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On constructing algebras of finite range

Votiakova L.A.[1], Fryz I.V.[2]

In the paper, a subalgebra whose elements are square matrices with real entries having the same sum of row entries is extracted from a complete matrix algebra. Using classical methods of matrix theory, the properties of constructed algebra are studied. This algebra is endowed with a norm that makes it possible to construct of elements of analysis in it by means of the matrix analysis methods. A new class of algebras of finite range is constructed, namely, an algebra of hypercomplex numbers, which is isomorphic to the corresponding matrix algebra. Thus, the obtained results for the matrices can be transferred to the elements of the isomorphic algebra of finite range, i.e. hypercomplex numbers. This lead to defining the functions of hypercomplex variable.

Key words and phrases: algebra of finite range, hypercomplex number, projector, matrix norm, function of a hypercomplex variable.

[1] Vinnytsia Mykhailo Kotsiubynskyi State Pedagogical University, 32 Ostrozhskogo str., 21100, Vinnytsia, Ukraine
[2] Vasyl' Stus Donetsk National University, 21 600-richchia str., 21021, Vinnytsia, Ukraine
E-mail: tomusiak.lesia@gmail.com (Votiakova L.A.), iryna.fryz@ukr.net (Fryz I.V.)

Introduction

The classic technique for constructing algebras of finite range, whose carriers are some linear spaces of finite dimension over a field with the given basis, is based on the selection of structural constants [7], which means multiplication of the basis vectors, and so the elements of an algebra, and it inherits some properties, for instance, the operations are associative, multiplication is distributive over addition, each operation has an identity element in the set etc.

Among all associative algebras of finite range, complete matrix algebras $\mathbf{M}_n = \mathbf{M}_n(\mathrm{P})$ of order n over a field P play an important role, which is similar to the role of symmetric groups in the set of finite groups [8]. Namely, each associative algebra of finite range over a field P allows monomorphic embedding in a complete matrix algebra over the same field, which is effectively the same as that each finite group may be monomorphically embeddable in the corresponding symmetric group. Thus, each algebra of finite range allows a matrix representation. Clearly, the inverse statement is true, that is each subalgebra of a complete matrix algebra is a matrix representation of some algebra of finite range.

On the other hand, since we do not have a list of all subalgebras of complete matrix algebras even of small orders, finding such an algebra provides an opportunity to obtain an algebra of finite range and besides we can use the matrix analysis tools for their study. These subalgebras can be used for construction of hypercomplex number systems, which were studied starting from the papers of W.R. Hamilton. A good review of the development of the theory of hypercomplex numbers and the corresponding algebras is given in [4]. The most common

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types of generalized complex numbers are quaternions, octonions (also known as Cayley numbers), dual numbers, split-complex numbers, biquaternions etc. (see, for example, [1,2,5,9,14]). Specific role takes the corresponding algebras in sense of application, in particular, machine learning, digital signal and image processing etc. (see, for example, [4, 12, 14] and references therein).

In Section 1, a subalgebra $\mathbf{M}_n^{\mathsf{ch}}$ is extracted from complete matrix algebra $\mathbf{M}_n(\mathbb{R})$ defined over the field of real numbers $\mathbb{R}$. The carrier of $\mathbf{M}_n^{\mathsf{ch}}$ is a linear matrix space each matrix of which has the same sum of elements in its rows, i.e. it is a generalization of a semi-stochastic matrix. The proposed method enables to construct a new hypercomplex systems. In Section 2, we specify some properties for subalgebra $\mathbf{M}_2^{\mathsf{ch}}$. In Section 3, we construct isomorphic algebra V_3 to $\mathbf{M}_n^{\mathsf{ch}}$, which is an algebra of hypercomplex numbers and we define functions on the set of hypercomplex numbers, and so construct the elements of analysis.

1 Algebras $\mathbf{M}_n^{\mathsf{ch}}$ of finite range

In this section, we will consider a subalgebra of associative complete matrix algebra $\mathbf{M}_n(\mathbb{R})$ of order n over the field of real numbers $\mathbb{R}$.

Here, we propose a generalization of the concept of a semi-stochastic matrix. A square matrix A is said to be *semi-stochastic* if the sum of elements of each row equals 1, the family of all such matrices is given by

$$\left\{ A \in \mathbb{R}^{n\times n} \ : \ \sum_{j=1}^{n} a_{ij} = 1 \ \text{ for every } i = 1, \dots, n \right\}.$$

Semi-stochastic square matrices over the field of real numbers $\mathbb{R}$ were considered in [11], specifically some their characteristics were described.

If a sum of each row of a matrix A is c, then c will be called a *characteristic* of the matrix A and will be denoted by $\mathsf{ch}\, A := c$. In the case, when $\mathsf{ch}\, A = 1$, the matrix A is semi-stochastic.

We will reserve $\mathbf{M}_n^{\mathsf{ch}}(\mathbb{R})$ to denote a set of square matrices of order n, which satisfy the following condition: a matrix $A = \left(a_{ij}\right)_{i,j=\overline{1,n}}$ belongs to $\mathbf{M}_n^{\mathsf{ch}}(\mathbb{R})$ if and only if

$$\sum_{j=1}^{n} a_{1j} = \sum_{j=1}^{n} a_{2j} = \cdots = \sum_{j=1}^{n} a_{nj}.$$

It is clear that under matrix addition and scalar multiplication the set $\mathbf{M}_n^{\mathsf{ch}}(\mathbb{R})$ is a linear space of dimension $n^2 - n + 1$ over field $\mathbb{R}$, besides for all $\alpha, \beta \in \mathbb{R}$ and for all matrices $A, B \in \mathbf{M}_n^{\mathsf{ch}}$ the equality

$$\mathsf{ch}(\alpha A + \beta B) = \alpha \mathsf{ch}(A) + \beta \mathsf{ch}(B) \tag{1}$$

holds. The set $\mathbf{M}_n^{\mathsf{ch}}$ is closed under matrix multiplication. Indeed, if $A, B \in \mathbf{M}_n^{\mathsf{ch}}$, then $AB \in \mathbf{M}_n^{\mathsf{ch}}$. Moreover,

$$\mathsf{ch}(AB) = \mathsf{ch}(A) \cdot \mathsf{ch}(B). \tag{2}$$

According to (1) and (2), we have $\mathsf{ch}(AB - BA) = 0$.

Consequently, *the set $\mathbf{M}_n^{\mathsf{ch}}$ under the basic matrix operations is an algebra of range $n^2 - n + 1$ with the identity matrix I.*

Note that a subset $\mathbf{M}_n^0$, where $\mathbf{M}_n^0 = \{A : A \in \mathbf{M}_n^{\mathsf{ch}}, \mathsf{ch}A = 0\}$, is a subalgebra of algebra $\mathbf{M}_n^{\mathsf{ch}}$ with no identity element, and a subset $\mathbf{M}_n^1$, where $\mathbf{M}_n^1 = \{A : A \in \mathbf{M}_n^{\mathsf{ch}}, \mathsf{ch}A = 1\}$, is a semigroup of semi-stochastic matrices [11].

By the definition, each element of $\mathbf{M}_n^{\mathsf{ch}}$ is a matrix A in a characteristic c, which can be represented as

$$A = cI_n + A_0, \tag{3}$$

where I_n is the identity matrix of order n, matrix A_0 has the form

$$A_0 = \begin{pmatrix} b_1 & a_{12} & \dots & a_{1n} \\ a_{21} & b_2 & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & b_n \end{pmatrix},$$

where $b_1 = -a_{12} - \dots - a_{1n}$, $b_2 = -a_{21} - a_{23} - \dots - a_{2n}, \dots, b_n = -a_{n1} - a_{n2} - \dots - a_{n,n-1}$, and $\mathsf{ch}(A_0) = 0$. Then, taking into account (3), we get $|\lambda I_n - A| = |(\lambda - c)I_n - A_0|$. So we have $|\lambda I_n - A| = 0$, when $\lambda = c$, i.e. $\mathsf{ch}(A)$ is an eigenvalue of A. Therefore, if the matrix A is nonsingular, then $\mathsf{ch}(A) \neq 0$.

Theorem 1. *If a matrix $A \in \mathbf{M}_n^{\mathsf{ch}}$ is nonsingular, then $A^{-1} \in \mathbf{M}_n^{\mathsf{ch}}$. Additionally, the equality*

$$\mathsf{ch}\big(A^{-1}\big) = \mathsf{ch}^{-1}(A)$$

holds.

Proof. Let $A = (a_{ij})_{i,j=\overline{1,n}}$ be a nonsingular matrix from $\mathbf{M}_n^{\mathsf{ch}}$. Suppose that $\mathsf{ch}(A) = c$ and its inverse is $A^{-1} = (\hat{a}_{ij})_{i,j=\overline{1,n}}$. Then it is clear that the entries of kth row of $A^{-1}A = I_n$ for each $k = 1, \dots, n$ satisfy the following conditions:

$$\sum_{i=1}^{n} \hat{a}_{ki} a_{i1} = 0, \ \dots, \ \sum_{i=1}^{n} \hat{a}_{ki} a_{i,k-1} = 0, \ \sum_{i=1}^{n} \hat{a}_{ki} a_{ik} = 1, \ \sum_{i=1}^{n} \hat{a}_{ki} a_{i,k+1} = 0, \ \dots, \ \sum_{i=1}^{n} \hat{a}_{ki} a_{in} = 0.$$

Since $\mathsf{ch}(A) = c$, which means that

$$\sum_{j=1}^{n} a_{ij} = c \quad \text{for every } i = 1, \dots, n,$$

adding these equalities results in

$$c \sum_{j=1}^{n} \hat{a}_{ij} = 1.$$

Hence, a characteristic of the inverse matrix A^{-1} is

$$\mathsf{ch}\big(A^{-1}\big) = \frac{1}{c},$$

because a characteristic of a nonsingular matrix is not zero. This completes the proof. □

Recall that a square matrix A^- is called *semi-inverse* [13] to a matrix A if

$$AA^-A = A, \qquad A^-AA^- = A^-. \tag{4}$$

A square $n \times n$ matrix Π_A is called a *projection matrix* or *projector* [10, p. 69] of a matrix A if

$$\Pi_A^2 = \Pi_A, \qquad A\Pi_A = \Pi_A A = 0_n,$$

where 0_n is a zero matrix of order n. If Π_A is a projection matrix of the matrix A, then $A + \Pi_A$ is not singular [10, p. 69] and A^- is semi-inverse to the matrix A [13], where

$$A^- = (A + \Pi_A)^{-1} - \Pi_A.$$

Note that a projector is the zero matrix for an invertible matrix. The latter equality implies that if a matrix is invertible, then its semi-inverse matrix coincides with its inverse matrix.

Theorem 2. *If $A \in \mathbf{M}_n^{\mathsf{ch}}$, then Π_A, $A^- \in \mathbf{M}_n^{\mathsf{ch}}$. Additionally,*

$$\mathsf{ch}\,(\Pi_A) = 0, \quad \mathsf{ch}\left(A^-\right) = \frac{1}{c}, \qquad \textit{if} \quad c \neq 0,$$
$$\mathsf{ch}\,(\Pi_A) = 1, \quad \mathsf{ch}\left(A^-\right) = 0, \qquad \textit{if} \quad c = 0.$$

Proof. Suppose that $\Pi_A = \left(\pi_{ij}\right)_{i,j=\overline{1,n}}$ is a projection matrix of the matrix $A \in \mathbf{M}_n^{\mathsf{ch}}$. Then the equality $\Pi_A \cdot A = 0_n$ implies that the entries of kth row are

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\pi_{kj}a_{ij} = 0 \qquad \text{or} \qquad c\sum_{j=1}^{n}\pi_{kj} = 0 \qquad \text{for each} \qquad k = 1, \ldots, n.$$

If $c \neq 0$, then $\sum\limits_{j=1}^{n} \pi_{kj} = 0$ for each $k = 1, \ldots, n$. Therefore,

$$\mathsf{ch}(\Pi_A) = 0, \quad \text{then} \quad \mathsf{ch}(A + \Pi_A) = c \quad \text{and} \quad \mathsf{ch}\left(A^-\right) = \mathsf{ch}\left((A + \Pi_A)^{-1} - \Pi_A\right) = \frac{1}{c}.$$

If $c = 0$, then $\sum\limits_{j=1}^{n} \pi_{kj}$ can be considered to be equal to any nonzero real number. In order to provide that the equalities (4) are satisfied, let us suppose $\sum\limits_{j=1}^{n} \pi_{kj} = 1$ for each $k = 1, \ldots, n$. Then

$$\mathsf{ch}\,(A + \Pi_A) = 1, \qquad \mathsf{ch}\left(A^-\right) = \mathsf{ch}\left((A + \Pi_A)^{-1} - \Pi_A\right) = 0,$$

which completes the proof. □

Theorem 1 and Theorem 2 imply that if the matrix $A \in \mathbf{M}_n^{\mathsf{ch}}$ is invertible or semi-invertible, then it generates a cyclic group with the identity element $I_n - \Pi_A$ and its inverse matrix is $A^- = A^{-1}$ (because as we mentioned above if A is nonsingular, then Π_A is the zero matrix).

Using Hilbert-Schmidt norm (Euclidean matrix norm) [6, p. 341], the algebra $\mathbf{M}_n^{\mathsf{ch}}$ is endowed with the norm

$$\|A\| = \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}}.$$

As already mentioned earlier in this article, the matrix A in characteristic c can be represented by (3), consequently, its norm is

$$\|A\| = \left(nc^2 - 2c\sum_{k=1}^{n}\sum_{j\neq k}a_{kj} + 2\sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ki}a_{kj}\right)^{\frac{1}{2}}. \tag{5}$$

Theorem 3. *The norm* (5) *is generated by the scalar product.*

Proof. It is sufficient to show that for each matrices $A, B \in \mathbf{M}_n^{\mathsf{ch}}$ the parallelogram law $\|A+B\|^2 + \|A-B\|^2 = 2\big(\|A\|^2 + \|B\|^2\big)$ holds.

Let $\mathsf{ch}(A) = a$, $\mathsf{ch}(B) = b$. Then

$$\|A+B\|^2 \overset{(5)}{=} n(a+b)^2 - 2(a+b)\sum_{k=1}^{n}\sum_{j\neq k}(a_{kj}+b_{kj}) + 2\sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ki}+b_{ki})(a_{kj}+b_{kj}),$$
$$\|A-B\|^2 \overset{(5)}{=} n(a-b)^2 - 2(a-b)\sum_{k=1}^{n}\sum_{j\neq k}(a_{kj}-b_{kj}) + \sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ki}-b_{ki})(a_{kj}-b_{kj}).$$

Consequently,

$$\|A+B\|^2 + \|A-B\|^2 = 2na^2 + 2nb^2 - 4a\sum_{k=1}^{n}\sum_{j\neq k}a_{kj} - 4b\sum_{k=1}^{n}\sum_{j\neq k}b_{kj}$$
$$+ 4\sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ki}a_{kj} + 4\sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ki}b_{kj} \overset{(5)}{=} 2\left(\|A\|^2 + \|B\|^2\right).$$

□

Theorem 3 implies that *algebra $\mathbf{M}_n^{\mathsf{ch}}$ as a linear space is Euclidean $(n^2 - n + 1)$-space.*

Theorem 4. *If a function f defined on the spectrum of a matrix $A \in \mathbf{M}_n^{\mathsf{ch}}$, in addition, it is m_k times differentiable for each $k = 1, \dots, s$, where m_k is the multiplicity of a zero of the minimal polynomial of the matrix and s is the number of distinct roots of the polynomial, then $f(A) \in \mathbf{M}_n^{\mathsf{ch}}$, in particular*

$$\mathsf{ch}\big(f(A)\big) = f\big(\mathsf{ch}(A)\big). \tag{6}$$

Proof. Let λ_k denote a zero of the minimal polynomial of the matrix $A \in \mathbf{M}_n^{\mathsf{ch}}$ of multiplicity m_k for each $k = 1, \dots, s$.

Suppose that $\psi(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2}\dots(\lambda - \lambda_s)^{m_s}$ is a minimal polynomial of the matrix A. One of the roots of the polynomial is $\mathsf{ch}(A)$, say $\lambda_1 = \mathsf{ch}(A)$. By F.R. Gantmacher (see [3, p. 101–103]), $f(A)$ is defined by

$$f(A) = \sum_{k=1}^{s}\sum_{j=0}^{m_j-1}\alpha_{kj}(A - \lambda_k I_n)^j \psi_k(A),$$

where

$$\psi_k(A) = \frac{\psi(\lambda)}{(\lambda-\lambda_k)^{m_k}}, \qquad \alpha_{kj} = \frac{1}{j!}\left(\frac{f(\lambda)}{\psi_k(\lambda)}\right)^{(j)}_{\lambda=\lambda_k}.$$

In this case,

$$\mathsf{ch}(A - \lambda_1 I_n) = \mathsf{ch}(A) - \lambda_1 = 0, \tag{7}$$

and, so for each $k = 2, \dots, s$, we have

$$\mathsf{ch}\big(\psi_k(A)\big) = 0.$$

Hence,

$$\mathrm{ch}\big(f(A)\big) = \mathrm{ch}\bigg(\sum_{k=2}^{s}\sum_{j=2}^{m_j-1}\alpha_{kj}(A-\lambda_k I_n)^j\psi_k(A)\bigg) = 0.$$

According to (7), we have

$$\mathrm{ch}\bigg(\sum_{j=0}^{m_j-1}\alpha_{1j}(A-\lambda_k I_n)^j\psi_1(A)\bigg) = \mathrm{ch}\big(\alpha_{10}\psi_1(A)\big) = \mathrm{ch}\left(\frac{f(\lambda_1)}{\psi_1(\lambda_1)}\psi_1(A)\right).$$

Since

$$\mathrm{ch}\left(\frac{1}{\lambda_1} - \lambda_k(A-\lambda_k I_n)\right) = 1$$

for each $k = 2,\ldots,s$, the equality (6) holds. □

2 Matrix algebra M_2^{ch}

In this section, we describe a subalgebra M_2^{ch} of 2×2-matrices having a constant characteristic.

The algebra M_2^{ch} is an algebraic structure with a carrier

$$M_2^{\mathrm{ch}} = \left\{\begin{pmatrix} a-b & b \\ c & a-c \end{pmatrix} :\ a,b,c\in\mathbb{R}\right\} \tag{8}$$

and usual operations of addition and multiplication. If we choose the matrices

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad E_1 = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \tag{9}$$

as a basis, then each matrix from M_2^{ch} can be represented in such a way that

$$A = \begin{pmatrix} a-b & b \\ c & a-c \end{pmatrix} = aE_0 + bE_1 + cE_2.$$

It is obvious that $\mathrm{ch}(A) = a$. If $|A| = a(a-b-c) \neq 0$, then

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} a-c & -b \\ -c & a-b \end{pmatrix},$$

otherwise if $|A| = 0$, then we will deal with specific subcases of this situation.

Suppose that $a = 0$, $b+c \neq 0$, then each matrix from M_2^{ch} has the form

$$A = \begin{pmatrix} -b & b \\ c & -c \end{pmatrix} \tag{10}$$

and

$$\Pi_A = \frac{1}{b+c}\begin{pmatrix} c & b \\ c & b \end{pmatrix}, \qquad A^- = \frac{1}{(b+c)^2}\begin{pmatrix} -b & b \\ c & -c \end{pmatrix}. \tag{11}$$

Then according to (11), we have

$$I_2 - \Pi_A = -\frac{1}{b+c}A, \qquad A^n = (-1)^{n-1}(b+c)^{n-1}A,$$
$$\left(A^-\right)^n = \frac{1}{(b+c)^{2n}}A^n = \frac{(-1)^{n-1}}{(b+c)^{n+1}}A, \qquad n = 1,2,\ldots.$$

Therefore,

$$A^n\left(A^-\right)^n = \left(A^-\right)^n A^n = \frac{1}{(b+c)^2}A^2 = -\frac{1}{b+c}A = I_2 - \Pi_A,$$
$$\left(AA^-\right)^n = \left(A^-A\right)^n = (I_2 - \Pi_A)^n = I_2 - \Pi_A.$$

Thus, each matrix of the form (10) generates a cyclic group with the identity element $I_2 - \Pi_A$ and for each natural n the matrix $\left(A^-\right)^n$ is the inverse matrix of A^n.

Next, suppose that $a = b + c$, $b + c \neq 0$, then each matrix from M_2^{ch} has the form

$$A = \begin{pmatrix} c & b \\ c & b \end{pmatrix} \tag{12}$$

and

$$\Pi_A = \frac{1}{b+c}\begin{pmatrix} b & -b \\ -c & c \end{pmatrix}, \qquad A^- = \frac{1}{(b+c)^2}\begin{pmatrix} c & b \\ c & b \end{pmatrix}. \tag{13}$$

According to (13), we have

$$I_2 - \Pi_A = \frac{1}{b+c}A, \qquad A^n = (b+c)^{n-1}A, \qquad \left(A^-\right)^n = \frac{1}{(b+c)^{2n+1}}A, \qquad n = 1, 2, \dots .$$

Therefore,

$$A^n\left(A^-\right)^n = \left(A^-\right)^n A^n = \frac{1}{b+c}A = I_2 - \Pi_A,$$
$$\left(AA^-\right)^n = \left(A^-A\right)^n = (I_2 - \Pi_A)^n = I_2 - \Pi_A.$$

Consequently, each matrix of the form (12) under the condition $b + c \neq 0$ also generates a cyclic group with the identity element $I_2 - \Pi_A$ and inverse matrix $\left(A^-\right)^n$ of A^n.

Finally, if $a = 0$, $b + c = 0$, i.e. any matrix from M_2^{ch} has the form

$$A = \begin{pmatrix} -b & b \\ -b & b \end{pmatrix},$$

then there does not exist a matrix $X \in M_2^{\mathrm{ch}}$ such that it is a solution of the equations $AXA = A$, $XAX = X$.

Another property of elements of algebra M_2^{ch} is in the following statement.

Theorem 5. *Eigenvalues of a product of matrices from M_2^{ch} are equal to a product of the corresponding eigenvalues of these factors.*

Proof. Suppose that

$$A_1 = \begin{pmatrix} a_1 - b_1 & b_1 \\ c_1 & a_1 - c_1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} a_2 - b_2 & b_2 \\ c_2 & a_2 - c_2 \end{pmatrix}$$

are matrices from M_2^{ch}. Then $\lambda_1^{(1)} = a_1$, $\lambda_2^{(1)} = a_1 - b_1 - c_1$ and $\lambda_1^{(2)} = a_2$, $\lambda_2^{(2)} = a_2 - b_2 - c_2$ are eigenvalues of A_1 and A_2, respectively. Since the equation $|\lambda I_2 - A_1A_2| = 0$ can be written in the form

$$(\lambda - a_1a_2)^2 + (\lambda - a_1a_2)(b_1a_2 + a_1b_2 - b_1b_2 - b_1c_2 + c_1a_2 + a_1c_2 - c_1b_2 - c_1c_2) = 0,$$

the eigenvalues of matrix A_1A_2 are

$$\lambda_1 = a_1a_2, \qquad \lambda_2 = a_1a_2 - b_1a_2 - a_1b_2 + b_1b_2 + b_1c_2 - c_1a_2 - a_1c_2 + c_1b_2 + c_1c_2$$
$$= (a_1 - b_1 - c_1)(a_2 - b_2 - c_2).$$

□

We endow algebra M_2^{ch} with the norm

$$\|A\| = \left\| \begin{pmatrix} a-b & b \\ c & a-c \end{pmatrix} \right\| = \sqrt{2}\left(a^2+b^2+c^2-ab-ac\right)^{\frac{1}{2}}, \tag{14}$$

which is generated by the scalar product

$$\begin{aligned}(A_1, A_2) := \left(\begin{pmatrix} a_1-b_1 & b_1 \\ c_1 & a_1-c_1 \end{pmatrix}, \begin{pmatrix} a_2-b_2 & b_2 \\ c_2 & a_2-c_2 \end{pmatrix} \right) = \\ = b_1b_2 + c_1c_2 + (a_1-b_1)(a_2-b_2) + (a_1-c_1)(a_2-c_2).\end{aligned} \tag{15}$$

By view of (14) and (15), we have

$$\|E_0\| = \|E_1\| = \|E_2\| = \sqrt{2}, \qquad (E_0, E_1) = (E_0, E_2) = -1, \qquad (E_1, E_2) = 0$$

for basis (9). Consequently, the algebra M_2^{ch} as a linear space is a Euclidean one with the basis vectors E_0, E_1, E_2 of the length $\sqrt{2}$, besides vectors E_1, E_2 are orthogonal, and the angles formed by E_0, E_1 and E_0, E_2 are $\dfrac{2\pi}{3}$.

3 Constructing an algebra of hypercomplex numbers

In this section, we consider a method for constructing a noncommutative algebra of hypercomplex numbers of range 3 by means of the considered algebra M_2^{ch} and we construct the functions of hypercomplex variable on this algebra.

Let V_3 be a linear space of dimension 3 over the field $\mathbb{R}$ and let $\bar{e}_0$, $\bar{e}_1$, $\bar{e}_2$ be its basis, i.e.

$$V_3 = \{a\bar{e}_0 + b\bar{e}_1 + c\bar{e}_2 : \ a, b, c \in \mathbb{R}\}.$$

Let us define the operation of multiplication in V_3 by the following Cayley table

$\cdot$	$\bar{e}_0$	$\bar{e}_1$	$\bar{e}_2$
$\bar{e}_0$	$\bar{e}_0$	$\bar{e}_1$	$\bar{e}_2$
$\bar{e}_1$	$\bar{e}_1$	$-\bar{e}_1$	$-\bar{e}_1$
$\bar{e}_2$	$\bar{e}_2$	$-\bar{e}_2$	$-\bar{e}_2$

One can easily verify that V_3 is an algebra of hypercomplex numbers of range 3 and the mapping $\varphi : M_2^{\mathrm{ch}} \to V_3$, which defines by the law

$$A = \begin{pmatrix} a-b & b \\ c & a-c \end{pmatrix} \longmapsto (a, b, c) := a\bar{e}_0 + b\bar{e}_1 + c\bar{e}_2,$$

where $\varphi(E_0) = (1,0,0) = \bar{e}_0$, $\varphi(E_1) = (0,1,0) = \bar{e}_1$, $\varphi(E_2) = (0,0,1) = \bar{e}_2$, is one-to-one correspondence. Moreover, the equalities

$$\varphi(\alpha A_1 + \beta A_2) = \alpha\varphi(A_1) + \beta\varphi(A_2), \qquad \varphi(A_1A_2) = \varphi(A_1)\varphi(A_2)$$

hold for all $\alpha, \beta \in \mathbb{R}$ and $A_1, A_2 \in M_2^{\mathrm{ch}}$. It means that *algebras M_2^{ch} and V_3 are isomorphic.*

Consequently, the algebra M_2^{ch} is a matrix representation of algebra of hypercomplex numbers V_3 in which the subset $\{(a, 0, 0) : \ a \in \mathbb{R}\}$ is a field being isomorphic to the field of real numbers $\mathbb{R}$, i.e. the field $\mathbb{R}$ is monomorphically embedded into this subset.

Using such matrix representation, each hypercomplex number $\mathbf{v} = a\bar{e}_0 + b\bar{e}_1 + c\bar{e}_2$ or its coordinate representation (a, b, c) are associated with the following characteristics:

– characteristic of a hypercomplex number (i.e. its real part) $\mathrm{ch}\mathbf{v} = a$;

– determinant of a hypercomplex number $\det\mathbf{v} = a(a - b - c)$;

– norm of a hypercomplex number, which is defined by $\|\mathbf{v}\| := \sqrt{2}\left(a^2 + b^2 + c^2 - ab - ac\right)^{\frac{1}{2}}$.

Let us define a semi-inverse (inverse) element in V_3. If $a \neq 0$ and $a \neq b + c$, then there exists an element $\mathbf{v}^{-1} \in V_3$ such that $\mathbf{v}\mathbf{v}^{-1} = \mathbf{v}^{-1}\mathbf{v} = \bar{e}_0$. Namely, if $\mathbf{v} = (a, b, c)$, then

$$\mathbf{v}^{-1} = (a, b, c)^{-1} = \frac{1}{\det\mathbf{v}}(a - b - c, -b, -c).$$

For a hypercomplex number $\mathbf{v}$, there exists a semi-inverse $\mathbf{v}^{-}$, which has the form

$$\mathbf{v}^{-} = (a, b, c)^{-} = \begin{cases} \dfrac{1}{(b+c)^2}(0, b, c), & \text{if } a = 0,\ b + c \neq 0, \\ \dfrac{1}{a^2}(a, b, c), & \text{if } a - b - c = 0,\ b + c \neq 0. \end{cases}$$

Note that the hypercomplex numbers of the form $\mathbf{v} = (0, b, -b)$ are zero divisors.

According to isomorphism of algebras V_3 and M_2^{ch} and Theorem 4, a one-to-one correspondence of V_3 onto itself (hypercomplex functions) may be constructed.

Let f be a real function with a domain $D(f)$ and it is differentiable in its domain.

Theorem 6. *To a function $f : D(f) \to \mathbb{R}$, there corresponds a mapping $\bar{f}$ of V_3 onto itself, which is defined on the set of hypercomplex numbers*

$$D(\bar{f}) = \{(a, b, c) :\ a,\ a - b - c \in D(f)\},$$

and the function $\bar{f}$ is defined by

$$\bar{f}\big((a, b, c)\big) = \Big(f(a), \frac{b}{b+c}\big(f(a) - f(a - b - c)\big), \frac{c}{b+c}\big(f(a) - f(a - b - c)\big)\Big), \tag{16}$$

if $b + c \neq 0$; and by

$$\bar{f}\big((a, b, -b)\big) = \big(f(a), bf'(a), -bf'(a)\big), \tag{17}$$

if $b + c = 0$.

Proof. Suppose that (a, b, c) is a hypercomplex number such that a, $a - b - c \in D(f)$ and $b + c \neq 0$. Then its matrix representation in algebra M_2^{ch} is the matrix

$$A = \begin{pmatrix} a - b & b \\ c & a - c \end{pmatrix},$$

which has different eigenvalues $\lambda_1 = a$, $\lambda_2 = a - b - c$. Therefore according to Theorem 4, we get

$$\begin{aligned} f(A) &= \frac{f(\lambda_1)}{\lambda_1 - \lambda_2}(A - \lambda_2 I_2) + \frac{f(\lambda_2)}{\lambda_2 - \lambda_1}(A - \lambda_1 I_2) \\ &= \frac{1}{b+c}\begin{pmatrix} cf(a) + bf(a - b - c) & bf(a) - bf(a - b - c) \\ cf(a) - cf(a - b - c) & bf(a) + cf(a - b - c) \end{pmatrix}. \end{aligned}$$

Note that the latter matrix is a representation of hypercomplex number (16).

If $b+c=0$, i.e. the hypercomplex number has the form $(a,b,-b)$, then its matrix representation in algebra M_2^{ch} is the matrix

$$A=\begin{pmatrix} a-b & b \\ -b & a+b \end{pmatrix},$$

which has multiple eigenvalue $\lambda=a$. Therefore according to Theorem 4, we get

$$f(A)=f(a)I_2+f'(a)(A-aI_2)=\begin{pmatrix} f(a)-bf'(a) & bf'(a) \\ -bf'(a) & f(a)+bf'(a) \end{pmatrix}.$$

The obtained matrix is a representation of hypercomplex number (17). □

For example,

$$\sqrt{(3,2,-1)} \overset{(16)}{=} \left(\sqrt{3}, 2\sqrt{3}-2\sqrt{2}, -\sqrt{3}+\sqrt{2}\right),$$

$$\sqrt{(3,2,-2)} \overset{(17)}{=} \left(\sqrt{3}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right).$$

By Theorem 6, the basic elementary hypercomplex functions (functions on V_3) can be defined in such a way

$$e^{\mathbf{v}}=e^{(a,b,c)}=\begin{cases} \left(e^a, \dfrac{b}{b+c}\left(e^a-e^{a-b-c}\right), \dfrac{c}{b+c}\left(e^a-e^{a-b-c}\right)\right), & \text{if } b+c\neq 0, \\ \left(e^a, be^a, -be^a\right), & \text{if } b+c=0; \end{cases}$$

$$\cos\mathbf{v}=\cos(a,b,c)$$
$$=\begin{cases} \left(\cos a, \dfrac{b}{b+c}\big(\cos a-\cos(a,b,c)\big), \dfrac{c}{b+c}\big(\cos a-\cos(a,b,c)\big)\right), & \text{if } b+c\neq 0, \\ \left(\cos a, -b\sin a, b\sin a\right), & \text{if } b+c=0; \end{cases}$$

$$\sin\mathbf{v}=\sin(a,b,c)$$
$$=\begin{cases} \left(\sin a, \dfrac{b}{b+c}\big(\sin a-\sin(a,b,c)\big), \dfrac{c}{b+c}\big(\sin a-\sin(a,b,c)\big)\right), & \text{if } b+c\neq 0, \\ \left(\sin a, b\cos a, -b\cos a\right), & \text{if } b+c=0. \end{cases}$$

For all $(a,b,c)\in V_3$, which satisfy the conditions $a>-1$, $a-b-c>-1$, the following function is defined

$$\ln(e_0+\mathbf{v})=\ln((1+a,b,c))$$
$$=\begin{cases} \left(\ln(1+a), \dfrac{b}{b+c}\ln\dfrac{1+a}{1+a-b-c}, \dfrac{c}{b+c}\ln\dfrac{1+a}{1+a-b-c}\right), & \text{if } b+c\neq 0, \\ \left(\ln(1+a), \dfrac{b}{1+a}, -\dfrac{b}{1+a}\right), & \text{if } b+c=0. \end{cases}$$

Since function $f(\lambda)=\lambda^n$ is defined on $\mathbb{R}$ for each $n\in\mathbb{N}$, for every $(a,b,c)\in V_3$ we can define the following function

$$(a,b,c)^n=\begin{cases} \left(a^n, \dfrac{b}{b+c}\left(a^n-(a-b-c)^n\right), \dfrac{c}{b+c}\left(a^n-(a-b-c)^n\right)\right), & \text{if } b+c\neq 0, \\ \left(a^n, nba^{n-1}, -nba^{n-1}\right), & \text{if } b+c=0. \end{cases} \tag{18}$$

Algebra of hypercomplex numbers V_3 is endowed with the norm

$$\|(a,b,c)\| = \sqrt{2}\big(a^2+b^2+c^2-ab-ac\big)^{\frac{1}{2}}.$$

Therefore, the concept of convergence of sequences and series can be defined on this algebra in a natural way, besides it is clear that the series

$$\sum_{n=1}^{\infty}\mathbf{v}_n = \sum_{n=1}^{\infty}(a_n,b_n,c_n)$$

is convergent if and only if the series $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} b_n$, $\sum\limits_{n=1}^{\infty} c_n$ are convergent.

Theorem 7. *If a real function $f(\lambda)$ is given by a convergent power series on an interval, that is*

$$f(\lambda) = \sum_{n=1}^{\infty} a_n\lambda^n,$$

and a, $a-b-c$ belongs to the interval of convergence, then the series of hyperpercomplex numbers $\sum\limits_{n=1}^{\infty} a_n(a,b,c)^n$ is convergent and its sum is $\bar{f}\big((a,b,c)\big)$.

Proof. According to (18) and Theorem 6, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n(a,b,c)^n &= \sum_{n=1}^{\infty} a_n\left(a^n, \frac{b}{b+c}\big(a^n-(a-b-c)^n\big), \frac{c}{b+c}\big(a^n-(a-b-c)^n\big)\right)\\
&= \left(\sum_{n=1}^{\infty} a_na^n, \frac{b}{b+c}\left(\sum_{n=1}^{\infty} a_na^n - \sum_{n=1}^{\infty} a_n(a-b-c)^n\right), \frac{c}{b+c}\left(\sum_{n=1}^{\infty} a_na^n - \sum_{n=1}^{\infty} a_n(a-b-c)^n\right)\right)\\
&= \left(f(a), \frac{b}{b+c}\big(f(a)-f(a-b-c)\big), \frac{c}{b+c}\big(f(a)-f(a-b-c)\big)\right) = \bar{f}\big((a,b,c)\big),
\end{aligned}$$

if $b+c\neq 0$; and

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n(a,b,-b)^n = \sum_{n=1}^{\infty} a_n\left(a^n, nba^{n-1}, -nba^{n-1}\right) &= \left(\sum_{n=1}^{\infty} a_na^n, b\sum_{n=1}^{\infty} na_na^{n-1}, -b\sum_{n=1}^{\infty} na_na^{n-1}\right)\\
&= \big(f(a), bf'(a), -bf'(a)\big) = \bar{f}\big((a,b,c)\big),
\end{aligned}$$

if $b+c=0$. □

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Вотякова Л.А., Фриз І.В. *Про побудову алгебр скінченного рангу* // Карпатські матем. публ. — 2025. — Т.17, №2. — С. 754–765.

В роботі з повної матричної алгебри виділяється підалгебра, елементами якої є квадратні матриці з дійсними елементами, сума рядків яких є однаковою. За допомогою класичних методів теорії матриць досліджуються властивості побудованої матричної алгебри. Ця алгебра наділяється нормою, що дає можливість будувати елементи аналізу в ній, використовуючи методи матричного аналізу. Будується новий клас алгебр скінченного рангу, а саме алгебра гіперкомплексних чисел, яка є ізоморфною відповідній матричній алгебрі. Таким чином, одержані для матриць результати переносяться на елементи ізоморфної алгебри скінченного рангу, тобто на гіперкомплексні числа. Це дозволило побудувати функції гіперкомплексної змінної.

Ключові слова і фрази: алгебра скінченного рангу, гіперкомплексне число, власний проектор, матрична норма, функція гіперкомплексної змінної.