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On the regular continued fractions of real algebraic irrational numbers

Yoshida H.

It is well known that an irrational number is quadratic if and only if its regular continued fraction expansion is ultimately periodic. However, no such characterization is known for other real irrational numbers. In 1949, A.Ya. Khinchin conjectured that partial denominators of the regular continued fractions of real algebraic numbers of degree higher than 2 are unbounded. In other words, if partial denominators of the regular continued fractions is bounded, then it is a quadratic number or a transcendental number.

In this paper, we observe the regular continued fractions of real algebraic numbers of degree higher than 2. More precisely, we give the minimal polynomials of the real algebraic numbers appearing in the regular continued fractions and establish their properties.

Key words and phrases: continued fraction, irrational number, transcendental number, diophantine approximation.

Graduate School of Science and Engineering, Kansai University, 3-3-35 Yamate-cho, Suita-shi, Osaka 564-8680, Japan
E-mail: k321930@kansai-u.ac.jp

1 Introduction

Let α be a real irrational number. We denote the regular continued fraction of α by

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} = [a_0; a_1, a_2, a_3, \ldots],$$

where a_0 is an integer and $a_1, a_2, a_3, \ldots$ are positive integers. If α is a quadratic number, its regular continued fraction expansion $\alpha = [a_0; a_1, a_2, a_3, \ldots]$ is ultimately periodic. Namely, there exists an integer N greater than or equal to 0 and a positive integer T such that $a_{n+T} = a_n$ for any integer $n \geq N$. Hence other cases are not ultimately periodic. As characterization of the regular continued fraction of real algebraic numbers of degree higher than 2, it is widely believed the following conjecture due to A.Ya. Khinchin.

Conjecture ([1]). *If α is a real algebraic number of degree higher than 2 (denote by $\deg \alpha > 2$), then partial denominators are unbounded. Namely,*

$$\deg \alpha > 2 \quad \Longrightarrow \quad \sup_{n \geq 1} a_n = \infty.$$

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In other words, if partial denominators are bounded, then α is a quadratic number or a transcendental number. This conjecture is called Khinchin's conjecture. This conjecture is partially solved. More precisely, B. Adamczewski and Y. Bugeaud showed that if $a_0, a_1, \cdots, a_n$ are palindrome (i.e. $a_j = a_{n-j}$ for every integer j with $0 \leq j \leq n$), then α is transcendental (see [2]).

Let α be a real algebraic number of degree $n \geq 3$, and let $f(x) := \sum_{k=0}^{n} b_k x^k$ be the minimal polynomial of α. We denote the ith convergent for regular continued fractions by p_i/q_i, that is

$$\frac{p_i}{q_i} = [a_0; a_1, a_2, a_3, \ldots, a_i],$$

where p_i and q_i are defined by

$$\begin{cases} p_{-1} = 1,\ p_0 = a_0,\ q_{-1} = 0,\ q_0 = 1, \\ p_i = a_i p_{i-1} + p_{i-2}, \quad i \geq 1, \\ q_i = a_i q_{i-1} + q_{i-2}, \quad i \geq 1. \end{cases}$$

We put $\alpha = [a_0; a_1, \ldots, \alpha_{i+1}]$ for any integer $i \geq 0$, namely $\alpha_{i+1} = [a_{i+1}; a_{i+2}, a_{i+3}, \ldots]$.

Theorem 1. *We have the following equality*

$$\alpha_{i+1} = \frac{(-1)^{i-1}}{p_i q_i f\left(\frac{p_i}{q_i}\right)} \left(\sum_{k=1}^{n-1} \sum_{l=1}^{n-k} b_{l+k} \left(\frac{p_i}{q_i}\right)^l \alpha^k - b_0 \right) - \frac{p_{i-1}}{p_i}.$$

From this theorem, we have that

$$\lim_{i \to \infty} \left| \frac{f\left(\frac{p_i}{q_i}\right)}{\alpha - \frac{p_i}{q_i}} \right| = |f'(\alpha)|.$$

Alternatively, a relationship between $|f(p_i/q_i)|$ and $|\alpha - p_i/q_i|$ is the following

$$\left| f\left(\frac{p_i}{q_i}\right) \right| \sim |f'(\alpha)| \left| \alpha - \frac{p_i}{q_i} \right| \quad \text{as } i \to \infty.$$

2 Preliminaries

Firstly, we prepare to prove Theorem 1. Let α be a real algebraic number of degree $n \geq 3$, and let $f(x) := \sum_{k=0}^{n} b_k x^k$ be the minimal polynomial of α. If we put $\alpha = [a_0; a_1, \ldots, a_i, \alpha_{i+1}]$, then we have

$$\alpha = \frac{p_i \alpha_{i+1} + p_{i-1}}{q_i \alpha_{i+1} + q_{i-1}}. \tag{1}$$

Then we see that

$$f(\alpha) = f\left(\frac{p_i \alpha_{i+1} + p_{i-1}}{q_i \alpha_{i+1} + q_{i-1}}\right) = \sum_{k=0}^{n} b_k \left(\frac{p_i \alpha_{i+1} + p_{i-1}}{q_i \alpha_{i+1} + q_{i-1}}\right)^k = 0.$$

Multiplying both sides of the last equality by $(q_i \alpha_{i+1} + q_{i-1})^n$, we get

$$\sum_{k=0}^{n} b_k (p_i \alpha_{i+1} + p_{i-1})^k (q_i \alpha_{i+1} + q_{i-1})^{n-k} = 0.$$

Here, the coefficient of α_{i+1}^n is given by

$$\sum_{k=0}^{n} b_k p_i^k q_i^{n-k} = q_i^n \sum_{k=0}^{n} b_k (p_i/q_i)^k = q_i^n f(p_i/q_i).$$

Since $f(p_i/q_i) \neq 0$, the minimal polynomial of α_{i+1} is given by

$$\sum_{k=0}^{n} b_k(p_i x + p_{i-1})^k (q_i x + q_{i-1})^{n-k}.$$

By equation (1), we obtain

$$q_i \alpha \alpha_{i+1} + q_{i-1}\alpha - p_i \alpha_{i+1} - p_{i-1} = 0. \tag{2}$$

By the above argument, since $\deg \alpha_{i+1} = n$, we can write $\alpha_{i+1} = \sum_{k=0}^{n-1} c_k^{(i+1)} \alpha^k$, where $c_{n-1}^{(i+1)} \neq 0$ and $c_0^{(i+1)}, c_1^{(i+1)}, \ldots, c_{n-1}^{(i+1)} \in \mathbb{Q}$. Substituting this into (2), we get

$$q_i \sum_{k=0}^{n-1} c_k^{(i+1)} \alpha^{k+1} + q_{i-1}\alpha - p_i \sum_{k=0}^{n-1} c_k^{(i+1)} \alpha^k - p_{i-1} = 0.$$

Comparing the coefficients of the left hand side and $f(\alpha) = \sum_{k=0}^{n} b_k \alpha^k$, we obtain

$$\begin{cases} b_n = s_{i+1} q_i c_{n-1}^{(i+1)}, \\ b_k = s_{i+1}(q_i c_{k-1}^{(i+1)} - p_i c_k^{(i+1)}), \quad 2 \leq k \leq n-1, \\ b_1 = s_{i+1}(q_i c_0^{(i+1)} + q_{i-1} - p_i c_1^{(i+1)}), \\ b_0 = -s_{i+1}(p_i c_0^{(i+1)} + p_{i-1}), \end{cases}$$

where s_{i+1} are non-zero rational numbers for $i \geq 0$. From these, the constants $c_k^{(i+1)}$ can be expressed as follows

$$\begin{cases} c_{n-1}^{(i+1)} = \dfrac{b_n}{s_{i+1} q_i}, \\ c_k^{(i+1)} = \dfrac{(-1)^{i-1} s_{i+1} q_i^{k-1} - \sum_{l=0}^{k} b_l p_i^l q_i^{k-l}}{s_{i+1} p_i^{k+1}}, \quad 1 \leq k \leq n-1, \\ c_0^{(i+1)} = -\dfrac{s_{i+1} p_{i-1} + b_0}{s_{i+1} p_i}. \end{cases}$$

From these, we see that

$$c_{n-1}^{(i+1)} = \frac{b_n}{s_{i+1} q_i} = \frac{(-1)^{i-1} s_{i+1} q_i^{n-2} - \sum_{l=0}^{n-1} b_l p_i^l q_i^{n-l-1}}{s_{i+1} p_i^n}.$$

By simple observation, we obtain

$$s_{i+1} = \frac{(-1)^{i-1}}{q_i^{n-1}} \sum_{l=0}^{n} b_l p_i^l q_i^{n-l} = (-1)^{i-1} q_i \sum_{l=0}^{n} b_l \left(\frac{p_i}{q_i}\right)^l = (-1)^{i-1} q_i f\left(\frac{p_i}{q_i}\right). \tag{3}$$

3 Properties of s_{i+1}

In this section, we introduce the properties regarding the sign and asymptotic behavior of s_{i+1}. There is the following well known relationship between an irrational number α and an ith convergents p_i/q_i of the regular continued fraction of α, namely

$$\frac{1}{q_i(q_{i+1} + q_i)} < \left|\alpha - \frac{p_i}{q_i}\right| < \frac{1}{q_i q_{i+1}}. \tag{4}$$

Especially,

$$\frac{p_{2k}}{q_{2k}} < \alpha < \frac{p_{2k-1}}{q_{2k-1}}$$

for any integer $k \geq 0$.

We denote the derivative of $f(x)$ by $f'(x)$. Then if $f'(\alpha) > 0$ (respectively, $f'(\alpha) < 0$), we have that

$$\begin{cases} f\left(\frac{p_i}{q_i}\right) < 0 \text{ (respectively } > 0), & \text{if } i \text{ even,} \\ f\left(\frac{p_i}{q_i}\right) > 0 \text{ (respectively } > 0), & \text{if } i \text{ odd.} \end{cases}$$

Therefore, we can state the following theorem.

Theorem 2. *For any integer $i \geq 0$, if $f'(\alpha) > 0$, we obtain that $s_{i+1} > 0$ and if $f'(\alpha) < 0$, we also have that $s_{i+1} < 0$.*

In order to investigate the asymptotic behavior of s_{i+1}, we establish inequalities involving $f(p_i/q_i)$.

Theorem 3. *The sequence $|f(p_i/q_i)|$ satisfies the following inequalities*

$$\frac{\left|\sum_{k=1}^{n} b_k \sum_{l=0}^{k-1} \alpha^l \left(\frac{p_i}{q_i}\right)^{k-l-1}\right|}{(2+a_{i+1})q_i^2} < \left|f\left(\frac{p_i}{q_i}\right)\right| < \frac{\left|\sum_{k=1}^{n} b_k \sum_{l=0}^{k-1} \alpha^l \left(\frac{p_i}{q_i}\right)^{k-l-1}\right|}{a_{i+1}q_i^2}. \tag{5}$$

Proof. We consider the absolute value of the difference between $f(\alpha)$ and $f(p_i/q_i)$. Then we obtain

$$\left|f\left(\frac{p_i}{q_i}\right)\right| = \left|f(\alpha) - f\left(\frac{p_i}{q_i}\right)\right| = \left|\sum_{k=0}^{n} b_k\alpha^k - \sum_{k=0}^{n} b_k \left(\frac{p_i}{q_i}\right)^k\right| = \left|\sum_{k=1}^{n} b_k \left(\alpha^k - \left(\frac{p_i}{q_i}\right)^k\right)\right|.$$

Since $\alpha^k - (p_i/q_i)^k = (\alpha - p_i/q_i)\sum_{l=0}^{k-1} \alpha^l (p_i/q_i)^{k-l-1}$, the rightmost side equals to

$$\left|\left(\alpha - \frac{p_i}{q_i}\right)\sum_{k=1}^{n} b_k \sum_{l=0}^{k-1} \alpha^l \left(\frac{p_i}{q_i}\right)^{k-l-1}\right| = \left|\alpha - \frac{p_i}{q_i}\right| \left|\sum_{k=1}^{n} b_k \sum_{l=0}^{k-1} \alpha^l \left(\frac{p_i}{q_i}\right)^{k-l-1}\right|.$$

By inequalities (4), we have that

$$\frac{\left|\sum_{k=1}^{n} b_k \sum_{l=0}^{k-1} \alpha^l \left(\frac{p_i}{q_i}\right)^{k-l-1}\right|}{q_i(q_{i+1}+q_i)} < \left|f\left(\frac{p_i}{q_i}\right)\right| < \frac{\left|\sum_{k=1}^{n} b_k \sum_{l=0}^{k-1} \alpha^l \left(\frac{p_i}{q_i}\right)^{k-l-1}\right|}{q_iq_{i+1}}.$$

By recurrence relation of q_i, since $q_{i+1} = a_{i+1}q_i + q_{i-1}$, then we get

$$q_i(q_{i+1}+q_i) = q_i((1+a_{i+1})q_i + q_{i-1}) < (2+a_{i+1})q_i^2$$

and

$$q_iq_{i+1} = a_{i+1}q_i^2 + q_iq_{i-1} > a_{i+1}q_i^2.$$

Therefore, we obtain the desired inequalities (5). □

By the equality (3) and this theorem, we have that

$$\frac{\left|\sum_{k=1}^{n} b_k \sum_{l=0}^{k-1} \alpha^l \left(\frac{p_i}{q_i}\right)^{k-l-1}\right|}{(2+a_{i+1})q_i} < |s_{i+1}| < \frac{\left|\sum_{k=1}^{n} b_k \sum_{l=0}^{k-1} \alpha^l \left(\frac{p_i}{q_i}\right)^{k-l-1}\right|}{a_{i+1}q_i}.$$

Since $\lim_{i\to\infty} p_i/q_i = \alpha$, we see that

$$\lim_{i\to\infty}\left|\sum_{k=1}^{n} b_k \sum_{l=0}^{k-1} \alpha^l \left(\frac{p_i}{q_i}\right)^{k-l-1}\right| = |f'(\alpha)| < \infty.$$

Therefore, we obtain $\lim_{i\to\infty} |s_{i+1}| = 0$. Furthermore, if partial denominators a_i are bounded, then there exists a positive integer N $(= \max_{i\geq 1} a_i)$ such that

$$0 < \frac{\left|\sum_{k=1}^{n} b_k \sum_{l=0}^{k-1} \alpha^l \left(\frac{p_i}{q_i}\right)^{k-l-1}\right|}{2+N} < q_i|s_{i+1}| < \left|\sum_{k=1}^{n} b_k \sum_{l=0}^{k-1} \alpha^l \left(\frac{p_i}{q_i}\right)^{k-l-1}\right|$$

for any positive integer i. Hence, we see that

$$\frac{|f'(\alpha)|}{2+N} \leq \limsup_{i\to\infty} q_i|s_{i+1}| \leq |f'(\alpha)| \qquad \text{and} \qquad \frac{|f'(\alpha)|}{2+N} \leq \liminf_{i\to\infty} q_i|s_{i+1}| \leq |f'(\alpha)|.$$

4 Proof of Theorem 1

In this section, we give a proof of our main theorem. By the argument in Section 2, since $\alpha_{i+1} = \sum_{k=0}^{n-1} c_k^{(i+1)} \alpha^k$ with

$$\begin{cases} c_k^{(i+1)} = \dfrac{(-1)^{i-1} s_{i+1} q_i^{k-1} - \sum_{l=0}^{k} b_l p_i^l q_i^{k-l}}{s_{i+1} p_i^{k+1}}, & 1 \leq k \leq n-1, \\ c_0^{(i+1)} = -\dfrac{s_{i+1} p_{i-1} + b_0}{s_{i+1} p_i}, \end{cases}$$

we obtain that

$$\alpha_{i+1} = \sum_{k=1}^{n-1} \frac{(-1)^{i-1} s_{i+1} q_i^{k-1} - \sum_{l=0}^{k} b_l p_i^l q_i^{k-l}}{s_{i+1} p_i^{k+1}} \alpha^k - \frac{s_{i+1} p_{i-1} + b_0}{s_{i+1} p_i}.$$

From (3), we can transform the following

$$\begin{aligned} \alpha_{i+1} &= \sum_{k=1}^{n-1} \frac{q_i^k f(p_i/q_i) - q_i^k \sum_{l=0}^{k} b_l (p_i/q_i)^l}{(-1)^{i-1} q_i f(p_i/q_i) p_i^{k+1}} \alpha^k - \frac{p_{i-1}}{p_i} - \frac{b_0}{(-1)^{i-1} q_i f(p_i/q_i) p_i} \\ &= \frac{(-1)^{i-1}}{p_i q_i f(p_i/q_i)} \left(\sum_{k=1}^{n-1} \left(\frac{q_i}{p_i}\right)^k \left(f\left(\frac{p_i}{q_i}\right) - \sum_{l=0}^{k} b_l \left(\frac{p_i}{q_i}\right)^l \right) \alpha^k - b_0 \right) - \frac{p_{i-1}}{p_i}. \end{aligned}$$

Since $f(p_i/q_i) - \sum_{l=0}^{k} b_l (p_i/q_i)^l = \sum_{l=k+1}^{n} b_l (p_i/q_i)^l$, then we see that

$$\begin{aligned} \alpha_{i+1} &= \frac{(-1)^{i-1}}{p_i q_i f(p_i/q_i)} \left(\sum_{k=1}^{n-1} \left(\frac{q_i}{p_i}\right)^k \sum_{l=k+1}^{n} b_l \left(\frac{p_i}{q_i}\right)^l \alpha^k - b_0 \right) - \frac{p_{i-1}}{p_i} \\ &= \frac{(-1)^{i-1}}{p_i q_i f(p_i/q_i)} \left(\sum_{k=1}^{n-1} \sum_{l=k+1}^{n} b_l \left(\frac{p_i}{q_i}\right)^{l-k} \alpha^k - b_0 \right) - \frac{p_{i-1}}{p_i} \\ &= \frac{(-1)^{i-1}}{p_i q_i f(p_i/q_i)} \left(\sum_{k=1}^{n-1} \sum_{l=1}^{n-k} b_{l+k} \left(\frac{p_i}{q_i}\right)^{l} \alpha^k - b_0 \right) - \frac{p_{i-1}}{p_i}. \end{aligned}$$

These complete the proof of Theorem 1.

Finally, we explain an asymptotic relation between $|f(p_i/q_i)|$ and $|\alpha - p_i/q_i|$. By Theorem 1, we obtain that

$$\left|\alpha_{i+1} + \frac{p_{i-1}}{p_i}\right| = \frac{1}{\left|p_i q_i f\left(\frac{p_i}{q_i}\right)\right|} \left|\sum_{k=1}^{n-1} \sum_{l=1}^{n-k} b_{l+k} \left(\frac{p_i}{q_i}\right)^l \alpha^k - b_0\right|. \tag{6}$$

Since $\alpha_{i+1} = (q_{i-1}\alpha - p_{i-1})/(q_i\alpha - p_i)$, we see that

$$\left|\alpha_{i+1} + \frac{p_{i-1}}{p_i}\right| = |\alpha| \left|\frac{1}{p_i(q_i\alpha - p_i)}\right|.$$

Then the equation (6) is transformed to

$$|\alpha| \left|\frac{f\left(\frac{p_i}{q_i}\right)}{\alpha - \frac{p_i}{q_i}}\right| = \left|\sum_{k=1}^{n-1}\sum_{l=1}^{n-k} b_{l+k}\left(\frac{p_i}{q_i}\right)^l \alpha^k - b_0\right|.$$

The limit of the right hand side is equal to

$$\left|\sum_{k=1}^{n-1}\sum_{l=1}^{n-k} b_{l+k}\alpha^{l+k} - b_0\right| = \left|\sum_{k=2}^{n}(k-1)\alpha^k - b_0\right| = \left|\sum_{k=1}^{n} k\alpha^k - \sum_{k=0}^{n} b_k\alpha^k\right| = |\alpha f'(\alpha)|.$$

Therefore, we have that

$$\lim_{i\to\infty}\left|\frac{f\left(\frac{p_i}{q_i}\right)}{\alpha - \frac{p_i}{q_i}}\right| = |f'(\alpha)|.$$

Alternatively, a relationship between $|f(p_i/q_i)|$ and $|\alpha - p_i/q_i|$ is the following

$$\left|f\left(\frac{p_i}{q_i}\right)\right| \sim |f'(\alpha)| \left|\alpha - \frac{p_i}{q_i}\right| \quad \text{as } i \to \infty.$$

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Йосіда Х. *Про регулярні ланцюгові дроби дійсних алгебраїчних ірраціональних чисел* // Карпатські матем. публ. — 2026. — Т.18, №1. — С. 258–263.

Добре відомо, що ірраціональне число є квадратичним тоді і тільки тоді, коли його розклад у регулярний ланцюговий дріб є зрештою періодичним. Однак подібної характеристики для інших дійсних ірраціональних чисел не відомо. У 1949 році А.Я. Хінчин висловив гіпотезу про те, що неповні частки регулярних ланцюгових дробів дійсних алгебраїчних чисел степеня, вищого за 2, є необмеженими. Іншими словами, якщо неповні частки регулярного ланцюгового дробу є обмеженими, то відповідне число є або квадратичним, або трансцендентним.

У цій роботі ми досліджуємо регулярні ланцюгові дроби дійсних алгебраїчних чисел степеня, вищого за 2. Зокрема, ми наводимо мінімальні многочлени дійсних алгебраїчних чисел, що виникають у регулярних ланцюгових дробах, та вивчаємо їхні властивості.

Ключові слова і фрази: ланцюговий дріб, ірраціональне число, трансцендентне число, діофантове наближення.