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Controllability results for fuzzy Hilfer fractional differential equations via measure of noncompactness

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This article focuses on the controllability results for fuzzy Hilfer fractional differential equations through the measure of noncompactness. The main results are established using Mönch's fixed point theorem, along with essential tools such as semigroup theory, fractional calculus, and the measure of noncompactness. Finally, the theoretical results are applied by providing an interesting example.

Key words and phrases: fuzzy Hilfer fractional derivative, controllability, measure of noncompactness, Mönch's fixed point theorem.

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1 Introduction

Fractional differential equations extend classical differential equations by incorporating derivatives of non-integer orders. This enables them to capture memory and hereditary effects within systems. Fractional differential equations are widely used in fields such as physics, biology, finance, engineering, control theory, and signal processing, and are particularly useful for modeling phenomena like anomalous diffusion and viscoelastic behavior. For further information, we refer the reader to the books and research articles [1–4].

The origins of fractional calculus can be traced back to the 17th century, with a correspondence between mathematicians Leibniz and L'Hopital discussing the possibility of non-integer derivatives. However, it was not until the 19th century, with the work of Riemann, Liouville, and Grünwald, that the formal foundations for fractional calculus were laid. Ordinary differential equations can be extended to arbitrarily high non-integer values through integration to produce fractional differential equations. A number of physical events can be explained in science and engineering by using differential equations with fractional-order derivatives, as demonstrated by recent research. Fractional-order ordinary differential equations and partial fractional differential equations have seen significant progress in recent years. For further details, the reader is referred to the books and research articles [5–8].

R. Hilfer [9] introduced the Hilfer fractional derivative, developed from the Riemann-Liouville and Caputo derivatives, which has garnered significant interest due to its wide-ranging applications. H. Gu and J.J. Trujillo [10] have explored its use in studying the existence of fractional evolution equations utilizing the measure of noncompactness. Other scholars

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have explored the exact controllability, and approximate controllability of the Hilfer fractional differential system. Additional information can be found in the articles [11–13].

The concept of fuzzy sets was introduced by L.A. Zadeh [14]. Fuzzy sets provide a powerful way to handle imprecise information and are particularly useful when dealing with vague or uncertain data. Fuzzy numbers, initially introduced by S.S.L. Chang and L.A. Zadeh [15], provide a valuable means of quantifying uncertainty within mathematical frameworks. The phrase fuzzy differential equation was first used in 1978 by A. Kandel and W.J. Byatt [16]. Researchers like O. Kaleva [17], S. Seikkala [18], C. Song and C. Wu [19] have all conducted extensive studies on the existence and uniqueness of solutions to fuzzy differential equations. Fuzzy partial differential equations are of great interest to the mathematical and engineering communities, and this is the initial work in this area. For further details, see the related articles [20, 21].

The concept of fuzzy fractional differential equations (FFDEs for short) was pioneered by R.P. Agarwal et al. [5]. S. Arshad [22] explored the properties of FFDEs, investigating their existence and uniqueness through fuzzy integral equivalent equations. S. Salahshour et al. [23] approached FFDEs from a different perspective, using the fuzzy Laplace transform of the Riemann-Liouville fractional H-derivative to address these equations.

Controllability is a central concept in control theory that determines whether a dynamical system can be driven from an initial state to a desired final state using admissible control inputs over a finite time interval. It is essential for developing effective control strategies and ensuring the desired performance of systems across various fields, including robotics, economics, engineering, and biology. Controllability is crucial in system design, as it guarantees that goals such as stabilization or trajectory tracking can be achieved. This concept is closely linked to other properties, such as observability and stability, which form the basis for analyzing and designing dynamic systems.

The system can produce any tiny neighborhood of the final state with controllability as well as any end state with absolute controllability. Furthermore, because controllability is frequently perfectly sufficient in applications, controllable systems are more common. For further information, see the research papers [11, 24]. V. Singh [25] addressed the challenges of establishing controllability for fractional systems where the domain of the governing operator is non-dense.

In [26], the focus was on demonstrating the existence of a mild solution, in an abstract sense, to the following Hilfer fractional differential equations of a given order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$

$$
\begin{aligned}
&{}^{H}D_{0^+}^{\alpha,\beta} x(s) = Ax(s) + f(s, x(s)), \quad s \in (0, T], \\
&I_{0_+}^{(1-\alpha)(1-\beta)} x(0) = x_0,
\end{aligned}
$$

where the Schauder fixed point theorem is applied and A is the almost sectorial operator of the analytic semigroup.

Inspired by the aforementioned work, no one had explored the controllability of fuzzy fractional differential equations using the measure of noncompactness. We prove the existence of a mild solution to fuzzy fractional differential equation

$$
\begin{cases}
D^{\gamma,\mu} y(t) = Ay(t) + Bu(t) + f(t, y(t)), \quad t \in U = [0, b], \\
I_{0_+}^{(1-\mu)(1-\gamma)} y(0) = y_0,
\end{cases}
\tag{1}
$$

where $D^{\gamma,\mu}$ is a fuzzy Hilfer fractional derivative of order $\gamma \in (0,1)$ and type $\mu \in [0,1]$. The state variable $y(\cdot)$ takes on values within the space of all triangular fuzzy numbers on a fuzzy number space $\mathbb{F}$. The operator $A : \mathrm{Dom}(A) \subseteq \mathbb{B} \to \mathbb{B}$ is the infinitesimal generator of a C_0-semigroup consisting of uniformly bounded linear operators $\{T(t)\}_{t\geq 0}$ on the space $\mathbb{B}$. The state variable $y(\cdot)$ takes its value in the complete metric space $\mathbb{B}$ equipped with the norm $\|y(\cdot)\| = d_\infty(y(\cdot), \hat{0})$. The operator $B : \mathbb{M} \to \mathbb{B}$ is a bounded linear operator and $u(\cdot) \in L^p(U, \mathbb{M})$ is a control fuction, where $\mathbb{M}$ is a closed subspace of $\mathbb{B}$.

The main contributions of this article can be summarized as follows. In this paper, we explore the existence of a mild solution for controllability results for fuzzy fractional differential equation utilizing measure of noncompactness. A mild solution to system (1) is constructed using the Laplace transform. Using Mönch's fixed point theorem, we proved the existence of a fixed point for a mild solution. A theoretical example is presented to illustrate the results.

The remainder of the paper is organized as follows. In Section 2, we present the basic concepts of fuzzy fractional calculus needed for our study. In Section 3, the existence and controllability results are established. An illustrative example is provided in Section 4 to demonstrate the applicability of the obtained results.

2 Preliminaries

Definition 1 ([27]). *Let us denote by $\mathbb{R}_{\mathfrak{F}}$ (called the space of fuzzy numbers) the class of fuzzy subsets of the real axis (i.e. mappings $y : \mathbb{R} \to [0,1]$) satisfying the following properties:*

1) *each $y \in \mathbb{R}_{\mathfrak{F}}$ is normal, i.e. there is a $t_0 \in \mathbb{R}$ such that $y(t_0) = 1$;*
2) *each $y \in \mathbb{R}_{\mathfrak{F}}$ is a fuzzy convex set, i.e. for all $\alpha \in [0,1]$ and $t_1, t_2 \in \mathbb{R}$ we have*
$$y(\alpha t_1 + (1-\alpha)t_2) \geq \min\{y(t_1), y(t_2)\};$$
3) *each $y \in \mathbb{R}_{\mathfrak{F}}$ is upper semicontinuous;*
4) *closure of $\{t \in \mathbb{R} : y(t) > 0\}$ is compact.*

In general, with $y \in \mathbb{R}_{\mathfrak{F}}$, there does not exist a $j \in \mathbb{R}_{\mathfrak{F}}$ such that $y \oplus j = 0$. We denote by $\mathbb{F}$ the set of all triangular fuzzy numbers in $\mathbb{R}_{\mathfrak{F}}$. The metric space $(\mathbb{R}_{\mathfrak{F}}, d_\infty)$ contains a subspace $(\mathbb{F}, d_\infty)$, which is a complete metric space.

Assume that $\mathbb{L}$ is a subset of $\mathbb{R}_{\mathfrak{F}}$ and that $\mathbb{F} \subset \mathbb{R}$. Let $\mathcal{C}(\mathbb{F}, \mathbb{L})$ denote the set of all continuous mappings $f : \mathbb{F} \to \mathbb{L}$.

The operations of addition and scalar multiplication of fuzzy numbers on $\mathbb{R}_{\mathfrak{F}}$ have the form
$$[h \oplus g]^\eta = [h]^\eta + [g]^\eta \qquad \text{and} \qquad [\rho \odot h]^\eta = \rho[h]^\eta, \quad \rho \in \mathbb{R},$$
where

(i) $[h]^\eta + [g]^\eta = \{u + v : u \in [h]^\eta, v \in [g]^\eta\}$ *is the Minkowski sum of* $[h]^\eta$ *and* $[g]^\eta$,

(ii) $\rho[h]^\eta = \{\rho u : u \in [h]^\eta\}$.

Definition 2 ([7]). *The following properties for the metric d_∞ hold:*

1) $d_\infty(h+k, g+k) = d_\infty(h,g)$ *for all* $h, g, k \in \mathbb{R}_{\mathfrak{F}}$;
2) $d_\infty(\rho h, \rho g) = |\rho|\, d_\infty(h,g)$ *for all* $\rho \in \mathbb{R}$, $h, g \in \mathbb{R}_{\mathfrak{F}}$;
3) $d_\infty(h+g, j+k) \leq d_\infty(h,j) + d_\infty(g,k)$ *for all* $h, g, k, j \in \mathbb{R}_{\mathfrak{F}}$.

Definition 3 ([28]). *For $h, g \in \mathbb{R}_{\mathfrak{F}}$, the gH-difference of h and g, denoted by $h \ominus_{gH} g$, is defined as a fuzzy integer $k \in \mathbb{R}_{\mathfrak{F}}$ such that*

$$h \ominus_{gH} g = k \quad \Leftrightarrow \quad \{(i) \;\; h = g + k \quad or \quad (ii) \;\; g = h + (-1)k\}.$$

Remark 1 ([29]). *In terms of η-level we have*

$$(h \ominus_{gH} y)^{\eta} = [\min\{\underline{h}(\eta) - \underline{g}(\eta), \bar{h}(\eta) - \bar{g}(\eta)\}, \max\{\underline{h}(\eta) - \underline{g}(\eta), \bar{h}(\eta) - \bar{g}(\eta)\}].$$

Definition 4 ([19]). *A map $\mathfrak{g} : H \times \mathbb{R}_{\mathfrak{F}} \to \mathbb{R}_{\mathfrak{F}}$ is called levelwise continuos at $(t_0, y_0) \in H \times \mathbb{R}_{\mathfrak{F}}$ provided for any fixed $\mathfrak{K} > 0$, there exist an $\mathfrak{J}(\mathfrak{K}) > 0$ such that*

$$d_{\infty}\big(\mathfrak{g}(t, y), \mathfrak{g}(t_0, y_0)\big) < \mathfrak{K},$$

whenever $|t - t_0| < \mathfrak{J}(\mathfrak{K})$ and $d_{\infty}(y, y_0) < \mathfrak{J}(\mathfrak{K})$ for all $t \in H, y \in \mathbb{R}_{\mathfrak{F}}$.

Definition 5 ([30]). *The generalized Hukuhara derivative $f'_{gH}(t_0)$ of a fuzzy-valued function $f : (a, b) \to \mathbb{R}_{\mathfrak{F}}$ at t_0 is defined as*

$$f'_{gH}(t_0) = \lim_{u \to 0} \frac{f(t_0 + u) \ominus_{gH} f(t_0)}{u}.$$

The notation $f'_{gH}(t_0) \in \mathbb{R}_{\mathfrak{F}}$ indicates that f is gH-differentiable at t_0.

Definition 6 ([31]). *The Riemann-Liouville integral of a fuzzy-valued function $y : [0, +\infty) \to \mathbb{F}$ is given by*

$${}^{RL}I^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s)\, ds, \quad 0 < s < t, \quad 0 < \alpha < 1.$$

Definition 7 ([9]). *The Hilfer fractional derivative of order $\gamma \in (0, 1)$ and type $\mu \in [0, 1]$ of the function $y : [0, +\infty) \to \mathbb{F}$ is given by*

$$D_{0^+}^{\gamma,\mu} y(t) = \left(I_{0^+}^{\mu(1-\gamma)} \frac{d}{dt} \big(I_{0^+}^{(1-\gamma)(1-\mu)} y \big) \right)(t), \quad t \in [0, +\infty),$$

where the Riemann-Liouville fractional integral of order b is defined by

$$I_{0^+}^{b} y(t) = \frac{1}{\Gamma(b)} \int_0^t (t - s)^{b-1} y(s)\, ds,$$

and $\Gamma(b) = \int_0^{\infty} e^{-t} t^{b-1}\, dt$ is the Gamma function. In fact, Hilfer fractional derivative can degenerate into the Riemann-Liouville fractional derivative and the Caputo derivative.

Remark 2 (see [9]). *If we take $\mu = 0$ and $\gamma \in (0, 1)$, then the Hilfer fractional derivative of the function y can be transformed into the Riemann-Liouville fractional order derivative as follows*

$$D_{0^+}^{\gamma,0} y(t) = \frac{d}{dt} \big(I_{0^+}^{(1-\gamma)} y(t) \big) = D_{0^+}^{\gamma} y(t).$$

If we take $\mu = 1$ and $\gamma \in (0, 1)$, then the Hilfer fractional derivative of the function y can be transformed into the Caputo fractional order derivative form as follows

$$D_{0^+}^{\gamma,1} y(t) = I_{0^+}^{(1-\gamma)} \frac{d}{dt} y(t) = {}^{C}D_{0^+}^{\gamma} y(t).$$

Lemma 1. *The Cauchy problem* (1) *is identical to an integral equation presented by*

$$y(t) = \frac{y_0 t^{(\mu-1)(\gamma-1)}}{\Gamma(\mu(1-\gamma)+\mu)} + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} [Ay(s) + f(s,y(s)) + Bu(s)]\, ds, \quad t \in U. \tag{2}$$

The wright function $\mathcal{M}_\gamma(\beta)$ is defind by

$$\mathcal{M}_\gamma(\beta) = \sum_{m=1}^{\infty} \frac{(-\beta)^{m-1}}{(m-1)!\Gamma(1-\gamma m)}, \quad \gamma \in (0,1), \quad \beta \in \mathbb{C},$$

which satisfies the following equality

$$\int_0^\infty s^\xi \mathcal{M}_\gamma(s) ds = \frac{\Gamma(1+\xi)}{\Gamma(1+\gamma\xi)}.$$

Let us define

$$\mathcal{V}_\gamma(t) = \int_0^\infty \gamma s \mathcal{M}_\gamma(s) T(t^\gamma s)\, ds, \qquad \mathcal{S}_\gamma(t) = t^{\gamma-1}\mathcal{V}_\gamma(t), \qquad \mathcal{H}_{\mu,\gamma}(t) = I_{0^+}^{\mu(1-\gamma)} \mathcal{S}_\gamma(t).$$

Lemma 2 ([32]). *For any fixed $t > 0$, $\mathcal{H}_{\mu,\gamma}(t)$ and $\mathcal{S}_\gamma(t)$ are linear operators. Then for any $y \in \mathbb{B}$ we have*

$$d_\infty(\mathcal{H}_{\mu,\gamma}(t)y, \hat{0}) \leq \frac{Mt^{(\mu-1)(\gamma-1)}}{\Gamma(\mu(1-\gamma)+\gamma)} d_\infty(y, \hat{0}), \qquad d_\infty(\mathcal{S}_\gamma(t)y, \hat{0}) \leq \frac{Mt^{(\gamma-1)}}{\Gamma(\gamma)} d_\infty(y, \hat{0}).$$

Lemma 3 ([32]). *The mild solution of the equation* (2) *is a function $y(t) \in C(U, \mathbb{F})$, that satisfies*

$$y(t) = \mathcal{H}_{\mu,\gamma}(t) y_0 + \int_0^t \mathcal{S}_\gamma(t-s) \odot Bu(s) ds + \int_0^t \mathcal{S}_\gamma(t-s) \odot f(s, y(s)) ds, \quad t \in U.$$

Lemma 4 ([32]). *Let $\{T(t)\}_{t>0}$ be a C_0-semigroup. Then $\{\mathcal{V}_\gamma(t)\}_{t>0}$, $\{\mathcal{S}_\gamma(t)\}_{t>0}$ and $\{\mathcal{H}_{\mu,\gamma}(t)\}_{t>0}$ are strongly continuous for any $y \in \mathbb{B}$ and $t_2, t_1 > 0$. Moreover, we have $d_\infty(\mathcal{V}_\gamma(t_2)y, \mathcal{V}_\gamma(t_1)y) \to 0$, $d_\infty(\mathcal{S}_\gamma(t_2)y, \mathcal{S}_\gamma(t_1)y) \to 0$, $d_\infty(\mathcal{H}_{\mu,\gamma}(t_2)y, \mathcal{H}_{\mu,\gamma}(t_1)y) \to 0$ as $t_2 \to t_1$.*

Definition 8 ([25]). *Let $\mathcal{K}$ be a bounded set in a Banach space $\mathbb{B}$. The Hausdorff measure of noncompactness ψ is defined as*

$$\psi(\mathcal{K}) = \inf\{\delta > 0 : \mathcal{K} \text{ can be covered by a finite number of balls with radii } \delta\}.$$

Lemma 5 ([6]). *Let $\mathcal{K}$ and $\mathcal{M}$ be two nonempty bounded subsets of a Banach space $\mathbb{B}$. Then the following conditions hold:*

1) *$\psi(\mathcal{K}) = 0$ if and only if $\mathcal{K}$ is relatively compact on $\mathbb{B}$,*
2) *$\psi(\{y\} \cup \mathcal{K}) = \psi(\mathcal{K})$ for every $y \in \mathbb{B}$,*
3) *$\psi(\mathcal{K}) \subset \psi(\mathcal{M})$ whenever $\mathcal{K} \subset \mathcal{M}$,*
4) *$\psi(\mathcal{K} + \mathcal{M}) \leq \psi(\mathcal{K}) + \psi(\mathcal{M})$, where $\mathcal{K} + \mathcal{M} = \{x + y : x \in \mathcal{K}, y \in \mathcal{M}\}$,*
5) *$\psi(\mathcal{K} + \mathcal{M}) \leq \max\{\psi(\mathcal{K}), \psi(\mathcal{M})\}$,*
6) *$\psi(h\mathcal{K}) = |h|\psi(\mathcal{K})$ for any $h \in \mathbb{R}$.*

Lemma 6 ([25]). *Let $\mathcal{K} \subset \mathbb{B}$ be equicontinuous and bounded. Then $\psi(\mathcal{K}(t))$ is continuous on U, and $\psi(\mathcal{K}) = \sup_{t\in[0,b]} \psi(\mathcal{K}(t))$, where $\mathcal{K}(t) = \{y(t) : y \in \mathcal{K}\}$.*

Theorem 1 ([33]). *Let $\{l_m\}_{m=1}^{\infty}$ be a sequence of Bochner integrable functions from U to $\mathbb{B}$ such that $\|l_m(t)\| \leq g(t)$ for every $m \leq 1$ and almost all $g \in L^1(U, \mathbb{R}^+)$. Then the function $\mathfrak{h}(t) = \psi\{l_m(t) : m \geq 1\}$ contained in $L^1(U, \mathbb{R}^+)$, and satisfy*

$$\psi\bigg(\int_0^t l_m(s)ds : m \geq 1\bigg) \leq 2 \int_0^t \mathfrak{h}(s)ds.$$

Lemma 7 ([34]). *Let $\mathcal{M}$ be a closed and convex subset of a Banach space $\mathbb{B}$ and $0 \in D$. Then a continuous mapping $\Delta : \mathcal{M} \to \mathbb{B}$ which satisfies Mönch condition (i.e. $\mathbb{M} \subseteq \mathcal{M}$ is countable and $\mathbb{M} \subseteq \overline{co}(\{0\} \cup \Delta(\mathbb{M})) \Rightarrow \overline{\mathbb{M}}$ is compact) has a fixed point in $\mathcal{M}$.*

3 Existence and controllabilty

The next findings will make use of the following hypotheses.

$(K1)$ For each $t \in U$, the function $f(t, \cdot) : \mathbb{F} \to \mathbb{F}$ is continuous and the function $f(\cdot, y(\cdot)) : U \to \mathbb{F}$ is strongly measurable for $y \in \mathbb{F}$.

$(K2)$ There exists a function $m \in L^{\frac{1}{q}}(U, \mathbb{R}^+)$ and $q \in (0, \gamma)$ such that $d_\infty\big(f\big(t, y(t)\big), \hat{0}\big) \leq m(t)$ for all $y \in \mathcal{C}(U, \mathbb{F})$ and $t \in U$.

$(K3)$ For a constant $q_1 \in (0, \gamma)$ and bounded subsets $\mathcal{M} \in \mathbb{F}$, there exists a function $n_1 \in L^{\frac{1}{q_1}}(U, \mathbb{R}^+)$ such that $\psi(f(t, \mathcal{M})) \leq n_1(t)[\psi(\mathcal{M})]$.

$(K4)$ The bounded linear operator $H : L^p(U, \mathbb{M}) \to \mathbb{B}$, defined by

$$Hu = \int_0^t \mathcal{S}_\gamma(t - s) \odot Bu(s)\, ds,$$

where H has an induced inverse operator H^{-1}, there exist a positive constants M_B, h_0 such that $d_\infty(B, \hat{0}) = M_B$ and $d_\infty(H^{-1}, \hat{0}) = h_0$.

$(K5)$ For a constant $q_2 \in (0, \gamma)$ and for every $\mathcal{M} \in \mathbb{B}$, there exists a function $n_2 \in L^{\frac{1}{q_2}}(U, \mathbb{R}^+)$ such that $\psi(H^{-1}(\mathcal{M})(t)) \leq n_2(t)\psi(\mathcal{M})$.

For functions $n_j \in L^{\frac{1}{q_j}}(U, \mathbb{R}^+)$, with $q_j \in (0, \gamma)$, $j = 1, 2$, because of Hölder's inequality, we get

$$\begin{aligned}\int_0^t (t-s)^{\gamma-1} n_j(s)ds &\leq \bigg(\int_0^t (t-s)^{\frac{\gamma-1}{1-q_j}} ds\bigg)^{1-q_j} \bigg(\int_0^t \||m_j(s)\|^{\frac{1}{q_j}} ds\bigg)^{q_j} \\ &\leq \bigg[\bigg(\frac{1-q_j}{\gamma-q_j}\bigg) b^{\frac{\gamma-q_j}{1-q_j}}\bigg]^{1-q_j} \|n_j\|_{L^{\frac{1}{q_j}}}.\end{aligned}$$

Thus, for conciseness, we denote

$$N_1 = c_1\|n_1\|_{L^{\frac{1}{q_1}}}, \qquad N_2 = c_2\|n_2\|_{L^{\frac{1}{q_2}}},$$

where

$$c_j = \bigg[\bigg(\frac{1-q_j}{\gamma-q_j}\bigg) b^{\frac{\gamma-q_j}{1-q_j}}\bigg]^{1-q_j}, \quad j = 1, 2.$$

Theorem 2. *Assume that the assumptions $(K1)-(K5)$ are satisfied. Then the fuzzy Hilfer fractional differential equation* (1) *is controllable on U if $\hat{\mathcal{Q}} < 1$, where*

$$\hat{\mathcal{Q}} = \left(\frac{2MN_1M_B}{\Gamma(\gamma)} + 1\right)\frac{2M}{\Gamma(\gamma)}N_2. \tag{3}$$

Proof. Using the assumption $(K4)$, we define the control $u(t)$ for an arbitrary function $y \in C(U, \mathbb{F})$ by

$$u(t) = H^{-1}\left[y_b - \mathcal{H}_{\mu,\gamma}(t)y_0 - \int_0^t \mathcal{S}_\gamma(t-s) \odot f(s, y(s))ds\right].$$

Let us consider an operator $\Delta : C(U, \mathbb{F}) \to C(U, \mathbb{F})$, defined by

$$\Delta y(t) = \mathcal{H}_{\mu,\gamma}(t)y_0 + \int_0^t \mathcal{S}_\gamma(t-s) \odot Bu(s)ds + \int_0^t \mathcal{S}_\gamma(t-s) \odot f(s, y(s))ds, \quad t \in U.$$

Let $y \in C(U, \mathbb{F})$. By hypothesis $(K1)$, we obtain that $f(s, y(s))$ is a measurable function on U. Let

$$\alpha = \frac{\gamma - 1}{1-q}, \qquad M_f = \|f\|_{L^{\frac{1}{q}}(U,\mathbb{F})}.$$

Applying Hölder's inequality, for all $t \in U$ we get

$$\begin{aligned}
d_\infty\left(\int_0^t \mathcal{S}_\gamma(t-s) \odot [Bu(s) \oplus f(s, y(s))]ds, \hat{0}\right) &\leq \int_0^t \frac{M(t-s)^{(\gamma-1)}}{\Gamma(\gamma)} \odot d_\infty(Bu(s) \oplus f(s, y(s)), \hat{0})ds \\
&\leq \int_0^t \frac{M(t-s)^{(\gamma-1)}}{\Gamma(\gamma)} \odot d_\infty(Bu(s), \hat{0})\, ds + \int_0^t \frac{M(t-s)^{(\gamma-1)}}{\Gamma(\gamma)} \odot d_\infty(f(s, y(s)), \hat{0})\, ds \\
&\leq \int_0^t \frac{M(t-s)^{(\gamma-1)}}{\Gamma(\gamma)} \odot d_\infty(f(s, y(s)), \hat{0})ds \\
&\quad + \int_0^t \frac{M(t-s)^{(\gamma-1)}}{\Gamma(\gamma)} d_\infty\big(H^{-1}B\big[y_b - \mathcal{H}_{\mu,\gamma}(t)y_0 - \int_0^t \mathcal{S}_\gamma(t-s) \odot f(s, y(s))\, ds\big], \hat{0}\big)\, ds.
\end{aligned}$$

Therefore, we obtain

$$d_\infty\left(\int_0^t \mathcal{S}_\gamma(t-s) \odot [Bu(s) \oplus f(s, y(s))]\, ds, \hat{0}\right) \leq I_1 + I_2.$$

Let us estimate I_1 and I_2 as follows

$$\begin{aligned}
I_1 = M\int_0^t \frac{(t-s)^{(\gamma-1)}}{\Gamma(\gamma)} d_\infty(f(s, y(s)), \hat{0})ds &\leq \frac{M}{\Gamma(\gamma)}\left(\int_0^t (t-s)^{\frac{\gamma-1}{1-q}}ds\right)^{1-q} \|f\|_{L^{\frac{1}{q}}(U,\mathbb{F})} \\
&\leq \frac{M}{\Gamma(\gamma)} \frac{M_f b^{(\alpha+1)(1-q)}}{(\alpha+1)^{1-q}},
\end{aligned}$$

$$\begin{aligned}
I_2 &= \int_0^t \frac{M(t-s)^{(\gamma-1)}}{\Gamma(\gamma)} d_\infty\big(H^{-1}B\big[y_b - \mathcal{H}_{\mu,\gamma}(t)y_0 - \int_0^t \mathcal{S}_\gamma(t-s) \odot f(s, y(s))ds\big], \hat{0}\big)\, ds \\
&\leq \frac{h_0MM_B}{\Gamma(\gamma)} \int_0^t (t-s)^{(\gamma-1)}\left[y_b - \mathcal{H}_{\mu,\gamma}(t)y_0 - \int_0^t \mathcal{S}_\gamma(t-s) \odot f(s, y(s))\, ds\right] ds \\
&\leq \frac{h_0MM_B}{\Gamma(\gamma)} \int_0^t (t-s)^{(\gamma-1)}\left[y_b - M\frac{b^{(1-\delta)}}{\Gamma(\delta)}\|y_0\| - \frac{M}{\Gamma(\gamma)}\frac{M_f b^{(\alpha+1)(1-q)}}{(\alpha+1)^{1-q}}\right] ds \\
&= \frac{h_0MM_B}{\Gamma(\gamma)} \int_0^t (t-s)^{(\gamma-1)}\, \Theta\, ds,
\end{aligned}$$

where

$$\Theta = y_b - M\frac{b^{(1-\delta)}}{\Gamma(\delta)}\|y_0\| - \frac{M}{\Gamma(\gamma)}\frac{M_f b^{(\alpha+1)(1-q)}}{(\alpha+1)^{1-q}}.$$

Therefore,

$$I_2 \leq \frac{h_0 M M_B \Theta}{\Gamma(\gamma)^2}\left(\int_0^t (t-s)^{\frac{(\gamma-1)}{1-q}} ds\right)^{1-q} \leq \frac{h_0 M M_B \Theta}{\Gamma(\gamma)}\frac{b^{(\alpha+1)(1-q)}}{(\alpha+1)^{1-q}}.$$

$$I_1 + I_2 \leq \frac{M}{\Gamma(\gamma)}\frac{M_f b^{(\alpha+1)(1-q)}}{(\alpha+1)^{1-q}} + \frac{h_0 M M_B \Theta}{\Gamma(\gamma)}\frac{b^{(\alpha+1)(1-q)}}{(\alpha+1)^{1-q}}, \quad \forall t \in [0,b].$$

Then $\int_0^t \mathcal{S}_\gamma(t-s) \odot [Bu(s) \oplus f(s,y(s))]ds$ is bounded for all $t \in [0,b]$.

Define the operator Δ by

$$\Delta y(t) = (\Delta_1 y)(t) + (\Delta_2 y)(t)$$

for any $y \in C(U,\mathbb{F})$, where

$$(\Delta_1 y)(t) = \mathcal{H}_{\mu,\gamma}(t) \odot y_0, \qquad (\Delta_2 y)(t) = \int_0^t \mathcal{S}_\gamma(t-s) \odot Bu(s)ds + \int_0^t \mathcal{S}_\gamma(t-s) \odot f(s,y(s))ds.$$

Set

$$r = \|y_b\| + \left(1 + \frac{h_0 M M_B}{\Gamma(\gamma)}\frac{b^{(\alpha+1)(1-q)}}{(\alpha+1)^{1-q}}\right)\Theta$$

and $B_r := \{y(\cdot) \in C(U,\mathbb{F}) : d_\infty(y(t),\hat{0}) \leq r, \forall\, t \in U\}$. Next, we will prove that Δ has a fixed point on B_r.

Step 1. We show that for every $y \in B_r$ we have $(\Delta y)(t) \in B_r$. Indeed, with $0 \leq t_1 \leq t_2 \leq b$, we have

$$\begin{aligned}
&d_\infty\left((\Delta_2 y)(t_2), (\Delta_2 y)(t_1)\right)\\
&\quad = d_\infty\left(\int_0^{t_2} \mathcal{S}_\gamma(t_2-s) \odot [Bu(s) \oplus f(s,y(s))]ds, \int_0^{t_1} \mathcal{S}_\gamma(t_1-s) \odot [Bu(s) \oplus f(s,y(s))]ds\right)\\
&\quad = d_\infty\left(\int_{t_1}^{t_2} \mathcal{S}_\gamma(t_2-s) \odot [Bu(s) \oplus f(s,y(s))]ds\right.\\
&\qquad\quad \left. + \int_0^{t_1} \mathcal{S}_\gamma(t_2-s) \odot [Bu(s) \oplus f(s,y(s))]ds - \int_0^{t_1} \mathcal{S}_\gamma(t_1-s) \odot [Bu(s) \oplus f(s,y(s))]ds, \hat{0}\right)\\
&\quad \leq d_\infty\left(\int_{t_1}^{t_2} \mathcal{S}_\gamma(t_2-s) \odot [Bu(s) \oplus f(s,y(s))]ds, \hat{0}\right)\\
&\qquad\quad + d_\infty\left(\int_0^{t_1} [\mathcal{S}_\gamma(t_2-s) - \mathcal{S}_\gamma(t_1-s)] \odot [Bu(s) \oplus f(s,y(s))]ds, \hat{0}\right) = I_3 + I_4,
\end{aligned}$$

where

$$\begin{aligned}
I_3 &= d_\infty\left(\int_{t_1}^{t_2} \mathcal{S}_\gamma(t_2-s) \odot [Bu(s) \oplus f(s,y(s))]ds, \hat{0}\right),\\
I_4 &= d_\infty\left(\int_0^{t_1} [\mathcal{S}_\gamma(t_2-s) - \mathcal{S}_\gamma(t_1-s)] \odot [Bu(s) \oplus f(s,y(s))]ds, \hat{0}\right).
\end{aligned}$$

We have

$$\begin{aligned}
I_3 &= d_\infty\bigg(\int_{t_1}^{t_2} \mathcal{S}_\gamma(t_2-s)\odot[Bu(s)\oplus f(s,y(s))]ds,\hat{0}\bigg)\\
&\leq \int_{t_1}^{t_2}\frac{M(t_2-s)^{\gamma-1}}{\Gamma(\gamma)}\odot d_\infty([f(s,y(s))\oplus Bu(s)],\hat{0})ds\\
&\leq \frac{MM_f}{\Gamma(\gamma)}\bigg(\int_{t_1}^{t_2}(t_2-s)^\alpha ds\bigg)^{1-q}+\frac{h_0MM_B}{\Gamma(\gamma)}\bigg(\int_{t_1}^{t_2}(t_2-s)^\alpha ds\bigg)^{1-q}\\
&\leq \frac{MM_f}{\Gamma(\gamma)}\frac{(t_2-t_1)^{(\alpha+1)(1-q)}}{(\alpha+1)^{1-q}}-\frac{h_0MM_B}{\Gamma(\gamma)}\frac{(t_2-t_1)^{(\alpha+1)(1-q)}}{(\alpha+1)^{1-q}}
\end{aligned}$$

and

$$\begin{aligned}
I_4 &= d_\infty\bigg(\int_0^{t_1}[\mathcal{S}_\gamma(t_2-s)-\mathcal{S}_\gamma(t_1-s)]\odot[Bu(s)\oplus f(s,y(s))]ds,\hat{0}\bigg)\\
&\leq \int_0^{t_1}\bigg(\frac{M(t_2-s)^{\gamma-1}}{\Gamma(\gamma)}-\frac{M(t_1-s)^{\gamma-1}}{\Gamma(\gamma)}\bigg)\odot d_\infty([f(s,y(s))\oplus Bu(s)],\hat{0})ds\\
&\leq \frac{MM_f}{\Gamma(\gamma)}\bigg[\bigg(\int_0^{t_1}(t_2-s)^\alpha ds\bigg)^{1-q}-\bigg(\int_0^{t_1}(t_1-s)^\alpha ds\bigg)^{1-q}\bigg]\\
&+\frac{h_0MM_B}{\Gamma(\gamma)}\bigg[\bigg(\int_0^{t_1}(t_2-s)^\alpha ds\bigg)^{1-q}-\bigg(\int_0^{t_1}(t_1-s)^\alpha ds\bigg)^{1-q}\bigg]\\
&\leq \frac{MM_f}{\Gamma(\gamma)}\bigg[\frac{(t_2-t_1)^{(\alpha+1)(1-q)}}{(\alpha+1)^{1-q}}-\frac{t_2^{(\alpha+1)(1-q)}}{(\alpha+1)^{1-q}}+\frac{t_1^{(\alpha+1)(1-q)}}{(\alpha+1)^{1-q}}\bigg]\\
&+\frac{h_0MM_B}{\Gamma(\gamma)}\bigg[\frac{(t_2-t_1)^{(\alpha+1)(1-q)}}{(\alpha+1)^{1-q}}-\frac{t_2^{(\alpha+1)(1-q)}}{(\alpha+1)^{1-q}}+\frac{t_1^{(\alpha+1)(1-q)}}{(\alpha+1)^{1-q}}\bigg].
\end{aligned}$$

Then I_3 and I_4 tend to 0 as $(t_2-t_1)\to 0$. So, $(\Delta_2 y)(t)$ is continuous on U. It is also easy to see that $(\Delta_1 y)(t)$ is continuous on U. Consequently, $\Delta y(t)$ is continuous on U. Moreover, for any $y\in B_r$ and $t\in U$, we have

$$d_\infty\big((\Delta_1 y)(t)+(\Delta_2 y)(t),\hat{0}\big)\leq \|y_b\|+\bigg(1+\frac{h_0MM_B}{(\Gamma(\gamma))}\frac{b^{(\alpha+1)(1-q)}}{(\alpha+1)^{1-q}}\bigg)\Theta\leq r.$$

Thus, $\Delta_1+\Delta_2$ is an operator from B_r into B_r. Therefore, Δ is a bounded operator.

Step 2. We show that Δ is continuous on B_r. Let $y^w, y\in B_r$ be such that $\lim\limits_{w\to\infty} y^w=y$. Then $\lim\limits_{w\to\infty} y^w(t)=y(t)$ for all $t\in U$ and

$$f(s,y^w(s))\to f(s,y(s))\quad\text{as}\ \ w\to+\infty.$$

Take

$$F^w(s)=f\big(s,y^w(s)\big)\qquad\text{and}\qquad F(s)=f\big(s,y(s)\big).$$

Then from hypotheses $(K1)$–$(K5)$ and Lebesgue dominated convergence theorem, we obtain

$$\int_0^t \mathcal{S}_\gamma(t-s) \odot d_\infty(F^w(s) - F(s), \hat{0})ds \to 0 \quad \text{as } \ w \to 0 \ \text{ for all } \quad t \in U. \tag{4}$$

Next, we have

$$u^w(t) = H^{-1}\Big[y_b - \mathcal{H}_{\mu,\gamma}(t)y_0 - \int_0^t \mathcal{S}_\gamma(t-s) \odot f(s, y^w(s))ds\Big],$$

$$d_\infty(u^w(t) - u(t), \hat{0}) = H^{-1}\Big[\int_0^t \mathcal{S}_\gamma(t-s) \odot d_\infty\Big(f(s, y^w(s)) - f(s, y(s)), \hat{0}\Big)ds\Big]. \tag{5}$$

From the equation (4) it follows that the above term converges to 0 as $w \to \infty$. Now,

$$d_\infty(\Delta y^w - \Delta y, \hat{0}) \le \int_0^t \mathcal{S}_\gamma(t-s) \odot \Big[d_\infty(F^w(s) - F(s), \hat{0}) + d_\infty(u^w(t) - u(t), \hat{0})\Big]ds.$$

Using (4) and (5), we obtain $d_\infty(\Delta y^w - \Delta y) \to 0$ as $w \to \infty$. Therefore, Δ is continuous on B_r.

Step 3. Let us show that Δ is equicontinuous. Let $y(t) \in B_r$, where $t \in [0, b]$.

For all $y(t) \in B_r$ and $0 \le t_1 \le t_2 \le b$, we have

$$d_\infty\big((\Delta y)(t_2), (\Delta y)(t_1)\big) = d_\infty\big((\Delta_1 y)(t_2), (\Delta_1 y)(t_1)\big) + d_\infty\big((\Delta_2 y)(t_2), (\Delta_2 y)(t_1)\big).$$

For the first addend, we get

$$\begin{aligned} d_\infty\big((\Delta_1 y)(t_2), (\Delta_1 y)(t_1)\big) &= d_\infty\Big(\mathcal{H}_{\mu,\gamma}(t_2)y_0 - \mathcal{H}_{\mu,\gamma}(t_1)y_0, \hat{0}\Big) \\ &\le M\frac{t_2^{(1-\delta)}}{\Gamma(\delta)}\|y_0\| - M\frac{t_1^{(1-\delta)}}{\Gamma(\delta)}\|y_0\|. \end{aligned}$$

It follows that this term converges to 0 as $(t_2 - t_1) \to 0$. Therefore Δ_1 is equicontinuos.

For the second addend, let $d_\infty\big((\Delta_2 y)(t_2), (\Delta_2 y)(t_1)\big) = I_3 + I_4$, where

$$\begin{aligned} I_3 &= d_\infty\bigg(\int_{t_1}^{t_2} \mathcal{S}_\gamma(t_2 - s) \odot [Bu(s) \oplus f(s, y(s))]ds, \hat{0}\bigg), \\ I_4 &= d_\infty\bigg(\int_0^{t_1} [\mathcal{S}_\gamma(t_2 - s) - \mathcal{S}_\gamma(t_1 - s)] \odot [Bu(s) \oplus f(s, y(s))]ds, \hat{0}\bigg). \end{aligned}$$

From the Step 1, it is easy to see that $\Delta_2(B_r)$ is equicontinuous independently of $y \in B_r$ as $t_2 \to t_1$. Thus, Δ_2 is equicontinuous on B_r. Therefore, Δ is equicontinuous on B_{h_0}.

Step 4. Let $\mathcal{M}$ be a countable subset of B_r and $\mathcal{M} \subseteq \overline{co}(\{0\} \cup \Delta(\mathcal{M}))$. Next, we prove that $\psi(\mathcal{M}) = 0$, where ψ denotes the Hausdorff measure of noncompactness. Without loss of generality, we may assume that $\mathcal{M} = \{y^w\}_{w=1}^{\infty}$. By Step 3, we get that Δy is equicontinuous on U, then $\mathcal{M} \subseteq \overline{co}(\{0\} \cup \Delta(\mathcal{M}))$ is also equicontinuous on U.

Therefore, by Theorem 1, we have

$$
\begin{aligned}
&\psi(\{\Delta y^{w}(t)\}_{w=1}^{\infty}) \\
&\quad\le \psi\left(\mathcal{H}_{\mu,\gamma}(t)y_0 + \left\{\int_0^t \mathcal{S}_\gamma(t-s)\odot[Bu^{w}(s)\oplus f(s,y^{w}(s))]ds\right\}_{w=1}^{\infty}\right) \\
&\quad\le \psi\left(\left\{\int_0^t \mathcal{S}_\gamma(t-s)\odot Bu^{w}(s)ds\right\}_{w=1}^{\infty}\right) \\
&\qquad+\psi\left(\left\{\int_0^t \mathcal{S}_\gamma(t-s)\odot f(s,y^{w}(s))ds\right\}_{w=1}^{\infty}\right) \\
&\quad\le \frac{2M}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}n_1(s)\psi(\{y^{w}\}_{w=1}^{\infty})ds \\
&\qquad+\frac{2MM_B}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\psi(\{u^{w}(s)\}_{w=1}^{\infty})ds \\
&\quad\le \frac{2M}{\Gamma(\gamma)}\int_0^b (b-s)^{\gamma-1}n_1(s)\psi(\{y^{w}\}_{w=1}^{\infty})ds \\
&\qquad+\frac{2MM_B}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\left[n_2(s)\frac{2MM_B}{\Gamma(\gamma)}\int_0^b (b-\xi)^{\gamma-1}n_1(\xi)\psi(\{y^{w}\}_{w=1}^{\infty})d\xi\right]ds \\
&\quad\le \frac{2M}{\Gamma(\gamma)}N_1\psi(\mathcal{M})+\frac{2MM_B}{\Gamma(\gamma)}N_2\frac{2M}{\Gamma(\gamma)}N_1\psi(\mathcal{M}) \\
&\quad\le \left(\frac{2MN_2M_B}{\Gamma(\gamma)}+1\right)\frac{2M}{\Gamma(\gamma)}N_1\psi(\mathcal{M}).
\end{aligned}
$$

Now, using Mönch condition, we deduce

$$\psi(\mathcal{M})\le\psi(\overline{co}(\{0\}\cup\Delta(\mathcal{M})))=\psi(\Delta(\mathcal{M}))\le\left(\frac{2MN_2M_B}{\Gamma(\gamma)}+1\right)\frac{2M}{\Gamma(\gamma)}N_1\psi(\mathcal{M}).$$

Thus,

$$\psi(\Delta(\mathcal{M}))\le\hat{\mathcal{Q}}\psi(\mathcal{M}),$$

where

$$\hat{\mathcal{Q}}=\left(\frac{2MN_3M_B}{\Gamma(\gamma)}+1\right)\frac{2M}{\Gamma(\gamma)}N_2.$$

This shows by the inequality (3) that $\psi(\mathcal{M})=0$. Now, by applying Lemma 7, we observe that Δ has a fixed point on B_r. Hence, the system (1) is controllable on U. This completes the proof. □

4 Example

Consider the following fuzzy Hilfer fractional differential system

$$
\begin{aligned}
&{}_{gH}D_{0^+}^{\gamma,\frac{2}{3}}y(t,\varpi)=\frac{\partial^2}{\partial\varpi^2}y(t,\varpi)+H\hat{\mathfrak{R}}(t,\varpi)+t^{\frac{1}{3}(1-\mu)}\frac{e^{-t}}{6}\sin(1+y(t,\varpi)),\\
&I_{0^+}^{(1-\gamma)(1-\frac{2}{3})}y(0,\varpi)=y_0(\varpi),\quad \varpi\in[0,\sigma],\\
&y(t,0)=y(t,\sigma)=0,\quad t\in U,
\end{aligned}
\tag{6}
$$

where $D_{0^+}^{\gamma,\frac{2}{3}}$ is the fuzzy fractional differential equation of order $\gamma\in(0,1)$ and type $\mu=\frac{2}{3}$.

Let $\mathbb{B} = \mathbb{M} = L^2[0, \sigma]$ be endowed with the norm $\|\cdot\|_{L^2}$ and $A : D(A) \subset \mathbb{B} \to \mathbb{B}$ be defined by $A\rho = \rho'$ with

$$D(A) = \left\{\rho \in \mathbb{F} : \rho, \frac{\partial}{\partial \rho} \text{ are absolutely continuous}, \frac{\partial^2}{\partial \rho^2} \in \mathbb{F}, \rho(0) = \rho(\sigma) = 0\right\}$$

and

$$A\rho = \sum_{w=1}^{\infty} w^2(\rho, \rho_w)\rho_w, \quad \rho \in D(A),$$

where

$$\rho_w(x) = \sqrt{\frac{2}{\sigma}} \sin(w\rho).$$

We have that A is the infinitesimal generator of the semigroup $\{T(t), t \geq 0\}$ in $\mathbb{B}$ and $T(t)$ is noncompact semigroup on $\mathbb{B}$ with $\psi(T(t)\mathcal{M}) \leq \psi(\mathcal{M})$, where the Hausdorff measure of noncompactness was represented by ψ and $\hat{Q} \leq 1$ is a constant, which satisfy

$$\sup_{t \in U} d_\infty(T(t), \hat{0}) \leq \hat{Q}.$$

Define,

$$\begin{aligned} y(t)(\varpi) &= y(t, \varpi) \\ f(t, y(t)) &= t^{\frac{1}{3}(1-\mu)} \frac{e^{-t}}{6} \sin(1 + y(t, \varpi)). \end{aligned}$$

Let $B : \mathbb{M} \to \mathbb{B}$ be defined by

$$(Bu)(\varpi) = H\hat{\mathfrak{R}}(t, \varpi), \quad 0 < \varpi < \sigma.$$

Given the entries A, B and f, equation (6) can be written as

$$\begin{aligned} &{}_{gH}D^{\gamma,\mu} y(t) = Ay(t) + Bu(t) + f(t, y(t)), \quad \mu = \frac{2}{3} \in (0, 1), \quad t \in U, \\ &I_{0_+}^{(1-\gamma)(1-\mu)} y(0) = y_0 \in B_r. \end{aligned}$$

Consequently, system (6) can be rewritten as system (1). The function f clearly satisfies the assumptions $(K1)$–$(K5)$. This implies that Theorem 2 is fully satisfied. Therefore, the fuzzy fractional differential equation (1) is controllable on U.

5 Conclusion

This work focused on the controllability of fuzzy Hilfer fractional differential equations using the measure of noncompactness. The main results were obtained through Caratheodory-type conditions, the measure of noncompactness, fixed-point techniques, semigroup theory, and fractional calculus. First, we demonstrated the controllability of the system by applying the Mönch fixed-point theorem. To illustrate the theory, we applied the findings to our system. Future work on fuzzy Hilfer fractional differential equations will explore various forms of controllability.

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У цій статті розглядаються результати щодо керованості нечітких диференціальних рівнянь Гільфера дробового порядку за допомогою міри некомпактності. Основні результати отримано із застосуванням теореми Мьонха про нерухому точку, а також таких важливих інструментів, як теорія напівгруп, дробове числення та міра некомпактності. Насамкінець теоретичні результати проілюстровано на цікавому прикладі.

Ключові слова і фрази: нечітка дробова похідна Гільфера, керованість, міра некомпактності, теорема Мьонха про нерухому точку.