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Explicit solutions for the Poisson problem in a ball for the logarithmic Laplacian

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In this note, we introduce explicit formulas for the solution of the Poisson problem in a ball for the logarithmic Laplacian by means of semigroup theory and the Fourier transform. In particular, the solution for such problem is closely related to Volterra functions, which arise, for instance, in some convolution-type integral equations with logarithmic kernels.

Key words and phrases: logarithmic Laplacian, fundamental solution, Volterra function.

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1 Introduction

In the last years nonlocal problems have attracted great attention, particularly, those driven by the well-known fractional Laplace operator $(-\Delta)^s$, an operator of order $2s$. Recently, nonlocal problems driven by a zero order kernel have been received increasing interest [4,6,11,21,22]. Of particular importance among these zero order operators is the so-called logarithmic Laplacian which arises, for instance, as a first order expansion of the fractional Laplacian $(-\Delta)^s$ as $s \to 0^+$. The logarithmic Laplacian, denoted in what follows by L_Δ, is a singular integral operator with Fourier symbol $2\ln|\xi|$. Up to our knowledge, this operator was introduced by H. Chen and T. Weth (see [6]), who derived the following pointwise integral representation

$$L_\Delta u(x) = c_N \int_{\mathbb{R}^N} \frac{u(x)\chi_{B_1(x)}(y) - u(y)}{|x-y|^N} dy + \rho_N\, u(x),$$

for $u \in C_c^\beta(\mathbb{R}^N)$ for some $\beta > 0$, being

$$c_N = \pi^{-\frac{N}{2}} \Gamma\left(\frac{N}{2}\right) \qquad \text{and} \qquad \rho_N = 2\ln(2) + \psi\left(\frac{N}{2}\right) - \gamma,$$

where γ denotes the Euler-Mascheroni constant and $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is the di-Gamma function. The former representation formula is obtained by computing the limit

$$\frac{\partial}{\partial s}\bigg|_{s=0} [(-\Delta)^s u] = \lim_{s\to 0^+} \frac{(-\Delta)^s u - u}{s}.$$

On the other hand, since the Fourier transform is a continuous map from $L^2(\mathbb{R}^N)$ to itself, we have

$$\widehat{L_\Delta u} = \lim_{s\to 0^+} \frac{\widehat{(-\Delta)^s u} - \hat{u}}{s} = \left(\lim_{s\to 0^+} \frac{|\xi|^{2s} - 1}{s}\right) \hat{u} = 2\ln(\xi)\hat{u}.$$

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Then, the operator L_Δ drives the first order asymptotic for the fractional Laplacian $(-\Delta)^s$ as $s \to 0^+$, namely, for $u \in C_c^2(\mathbb{R}^N)$ we have (cf. [6])

$$(-\Delta)^s u(x) = u(x) + sL_\Delta u(x) + o(s) \quad \text{as } s \to 0^+.$$

In terms of the spectrum $\{\lambda_{k,s}\}_{k\in\mathbb{N}}$ of the fractional Laplacian $(-\Delta)^s$, we have (cf. [13])

$$\lambda_{k,s} = 1 + s\lambda_{k,L_\Delta} + o(s) \quad \text{as } s \to 0^+,$$

where $\{\lambda_{k,L_\Delta}\}_{k\in\mathbb{N}}$ denotes the set of eigenvalues of L_Δ. Spectral properties, such as lower bounds for the first Dirichlet eigenvalue of L_Δ, are obtained in [16], while upper and lower bounds for the sum of the first k eigenvalues of L_Δ are obtained in [5]. We refer to [18] for other asymptotic phenomena in nonlinear problems, where the logarithmic operator L_Δ appears. The operator L_Δ also enjoys desirable properties such as the Maximum Principle or Faber-Krahn type inequalities (see [6]).

Among its key properties, the operator L_Δ allows to derive (cf. [15]) pointwise monotonicity properties of the solution map $s \to v(\cdot, s)$ and, as a consequence, explicit bounds for the corresponding Green operator norm, for the solutions $v(\cdot, s)$ of the Poisson problem

$$\begin{cases} (-\Delta)^s v = f & \text{in } \Omega, \\ v = 0 & \text{on } \Omega^c. \end{cases} \tag{1}$$

We focus now in the particular case $\Omega = B_r(x_0)$ and $f(x) = 1$. It is well known that in this case the solution $v(\cdot, s)$ to (1) is given by

$$v(x,s) = \gamma_{N,s}(r^2 - |x - x_0|^2)_+^s \qquad \text{with} \qquad \gamma_{N,s} = \frac{\Gamma\left(\frac{N}{2}\right)}{2^{2s}\Gamma(1+s)\Gamma\left(\frac{N}{2}+s\right)}. \tag{2}$$

Moreover, this formula holds for all $s > 0$ (see [1, Corollary 3.3]). Using (2), we derive next an explicit solution to the Poisson problem in a ball for the logarithmic Laplacian, namely

$$\begin{cases} L_\Delta u = 1 & \text{in } B_r(x_0), \\ u = 0 & \text{on } \mathbb{R}^N \backslash B_r(x_0). \end{cases} \tag{3}$$

To that end, we next introduce what are known as Volterra functions.

2 Volterra functions

The Volterra functions, which receive its name from Vito Volterra [20], arise as solutions of integral equations of convolution type with logarithmic kernels. These functions were deeply studied by S. Colombo (see [7–10]) in connection with the symbolic calculus and the Laplace transform. We refer to the extensive monographs by A. Apelblat [2, 3], where the connection of the Volterra functions with some elementary and special functions as well as their relation with functional transforms as Laplace, Mellin, Stieltjes, Hankel, among other transformations are also presented.

The Volterra functions are defined by the improper integrals

$$\begin{aligned} \nu(x) &= \int_0^\infty \frac{x^t}{\Gamma(t+1)} dt, \\ \nu(x,\alpha) &= \int_0^\infty \frac{x^{t+\alpha}}{\Gamma(t+\alpha+1)} dt, \\ \mu(x,\beta,\alpha) &= \int_0^\infty \frac{x^{t+\alpha} t^\beta}{\Gamma(t+1)\Gamma(\beta+1)} dt, \quad \Re(\beta) > -1. \end{aligned}$$

The Laplace transform of the Volterra functions is given in terms of the logarithm function

$$\mathcal{L}\{\mu(x,\beta,\alpha)\}(s) = \frac{1}{s^{\alpha+1}(\ln(s))^{\beta+1}}, \qquad \Re(\alpha) > -1,\ \Re(\beta) > -1,\ \Re(s) > 1.$$

By its very definition, it is easy to see that

$$\frac{d^n}{dx^n}\mu(x,\beta,\alpha) = \mu(x,\beta,\alpha-n) \quad \text{for } n = 1,2,3,\dots.$$

In particular, for $\beta = 0$, we have

$$\frac{d^n}{dx^n}\nu(x) = \nu(x,-n) \qquad \text{and} \qquad \frac{d^n}{dx^n}\nu(x,\alpha) = \nu(x,\alpha-n). \tag{4}$$

The next integral representation of the function $\nu(x)$ was provided by S. Ramanujan [14, XI]

$$\nu(x) = e^x - \int_0^\infty \frac{e^{-xt}}{t(\pi^2 + (\ln(t))^2)}dt.$$

The above integral also appears in problems related with heat conduction [17], neutron transport theory [12] or electron slow down theory [19].

3 Explicit solutions

Let $\chi_B(x)$ be the indicator function of the ball $B_r(x_0)$, $r > 0$, and consider the parabolic problem

$$\begin{cases} v_t(x,t) + L_\Delta v(x,t) = 0 & \text{in } \mathbb{R}^N \times \{t > 0\}, \\ v(x,0) = \chi_B(x) & \text{on } \mathbb{R}^N \times \{t = 0\}. \end{cases}$$

By applying the Fourier transform and the definition of the operator L_Δ, we get

$$\hat{v}(\xi,t) = \widehat{e^{-tL_\Delta}}\hat{\chi}_B(\xi) = e^{-2t\ln|\xi|}\hat{\chi}_B(\xi) = |\xi|^{-2t}\hat{\chi}_B(\xi) = \widehat{(-\Delta)^{-t}}\hat{\chi}_B(\xi),$$

that is $v(x,t) = (-\Delta)^{-t}\chi_B(x)$.

Equivalently, for each $t > 0$, the function $v(x,t)$ solves the problem $(-\Delta)^t v(x,t) = \chi_B(x)$. Next, we consider

$$u(x) := \int_0^\infty v(x,t)dt = \int_0^\infty (-\Delta)^{-t}\chi_B(x)dt = L_\Delta^{-1}\chi_B(x).$$

Thus, the function $u(x)$ solves $L_\Delta u(x) = \chi_B(x)$. Following this construction for the problem (1) in $B_r(x_0)$ with $f \equiv 1$, we have

$$v(x,t) = \frac{2^{-2t}\Gamma\left(\frac{N}{2}\right)}{\Gamma(1+t)\Gamma\left(\frac{N}{2}+t\right)}(r^2 - |x-x_0|^2)_+^t$$

and let us set

$$\Lambda(z;N) = \int_0^\infty \frac{2^{-2t}\Gamma\left(\frac{N}{2}\right)}{\Gamma(1+t)\Gamma\left(\frac{N}{2}+t\right)}z^t dt, \tag{5}$$

so, by the construction given above, the solution to problem (3) is given by the function

$$u(x) = \Lambda((r^2 - |x-x_0|^2)_+; N).$$

Next let us discuss some particular cases. For $N = 1$ we have

$$\Lambda(z;1) = \int_0^\infty \frac{2^{-2t}\Gamma\left(\frac{1}{2}\right)}{\Gamma(1+t)\Gamma\left(\frac{1}{2}+t\right)}z^t dt. \tag{6}$$

Using the duplication formula for the function Γ, we can write

$$\frac{2^{-2t}\Gamma\left(\frac{1}{2}\right)}{\Gamma(1+t)\Gamma\left(\frac{1}{2}+t\right)}z^t = \frac{1}{\Gamma(1+2t)}(\sqrt{z})^{2t}$$

so that

$$\Lambda(z;1) = \frac{1}{2}\nu(\sqrt{z}), \tag{7}$$

and thus, the solution to the one-dimensional case of problem (3) is given by

$$u(x) = \frac{1}{2}\nu(\sqrt{r^2 - |x-x_0|^2}).$$

On the other hand, let us write (6) as

$$\Lambda(z;1) = \int_0^\infty \frac{z^t}{\Gamma(1+t)} \frac{\Gamma\left(\frac{1}{2}\right)\left(\frac{1}{2^2}\right)^t}{\Gamma\left(\frac{1}{2}+t\right)}dt. \tag{8}$$

Using that (cf. [2, (40.2)])

$$\mathcal{L}\left\{\int_0^\infty \frac{z^t}{\Gamma(1+t)}f(t)dt\right\}(s) = \frac{1}{s}\mathcal{L}\{f\}(\ln(s)), \tag{9}$$

together with (cf. [2, (24.4)])

$$\mathcal{L}\left\{\frac{a^{x+\alpha}}{\Gamma(x+\alpha+1)}\right\} = e^{\alpha s}\nu(ae^{-s},\alpha), \tag{10}$$

the Laplace transform of (8) is given by

$$\mathcal{L}\left\{\Lambda(z;1)\right\}(s) = \frac{1}{s}\mathcal{L}\left\{\frac{\Gamma\left(\frac{1}{2}\right)\left(\frac{1}{2^2}\right)^x}{\Gamma\left(\frac{1}{2}+x\right)}\right\}(\ln(s)) = \frac{\sqrt{\pi}}{2}\frac{1}{s^{1+\frac{1}{2}}}\nu\left(\frac{1}{4s},-\frac{1}{2}\right).$$

So that, from (7), we conclude

$$\mathcal{L}\{\nu(\sqrt{z})\}(s) = \frac{\sqrt{\pi}}{s^{\frac{3}{2}}}\nu\left(\frac{1}{4s},-\frac{1}{2}\right),$$

which is a particular case of the identity (cf. [2, (70.4)])

$$\mathcal{L}\{\nu(a\sqrt{z},\alpha)\}(s) = \frac{a\sqrt{\pi}}{s^{\frac{3}{2}}}\nu\left(\frac{a^2}{4s},\frac{\alpha-1}{2}\right). \tag{11}$$

Next we consider the case $N \geq 2$. Let us write (5) as

$$\Lambda(z;N) = \int_0^\infty \frac{z^t}{\Gamma(1+t)}\frac{2^{-2t}\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N}{2}+t\right)}dt,$$

so that, because of (9), (10), we have

$$\mathcal{L}\{\Lambda(z;N)\}(s) = 4^{\frac{N}{2}-1}\Gamma\left(\frac{N}{2}\right)s^{\frac{N}{2}-2}\nu\left(\frac{1}{4s},\frac{N}{2}-1\right).$$

Then, by (11), we obtain

$$\mathcal{L}\{\Lambda(z;N)\}(s) = 2^{N-2}\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}s^{\frac{N-1}{2}}\mathcal{L}\{\nu(\sqrt{z},N-1)\}(s).$$

Thus, by the properties of the Laplace transform, we conclude

$$\Lambda(z;N) = 2^{N-2}\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{d}{dz}\right)^{\frac{N-1}{2}} \nu(\sqrt{z}, N-1).$$

For odd dimensions $N = 2k+1$ the above formula gives $\Lambda(z;N)$ as a weighted combination of Volterra functions $\nu(z,\alpha)$ by means of the identity (4). For instance,

$$\Lambda(z;5) = \frac{2}{3}\left(\frac{1}{(\sqrt{t})^2}\nu(\sqrt{z},2) - \frac{1}{(\sqrt{t})^3}\nu(\sqrt{z},3)\right).$$

In general, we have

$$\Lambda(z;N) = \sum_{j=\frac{N-1}{2}}^{N-2} c_j \frac{1}{(\sqrt{z})^j}\nu(\sqrt{z},j), \quad \text{with } c_j \in \mathbb{Q}.$$

For even dimensions $N = 2k$, we can write

$$\left(\frac{d}{dz}\right)^{\frac{N-1}{2}} = \left(\frac{d}{dz}\right)^{\frac{N}{2}} I^{\frac{1}{2}},$$

where I^{κ} denotes the Riemann-Liouville integral of order $\kappa > 0$, namely

$$I^{\kappa} f(x) = \frac{1}{\Gamma(\kappa)}\int_0^x f(t)(x-t)^{\kappa-1}dt.$$

Recall that $\mathcal{L}\{I^{\kappa} f\}(s) = s^{-\kappa}\mathcal{L}\{f\}(s)$. Thus, for even dimensions we have

$$\Lambda(z;N) = 2^{N-2}\Gamma\left(\frac{N}{2}\right)\left(\frac{d}{dz}\right)^{\frac{N}{2}} I^{\frac{1}{2}}[\nu(\sqrt{z},N-1)],$$

that is,

$$\Lambda(z;N) = 2^{N-2}\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{d}{dz}\right)^{\frac{N}{2}} \int_0^z \frac{\nu(\sqrt{t},N-1)}{\sqrt{z-t}}dt.$$

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Ортега А. *Явні розв’язки задачі Пуассона в кулі для логарифмічного лапласіана* // Карпатські матем. публ. — 2026. — Т.18, №1. — С. 111–116.

У цій замітці ми отримуємо явні формули для розв’язку задачі Пуассона в кулі для логарифмічного лапласіана за допомогою теорії напівгруп та перетворення Фур’є. Зокрема, розв’язок цієї задачі тісно пов’язаний із функціями Вольтерри, які виникають, наприклад, у деяких інтегральних рівняннях згорткового типу з логарифмічними ядрами.

Ключові слова і фрази: логарифмічний лапласіан, фундаментальний розв’язок, функція Вольтерри.