrow CB\,(\Omega)$ *and* $f : \Omega \longrightarrow \Omega$ *are defined by*

$$\Phi(x) = \begin{cases} \left[0, 3 - \frac{x}{2}\right], & x \in [0,3], \\ \left[4 - \frac{x}{3}, 5\right], & x \in (3,5], \end{cases} \qquad f(x) = 2 - \frac{x}{5}.$$

Now $f(4) \notin \Phi(4)$ *and* $D(f4, \Phi4) = D(1.2, [2.67, 5]) = 1.47 < 1.5 = \epsilon$. *For* $fv \in [2.67, 5]$, *we have*

$$D\,(fv, \Phi v) < D\,(f4, \Phi4)\,.$$

An orbital sequence $\{\lambda_n\}_{n=0}^{\infty}$ *such that* $\lambda_{n+1} = 2 - \frac{\lambda_n}{3}$ *can be constructed at a point* $\lambda = \lambda_0 = 2$, *as follows:*

$$\begin{aligned} &\lambda_1 = \frac{4}{3}, \quad \lambda_2 = \frac{14}{9}, \quad \lambda_3 = \frac{40}{27}, \quad \lambda_4 = \frac{122}{81}, \quad \lambda_5 = \frac{364}{243}, \\ &\lambda_6 = \frac{1094}{729}, \quad \lambda_7 = \frac{3280}{2187}, \quad \lambda_8 = \frac{9842}{6561}, \ldots \quad \textit{and} \\ &\frac{26}{15} = f\lambda_1 \in \Phi\lambda_0 = [0,2]\,, \quad \frac{76}{45} = f\lambda_2 \in \Phi\lambda_1 = \left[0, \frac{14}{6}\right], \\ &\frac{230}{135} = f\lambda_3 \in \Phi\lambda_2 = \left[0, \frac{40}{18}\right], \quad \frac{688}{405} = f\lambda_4 \in \Phi\lambda_3 = \left[0, \frac{122}{54}\right], \\ &\frac{2066}{1215} = f\lambda_5 \in \Phi\lambda_4 = \left[0, \frac{364}{162}\right], \ldots. \end{aligned}$$

Clearly, $f\lambda_{n+1} \in \Phi\lambda_n, n = 0, 1, 2, \ldots$, implying that f and Φ are pairwise eventually $D(\epsilon)$-restrictive. Also, there exists a monotone increasing subsequence $\{\lambda_{2n} : n = 1, 2, 3, \ldots\}$ converging to $\frac{3}{2}$. Thus all the conditions of the above theorem are satisfied and hence

$$f\left(\frac{3}{2}\right) = \frac{17}{10} \in \left[0, \frac{9}{4}\right] = R\left(\frac{3}{2}\right).$$

Theorem 4. *Let (Ω, d) be a compact metric space, $\Phi : \Omega \to CB(\Omega)$ be a H-continuous set valued map and $f : \Omega \to \Omega$ be a continuous map such that for each $v \in \Omega$, there exists $n \in \mathbb{N}$ for which $0 < D(fv, \Phi v) < \varepsilon$ implies $D(fv_n, \Phi v_n) < D(fv, \Phi v)$, where fv_n is the nth term of Orbital sequence*

$$\mathrm{Or}(v, \Phi, f) = \{v_0 = v,\ fv_{n+1} \in \Phi v_n,\ n = 0, 1, 2, \ldots\}.$$

Then Φ and f have a coincidence point.

Proof. Let $v_0 \in \Omega$. Since Ω is compact, the orbital sequence $\{fv_n\}$ has a limit point. So for given $\varepsilon > 0$, there exists $p \in \mathbb{N}$ such that $0 < D\left(fv_p, \Phi v_p\right) < \varepsilon$. Define the function $\psi : \Omega \to [0, \infty)$ by $\psi(v) = D(fv, \Phi v)$. Since d and D both are continuous, so ψ is continuous as well. Thus ψ achieves its least value at some point, say $\theta \in \Omega$.

Now $\psi(\theta) = D(f\theta, \Phi\theta) = \min\{D(fv, \Phi v) : v \in \Omega\} \leq D(fv_p, \Phi v_p) < \varepsilon$. If $D(f\theta, \Phi\theta) > 0$, then by hypothesis there exists $j \in \mathbb{N}$ such that $D\left(f\theta_j, \Phi\theta_j\right) < D(f\theta, \Phi\theta)$. This implies $\psi\left(\theta_j\right) < \psi(\theta)$, which is a contradiction to the choice of θ.

Hence $D(f\theta, \Phi\theta) = 0$, i.e. $f\theta \in \Phi\theta$. □

3 Applications in fuzzy fixed point results

Here, the results from previous section are applied to deduce their corresponding fuzzy fixed point theorems. In fact, all the results of this section are new and can be treated as independent. However, we decided to connect these two ideas using elementary technique to appreciate the fact that many results from the literature regarding contractive or non-expansive multi-functions have their fuzzy counterparts. To this end, we recall the necessary preliminaries of fuzzy sets and fuzzy mappings as follows.

Definition 11 ([18]). *Let Ω be a nonempty set. A fuzzy set in Ω is a function with domain Ω and values in $[0, 1] = I$. If A is a fuzzy set in Ω and $x \in \Omega$, then the function value $A(x)$ is called the grade of membership of $x \in A$. We denote the collection of all fuzzy sets in Ω by I^{Ω}. Let $A \in I^{\Omega}$ and $\alpha \in I$. The α-level set of A, denoted by $[A]_\alpha$, is defined by*

$$[A]_\alpha = \begin{cases} \overline{\{x \in \Omega : A(x) > 0\}}, & \text{if } \alpha = 0, \\ \{x \in \Omega : A(x) \geq \alpha\}, & \text{if } \alpha \in (0, 1], \end{cases}$$

where by $\overline{M}$ we mean the closure of the crisp set M.

We say that $A \in I_{CB}(\Omega)$ if $A \in I^{\Omega}$ and $[A]_\alpha$ is nonempty closed and bounded.

Definition 12 ([4,5]). *Let Ω be a nonempty set and Y be a metric space. A map $F : \Omega \longrightarrow I^Y$ is called a fuzzy mapping. A fuzzy mapping F is a fuzzy subset of $\Omega \times Y$ with membership value $F(x)(y)$.*

Definition 13 ([1]). *Let F and T be two fuzzy mappings from Ω into I^{Ω}. A point $v \in \Omega$ is called a fuzzy fixed point of F if there exists $\alpha \in (0,1]$ such that $v \in [Fv]_\alpha$. The point u is said to be a common fuzzy fixed point of F and T if $v \in [Fv]_\alpha \cap [Tv]_\alpha$ for some $\alpha \in (0,1]$.*

To discuss the main results of this section, we introduce the following auxiliary concepts.

Definition 14. *Let (Ω, d) be a metric space and $F : \Omega \longrightarrow I_{CB}(\Omega)$ be a fuzzy multi-valued map. An F-orbital sequence at a point $v \in \Omega$ is a set $\mathrm{Or}(v, F)$ of points in Ω defined by*

$$\mathrm{Or}(v,F) = \{v_0 = v,\ v_{n+1} \in [Fv_n]_\alpha,\ n = 0,1,2,\ldots,\ \alpha \in (0,1]\}.$$

Definition 15. *Let (Ω, d) be a metric space. A fuzzy multi-valued map $F : \Omega \longrightarrow I_{CB}(\Omega)$ is said to be eventually D-restrictive if there exists $k \in \mathbb{N}$ such that $D(v_n, [Fv_n]_\alpha) < D(v, [Fv]_\alpha)$ for all $n \geq k$, whenever $D(v, [Fv]_\alpha) > 0$ for some $v \in \Omega$ and $\alpha \in (0,1]$, where $v_n \in \mathrm{Or}(v, F)$.*

Definition 16. *Let (Ω, d) be a metric space. A fuzzy multi-valued map $F : \Omega \longrightarrow I_{CB}(\Omega)$ is said to be p-orbitally continuous if whenever a subsequence $\{v_{n_i}\}$ of a sequence $\{v_n\}$ converges to some $v \in \Omega$, i.e. if $v_{n_i} \to v$, then for any fixed $p \in \mathbb{N}$, we have $v_{n_i+p} \to v_p$ and $[Fv_{n_i+p}]_\alpha \to [Fv]_\alpha$ for some $\alpha \in (0,1]$, where v_p is the pth term of the F-orbital sequence $\mathrm{Or}(v, F)$.*

Definition 17. *Let (Ω, d) be a metric space and $F, T : \Omega \longrightarrow I_{CB}(\Omega)$ be any two fuzzy multi-valued maps. An (F, T)-orbital sequence of F and T at a point $v \in \Omega$ is a set $\mathrm{Or}(v, F, T)$ of points in Ω defined by*

$$\mathrm{Or}(v,F,T) = \{v_0 = v,\ v_{2n+1} \in [Fv_{2n}]_\alpha,\ v_{2n+2} \in [Tv_{2n+1}]_\alpha,\ n = 0,1,2,\ldots,\ \alpha \in (0,1]\}.$$

Definition 18. *Let (Ω, d) be a metric space. Two fuzzy multi-valued maps $F, T : \Omega \longrightarrow I^{\Omega}$ are said to be pairwise eventually H-restrictive, if there exists $k \in \mathbb{N}$ such that*

$$H([Fv_{2n}]_\alpha, [Tv_{2n+1}]_\alpha) < H([Fv]_\alpha, [Tv]_\alpha)$$

for all $n \geq k$, whenever $H([Fv]_\alpha, [Tv]_\alpha) > 0$ for some $v \in \Omega$ and $\alpha \in (0,1]$, where $v_n \in \mathrm{Or}(v, F, T)$.

Theorem 5. *Let (Ω, d) be a metric space and $F, T : \Omega \longrightarrow I^{\Omega}$ be any two pairwise H-restrictive p-orbitally continuous fuzzy multi-valued maps. If for some $v_0 \in \Omega$, the Orbital sequence $\mathrm{Or}(v, F, T)$ of the pair (F, T) has a subsequence $\{v_{n_i}\}$ converging to v, then $v \in coin(F, T)$.*

Proof. Let $\Phi, \Psi : \Omega \longrightarrow 2^{\Omega}$ be any two multi-valued maps, respectively defined by

$$\Phi x = [Fx]_\alpha \quad \text{and} \quad \Psi x = [Tx]_\alpha$$

for all $x \in \Omega$ and some $\alpha \in (0,1]$. It suffices to show that the pair (Φ, Ψ) is pairwise eventually H-restrictive. Since the pair (F, T) of fuzzy multi-valued maps is eventually H-restrictive, there exist $k \in \mathbb{N}$, $v \in \Omega$ and a sequence $v_n \in \mathrm{Or}(v, F, T)$ such that for all $n \geq k$, we have

$$H(\Phi u, Su) = H([Fu]_\alpha, [Tu]_\alpha) > H([Fv_{2n}]_\alpha, [Tv_{2n+1}]_\alpha) = H(\Phi v_{2n}, \Psi v_{2n+1})$$

for some $\alpha \in (0,1]$. Consequently, all the conditions of Theorem 1 are satisfied. Hence, it can be applied to find $v \in \Omega$ such that $v \in coin(F, T)$. $\square$

Definition 19. *Let (Ω, d) be a metric space. A fuzzy multi-valued map $F : \Omega \longrightarrow I_{CB}(\Omega)$ is said to be H-continuous at a point $v_0 \in \Omega$, if for each sequence $\{v_n\}$ in Ω such that $\lim_{n\to\infty} d(v_n, v_0) = 0$, there exists $\alpha \in (0,1]$ such that*

$$\lim_{n\to\infty} H([Fv_n]_\alpha, [Fv_0]_\alpha) = 0.$$

Equivalently, F is H-continuous at a point $v_0 \in \Omega$, if for every $\epsilon > 0$, there exists $\eta > 0$ such that $d(v_n, v_0) < \eta$ implies $H([Fv_n]_\alpha, [Fv_0]_\alpha) < \epsilon$ for some $\alpha \in (0,1]$. A map F is said to be continuous if it is continuous at each point of Ω.

Example 3. *Let $\Omega = [0, \infty)$ and $d(v, w) = |v - w|$ for all $v, w \in \Omega$. Define $F : \Omega \longrightarrow I_{CB}(\Omega)$ by*

$$F(v)(t) = \begin{cases} \frac{1}{2}, & \text{if } 0 \le t \le v+3, \\ \frac{2}{9}, & \text{elsewhere.} \end{cases}$$

Suppose $\alpha = 0.3$. Then for all $v \in \Omega$ we have $[Fv]_\alpha = [0, v+3]$. For $\epsilon > 0$, take $\eta = \frac{\epsilon}{6}$, then $d(v, w) < \eta$ implies $H([Fv]_\alpha, [Fw]_\alpha) < \epsilon$. Thus, F is H-continuous.

Lemma 1. *Every H-continuous fuzzy multi-valued map with closed α-level set has a corresponding H-continuous multi-valued map.*

Proof. Let $F : \Omega \longrightarrow I_{CB}(\Omega)$ be an H-continuous fuzzy multi-valued map. Then for every $\epsilon > 0$ there exists $\eta > 0$ such that whenever $d(v_n, v_0) < \eta$ we have $H([Fv_n]_\alpha, [Fv]_\alpha) < \epsilon$ for some $\alpha \in (0,1]$. Consider a multi-valued map $\Phi : \Omega \longrightarrow CB(\Omega)$ defined by $\Phi v = [Fv]_\alpha$ for all $v \in \Omega$ and some $\alpha \in (0,1]$, then $H(\Phi v_n, \Phi v) = H([Fv_n]_\alpha, [Fv]_\alpha) < \epsilon$, that is, Φ is H-continuous. □

Theorem 6. *Let (Ω, d) be a compact metric space, $F : \Omega \longrightarrow I_{CB}(\Omega)$ be an H-continuous fuzzy multi-valued map and $f : \Omega \longrightarrow \Omega$ be a continuous map such that for each $v \in \Omega$, there exists $n \in \mathbb{N}$ and $\alpha \in (0,1]$ for which $D(fv, [Fv]_\alpha) > 0$ implies $D(fv_n, [Fv_n]_\alpha) < D(fv, [Fv]_\alpha)$, where fv_n is the nth term of the orbital sequence*

$$\mathrm{Or}(v, F, f) = \{v_0 = v,\ fv_{n+1} \in [Fv_n]_\alpha,\ n = 0, 1, 2, \dots\}.$$

Then F and f have a coincidence point.

Proof. By taking $\Phi v = [Fv]_\alpha$ for all $v \in \Omega$ and some $\alpha \in (0,1]$, we get that all the hypotheses of Theorem 2 coincide with that of Theorem 6. Hence, f and F have a coincidence point in Ω. □

Definition 20. *Let (Ω, d) be a metric space. A fuzzy multi-valued map $F : \Omega \longrightarrow I_{CB}(\Omega)$ and a mapping $f : \Omega \longrightarrow \Omega$ are said to be pairwise eventually $D(\epsilon)$-restrictive if there exist $\epsilon > 0$ and $\alpha \in (0,1]$ such that $D(fv, [Fv]_\alpha) < D(f\gamma, [F\gamma]_\alpha)$ for all $fv \in [F\gamma]_\alpha$, whenever $0 < D(f\gamma, [F\gamma]_\alpha) < \epsilon$ for some $\gamma \in \Omega$.*

Theorem 7. *Let (Ω, d) be a metric space. Let $F : \Omega \to I_{CB}(\Omega)$ be an H-continuous fuzzy multi-valued map and $f : \Omega \to \Omega$ be a continuous map. If f and F are pairwise $D(\epsilon)$-restrictive and the orbital sequence*

$$\mathrm{Or}(v_0, F, f) = \{v_0 \in \Omega,\ fv_{n+1} \in [Fv_n]_\alpha\}$$

contains a convergent subsequence $\{v_{n_i}\}$, which converges to θ, then $f\theta \in [F\theta]_\alpha$, i.e. θ is a coincidence point of f and F.

Proof. By Lemma 1, for F there exists a corresponding H-continuous multi-valued map, say $\Phi : \Omega \to CB(\Omega)$. For all $v \in \Omega$ and some $\alpha \in (0,1]$, take $\Phi(v) = [Fv]_\alpha$, then Theorem 3 can be applied to find the coincidence point of f and F. $\square$

The following example is given in support of Theorem 4.

Example 4. *Let $\Omega = [0,1]$ and d be the usual metric on Ω. Let $F : \Omega \longrightarrow I_{CB}(\Omega)$ be a fuzzy muti-valued map defined by*

$$F(z)(t) = \begin{cases} \frac{\alpha}{6}, & 0 \le t < \frac{z}{17}, \\ \alpha, & \frac{z}{17} \le t \le z, \end{cases}$$

for some $\alpha \in (0,1]$. Define a continuous function $f : \Omega \longrightarrow \Omega$ by $f(z) = \frac{z}{2}$. Now for a given $\epsilon > 0$, $d(z_1, z_2) < \eta = \frac{\epsilon}{2}$ implies

$$H\left([Fz_1]_\alpha, [Fz_2]_\alpha\right) = H\left(\left[\frac{z_1}{17}, \frac{z_1}{10}\right], \left[\frac{z_2}{17}, \frac{z_2}{10}\right]\right) \le \left|\frac{z_1 - z_2}{10}\right| < |z_1 - z_2| < \eta < \epsilon.$$

Hence, F is H-continuous.

Suppose $\gamma = 0.2$. Then $[F\gamma]_\alpha = [0.01176, 0.02]$ and $0 < D(f\gamma, [F\gamma]_\alpha) = 0.08 < 0.1$. Also, for all $fv \in [F\gamma]_\alpha$, we have

$$D(fv, [Fv]_\alpha) < D(f\gamma, [F\gamma]_\alpha),$$

which implies that F and f are pairwise $D(0.1)$-restrictive. Now consider the Orbital sequence

$$\text{Or}(0.9, F, f) = \{v_n \in \Omega : fv_{n+1} \in [Fv_n]_\alpha\} = \{0.9, 0.12, 0.016, 0.002, \dots\},$$

which contains a subsequence converging to 0, then $f0 \in [F0]_\alpha$, that is 0 is the coincidence point of the mappings F and f.

Conclusion. *In this article, we discussed the coincidence points of a pair of multi-valued mappings by introducing a notion of pairwise eventually H-restrictive mappings. This restrictive condition is inspired by the Edelstein-type contractive conditions. We evaluated the coincidence points of a pair of multi-valued and single valued mappings by utilizing the concept of pairwise eventually $D(\varepsilon)$-restrictive mappings. By assuming $f = I$ we can easily achieve [11, Theorem 3]. In addition, our main results are further applied to deduce their corresponding fuzzy fixed point theorems. The latter technique is in appreciation of the fact that many results from the existing literature on contractive or nonexpansive multifunctions have their fuzzy counterparts.*

The study of common fixed points for such H-restrictive maps is suggested for future work.

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Рашид М., Азам А. *Точки збігу попарно зрештою H-обмежувальних багатозначних відображень та їх застосування до результатів про нечіткі нерухомі точки* // Карпатські матем. публ. — 2026. — Т.18, №1. — С. 318–327.

У цій статті вводиться узагальнене поняття попарно зрештою H-обмежувальних багатозначних відображень, що ґрунтується на обмежувальних умовах, описаних у [Symmetry 2020, **12** (1), 127]. Також використовується поняття $D(\epsilon)$-обмежувальних відображень для дослідження точок збігу пари неперервного однозначного відображення та H-неперервного багатозначного відображення. Основні результати далі застосовуються для отримання відповідних теорем про нечіткі нерухомі точки. Такий підхід зумовлений тим, що багато результатів, відомих у літературі для стискаючих або нерозширювальних багатозначних відображень, мають свої аналоги в теорії нечітких множин. Насамкінець, для підтвердження коректності отриманих результатів наведено кілька показових прикладів.

Ключові слова і фрази: H-обмежувальне відображення, $D(\epsilon)$-обмежувальне відображення, точка збігу.