ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2025, **17** (2), 406–415
doi:10.15330/cmp.17.2.406-415

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2025, Т.17, №2, С.406–415

On the Horadam sequence of order three

Amrouche S.[1], Belbachir H.[1], Cesarano C.[2,✉], Ricci P.E.[2]

In this paper, we define a generalized Horadam triangle in which the sum of elements located along the diagonal lines corresponds to the terms of generalized Horadam sequence of order three. We establish a relation between generalized Horadam triangle and the generalized Delannoy triangle. Additionally, we define the q-analogue of the generalized Tribonacci sequence and generalized Horadam sequence of order three.

Key words and phrases: Horadam sequence, q-analogue, recurrence relation.

[1] University of Science and Technology Houari Boumediene, P.B. 32, El Alia, 16111 Bab Ezzouar, Algeria
[2] International Telematic University Uninettuno, Corso Vittorio Emanuele II, 39, 00186 Roma, Italy
✉ Corresponding author
E-mail: samrouche@usthb.dz, saidamrouchee@gmail.com (Amrouche S.), hbelbachir@usthb.dz, hacenebelbachir@gmail.com (Belbachir H.), c.cesarano@uninettunouniversity.net (Cesarano C.), paoloemilioricci@gmail.com (Ricci P.E.)

Introduction

The Fibonacci sequence $(F_n)_{n\geq 0}$ is a well-known as a mathematical sequence that satisfies the recurrence relation $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, with the initial conditions $F_0 = 0, F_1 = 1$. The terms of the Fibonacci sequence can be obtained by summing the elements located along the diagonal lines of Pascal's triangle. Consequently, the precise representation of this relationship is

$$F_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}, \quad n \geq 0.$$

Numerous generalizations of the Fibonacci sequence have been explored, one well-known is the Horadam sequence $(H_n)_{n\geq 0}$, which satisfies the following recurrence relation $H_n = \alpha_1 H_{n-1} + \alpha_2 H_{n-2}$, $n \geq 2$, where $\alpha_1, \alpha_2 \in \mathbb{Z}$, with the initial conditions $H_0 = a$, $H_1 = b$, where $a, b \in \mathbb{Z}$ within $b \neq 0$. The Horadam sequence is widely used and has applications in various fields of mathematical sciences (see [9, 10] for more details). Another extension of the Fibonacci sequence is the generalized Tribonacci sequence $(\mathcal{T}_n)_{n\geq 0}$, which satisfies the recurrence relation $\mathcal{T}_{n+1} = \alpha_1 \mathcal{T}_n + \alpha_2 \mathcal{T}_{n-1} + \alpha_3 \mathcal{T}_{n-2}$, $n \geq 2$, where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}$, with the initial conditions $\mathcal{T}_0 = 0$, $\mathcal{T}_1 = 1$, $\mathcal{T}_2 = \alpha_1$. The generalized Tribonacci numbers can be obtained by summing the elements located along the diagonal lines of the generalized Delannoy triangle, namely

$$\mathcal{T}_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \mathcal{D}(n-k, k), \quad n \geq 0. \tag{1}$$

УДК 511.176
2020 *Mathematics Subject Classification:* 05A10, 11B39, 11B37.

For $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}$ the coefficients of the generalized Delannoy triangle $\{\mathcal{D}(n,k)\}_{n,k\geq 0}$ satisfy the following recurrence relation (see [8])

$$\mathcal{D}(n,k) = \alpha_1\mathcal{D}(n-1,k) + \alpha_2\mathcal{D}(n-1,k-1) + \alpha_3\mathcal{D}(n-2,k-1)$$

with $\mathcal{D}(0,0) = 1$. For $n < 0, k < 0$ or $k > n$, we use the convention that $\mathcal{D}(n,k) = 0$.

The explicit formula of the Delannoy numbers $\mathcal{D}(n,k)$ is given by

$$\mathcal{D}(n,k) = \sum_{j=0}^{k} \binom{k}{j}\binom{n-j}{k} \alpha_1^{n-k-j}\alpha_2^{k-j}\alpha_3^{j}. \tag{2}$$

The generating function of the generalized Delannoy numbers is $g_k(x) = \sum_{n\geq 0} \mathcal{D}(n,k)x^n$, then

$$g_k(x) = \frac{(\alpha_2 x + \alpha_3 x^2)^k}{(1-\alpha_1 x)^{k+1}}. \tag{3}$$

In a recent proposal, S. Amrouche et. al. [3] have extended the relation (1) by considering the sum of elements located along the finite direction (r,q) of the generalized Delannoy triangle, where $r + q > 0$, $q \in \mathbb{N}$, $0 \leq p < q$ and $r \in \mathbb{Z}$. This extension resulted in the establishment of the next theorem.

Theorem 1 ([3]). *For $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}$, the sequence*

$$\mathcal{T}_{n,q,r,p}^{(\alpha_1,\alpha_2,\alpha_3)} = \sum_{k=0}^{(n-p)/(q+r)} \mathcal{D}(n-qk, p+rk)$$

satisfies the following linear recurrence relation

$$\sum_{s=0}^{r} (-\alpha_2)^s \binom{r}{s} \mathcal{T}_{n-s,q,r,p}^{(\alpha_1,\alpha_2,\alpha_3)} = \sum_{s=0}^{r} \alpha_1^{r-s}\alpha_3^{s}\binom{r}{s}\mathcal{T}_{n-r-q-s,q,r,p}^{(\alpha_1,\alpha_2,\alpha_3)}.$$

1 The q-binomial coefficient

The q-binomial coefficient or the gaussian binomial number is a polynomial in q with integer coefficient defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad 1 \leq k \leq n,$$

where $[n]_q! = [1]_q[2]_q \ldots [n]_q$ and $[n]_q = 1 + q + q^2 + \ldots + q^{n-1}$. In [7], J.A. Cigler considered the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}^* = \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}},$$

which satisfies the following recurrence relations

$$\begin{bmatrix} n \\ k \end{bmatrix}_q^* = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q^* + q^{n-1}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q^* \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_q^* = q^k\begin{bmatrix} n-1 \\ k \end{bmatrix}_q^* + q^{k-1}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q^*.$$

Here we use the convention that
$$\begin{bmatrix} n \\ k \end{bmatrix}_q^* = 0$$
for the cases $n < 0$, $k < 0$ or $k > n$. Expanding on this, the generating functions of the q-binomial coefficients are given by

$$\sum_{n\geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_q^* x^n = \frac{x^k q^{\binom{k}{2}}}{(1-x)(1-qx)\cdots(1-q^k x)} \tag{4}$$

and

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^* x^k = (1+x)(1+qx)(1+q^2x)\cdots(1+q^{n-1}x). \tag{5}$$

This work is organized as follows. In Section 2, we introduced generalized Horadam triangle and provided an explicit formula for its coefficients. We demonstrated how the sums of the elements crossing the diagonal rays produced the terms of generalized Horadam sequence of order three. Additionally, we explored the relation between generalized Horadam triangle and the generalized Delannoy triangle. In Section 3, we discussed the recurrence relations obtained by summing the elements lying over all finite rays of generalized Horadam triangle. Finally, we introduced the q-analogues of generalized Horadam and Tribonacci sequences in Section 4.

2 Generalized Horadam trianle

Definition 1. *For $\alpha_1, \alpha_2, \alpha_3, a, b, c \in \mathbb{Z}$, the generalized Horadam triangle is defined by the following recurrence relation*

$$\mathcal{A}(n,k) = \alpha_1 \mathcal{A}(n-1,k) + \alpha_2 \mathcal{A}(n-1,k-1) + \alpha_3 \mathcal{A}(n-2,k-1), \quad n \geq 2,\ k \geq 1, \tag{6}$$

with initial conditions $\mathcal{A}(0,0) = a$, $\mathcal{A}(1,0) = b$, $\mathcal{A}(1,1) = c - \alpha_1 b$.

Here we use the convention that $\mathcal{A}(n,k) = 0$, if $k < n$, $k < 0$ or $n < 0$. The main advantage of such convention of the generalized Horadam coefficient $\mathcal{A}(n,k)$ is, for example, that one can omit the use of exact limits in sums like $\sum_{k=0}^{n} \mathcal{A}(n,k)$ by simply writing $\sum_k \mathcal{A}(n,k)$. In the sequel, for the sake of convenience, we exploit this kind of allowance. One can see [4] for more details and the usefulness of such convention.

Theorem 2. *Let $F_k(x) := \sum_{n\geq 0} \mathcal{A}(n,k)x^n$, $k \geq 0$, be the generating function of generalized Horadam coefficient. Then*

$$F_0(x) = \frac{a + x(b - a\alpha_1)}{1 - \alpha_1 x}$$

and

$$F_k(x) = \frac{(\alpha_2 x + \alpha_3 x^2)^k}{(1-\alpha_1 x)^{k+1}} \left[a + x(b - a\alpha_1) + \frac{x(c - b\alpha_1 - a\alpha_2)(1-\alpha_1 x)}{(\alpha_2 x + \alpha_3 x^2)} \right], \quad k \geq 1. \tag{7}$$

Proof. Let k be an arbitrary natural number. From relation (6) it follows that

$$F_k(x) = x\alpha_1 F_k(x) + \alpha_2 x F_{k-1}(x) + \alpha_3 x^2 F_{k-1}(x),$$

the repeated applications of this recurrence gives

$$F_k(x) = \Big(\frac{\alpha_2 x + \alpha_3 x^2}{1-\alpha_1 x}\Big)^{k-1} F_1(x). \tag{8}$$

Let us calculate $F_1(x)$. We have

$$\begin{aligned}
F_1(x) &= \mathcal{A}(0,1)x^0 + \mathcal{A}(1,1)x + \sum_{n\geq 2}\mathcal{A}(n,1)x^n \\
&= (c-\alpha_1 b)x + \alpha_1 \sum_{n\geq 2}\mathcal{A}(n-1,1)x^n + \alpha_2 \sum_{n\geq 2}\mathcal{A}(n-1,0)x^n + \alpha_3 \sum_{n\geq 2}\mathcal{A}(n-2,0)x^n \\
&= (c-\alpha_1 b)x + \alpha_1 x \sum_{n'\geq 1}\mathcal{A}(n',1)x^{n'} + \alpha_2 x \sum_{n'\geq 1}\mathcal{A}(n',0)x^{n'} + \alpha_3 x^2 \sum_{n'\geq 0}\mathcal{A}(n',0)x^{n'} \\
&= (c-\alpha_1 b)x + \alpha_1 x \sum_{n'\geq 0}\mathcal{A}(n',1)x^{n'} - \alpha_2 x\mathcal{A}(0,0) \\
&\qquad + \alpha_2 x \sum_{n'\geq 0}\mathcal{A}(n',0)x^{n'} + \alpha_3 x^2 \sum_{n'\geq 0}\mathcal{A}(n',0)x^{n'} \\
&= (c-\alpha_1 b)x + \alpha_1 x F_1(x) + (\alpha_2 x + \alpha_3 x^2)F_0(x) - a\alpha_2 x \\
&= \frac{\alpha_2 x + \alpha_3 x^2}{1-\alpha_1 x}F_0(x) + \frac{(c-\alpha_1 b - a\alpha_2)x}{1-\alpha_1 x}.
\end{aligned}$$

Similarly, we calculate $F_0(x)$,

$$\begin{aligned}
F_0(x) &= \mathcal{A}(0,0)x^0 + \mathcal{A}(1,0)x + \alpha_1 \sum_{n\geq 2}\mathcal{A}(n-1,0)x^n \\
&= a + bx + \alpha_1 x F_0(x) - a\alpha_1 x = \frac{a + x(b-a\alpha_1)}{1-\alpha_1 x}.
\end{aligned}$$

Thus,

$$F_1(x) = \frac{(\alpha_2 x + \alpha_3 x^2)(a + x(b-a\alpha_1))}{(1-\alpha_1 x)^2} + \frac{(c-\alpha_1 b - a\alpha_2)x}{(1-\alpha_1 x)}. \tag{9}$$

Finally, we replace the relation (9) with the relation (8), and we get (7). □

Theorem 3. *For $n \geq 0$ and $k \geq 0$ the relation between generalized Horadam coefficient $\mathcal{A}(n,k)$ and the generalized Delannoy coefficient $\mathcal{D}(n,k)$ is given by*

$$\mathcal{A}(n,k) = a\mathcal{D}(n,k) + (b-a\alpha_1)\mathcal{D}(n-1,k) + (c - b\alpha_1 - a\alpha_2)\mathcal{D}(n-1,k-1). \tag{10}$$

Proof. For $k=0$ the relation (10) is obvious. By Theorem 2, for any $k \geq 1$ we have

$$\begin{aligned}
F_k(x) = \sum_{n\geq 0}\mathcal{A}(n,k)x^n &= \frac{(\alpha_2 x + \alpha_3 x^2)^k}{(1-\alpha_1 x)^{k+1}}\Big[a + x(b-a\alpha_1) + \frac{x(c-b\alpha_1-a\alpha_2)(1-\alpha_1 x)}{(\alpha_2 x + \alpha_3 x^2)}\Big] \\
&= a\frac{(\alpha_2 x + \alpha_3 x^2)^k}{(1-\alpha_1 x)^{k+1}} + x(b-a\alpha_1)\frac{(\alpha_2 x + \alpha_3 x^2)^k}{(1-\alpha_1 x)^{k+1}} + x(c-b\alpha_1-a\alpha_2)\frac{(\alpha_2 x + \alpha_3 x^2)^{k-1}}{(1-\alpha_1 x)^k},
\end{aligned}$$

and by relation (3), we obtain

$$F_k(x) = \sum_{n\geq 0}(a\mathcal{D}(n,k) + (b-a\alpha_1)\mathcal{D}(n-1,k) + (c-b\alpha_1-a\alpha_2)\mathcal{D}(n-1,k-1))x^n.$$

Therefore, (10) follows. □

Theorem 4. *For $n \geq 0$ and $k \geq 0$ the explicit formula of generalized Horadam coefficient is given by*

$$\mathcal{A}(n,k) = \sum_{j=0}^{k} \binom{k}{j}\binom{n-j}{k}\left[a + \frac{(b-a\alpha_1)(n-j-k)}{\alpha_1(n-j)} + \frac{(c-b\alpha_1-a\alpha_2)(k-j)}{\alpha_2(n-j)}\right]\alpha_1^{n-k-j}\alpha_2^{k-j}\alpha_3^{j}.$$

Proof. One can deduce this result by Theorem 3 and relation (2). □

3 Recurrence relations

The purpose of this section is to develop a recurrence relation associated with the sum of elements along lines in generalized Horadam triangle of direction (α, r) in generalized Horadam trinagle. For further details concerning the concept of direction see [1,5].

We begin by considering the principal direction $(1,1)$. Let $(\mathcal{H}_n)_{n\geq 0}$ be the sequence obtained by summing the elements lying along the principal diagonal rays in generalized Horadam triangle. Then

$$\mathcal{H}_{n+1} = \sum_{k\geq 0} \mathcal{A}(n-k,k). \tag{11}$$

The following result shows that the sequence $(\mathcal{H}_n)_{n\geq 0}$ is the generalized Horadam sequence of order three.

Theorem 5. *For $n \geq 0$, $\mathcal{H}_n$ satisfies the recurrence relation*

$$\mathcal{H}_{n+1} = \alpha_1\mathcal{H}_n + \alpha_2\mathcal{H}_{n-1} + \alpha_3\mathcal{H}_{n-2}$$

with $\mathcal{H}_0 = a$, $\mathcal{H}_{-1} = b$, $\mathcal{H}_{-2} = c$.

Proof. Using the recurrence relation (6), from (11) we obtain

$$\begin{aligned}\mathcal{H}_{n+1} &= \alpha_1\sum_{k}\mathcal{A}(n-k-1,k) + \alpha_2\sum_{k}\mathcal{A}(n-k-1,k-1) + \alpha_3\sum_{k}\mathcal{A}(n-k-2,k-1)\\ &= \alpha_1\sum_{k}\mathcal{A}(n-k-1,k) + \alpha_2\sum_{k'}\mathcal{A}(n-k'-2,k') + \alpha_3\sum_{k'}\mathcal{A}(n-k'-3,k')\\ &= \alpha_1\mathcal{H}_n + \alpha_2\mathcal{H}_{n-1} + \alpha_3\mathcal{H}_{n-2},\end{aligned}$$

which had to be proven. □

The following result establishes the link between generalized Horadam sequence $\mathcal{H}_n$ and the generalized Tribonacci sequence $\mathcal{T}_n$.

Theorem 6. *For $n \geq 2$, the relation between $\mathcal{H}_n$ and $\mathcal{T}_n$ is given by*

$$\mathcal{H}_{n+1} = a\mathcal{T}_{n+1} + (b-a\alpha_1)\mathcal{T}_n + (c-b\alpha_1-a\alpha_2)\mathcal{T}_{n-1}. \tag{12}$$

Proof. By Theorem 3, from (11) we obtain

$$\begin{aligned}\mathcal{H}_{n+1} &= a\sum_{k}\mathcal{D}(n-k,k) + (b-a\alpha_1)\sum_{k}\mathcal{D}(n-k-1,k) + (c-b\alpha_1-a\alpha_2)\sum_{k}\mathcal{D}(n-k-1,k-1)\\ &= a\sum_{k}\mathcal{D}(n-k,k) + (b-a\alpha_1)\sum_{k}\mathcal{D}(n-k-1,k) + (c-b\alpha_1-a\alpha_2)\sum_{k'}\mathcal{D}(n-k'-2,k').\end{aligned}$$

Then, by relation (1), it follows (12). □

Now, let us consider

$$\mathcal{H}_{n+1}^{(\alpha,\beta,r)} = \sum_{k=0}^{\lfloor (n-\beta)/(r+\alpha) \rfloor} \mathcal{A}(n-rk, \beta+\alpha k).$$

Theorem 7. *For $n \geq \alpha + r$, the sequence $(\mathcal{H}_{n+1}^{(\alpha,\beta,r)})_{n\geq 0}$ satisfies the following linear recurrence relation*

$$\sum_{i=0}^{\alpha} (-\alpha_2)^i \binom{\alpha}{i} \mathcal{H}_{n-i}^{(\alpha,\beta,r)} = \sum_{i=0}^{\alpha} \alpha_1^{r-i} \alpha_3^i \binom{\alpha}{i} \mathcal{H}_{n-\alpha-r-i}^{(\alpha,\beta,r)}. \quad (13)$$

Proof. We have

$$\sum_{i=0}^{\alpha} (-\alpha_2)^i \binom{\alpha}{i} \mathcal{H}_{n-i}^{(\alpha,\beta,r)} = \sum_{i=0}^{\alpha} (-\alpha_2)^i \binom{\alpha}{i} \sum_{k} \mathcal{A}(n-rk-i, \beta+\alpha k).$$

Then, by Theorem 3, we get

$$\begin{aligned}\sum_{i=0}^{\alpha} (-\alpha_2)^i \binom{\alpha}{i} \mathcal{H}_{n-i}^{(\alpha,\beta,r)} &= a \sum_{i=0}^{\alpha} (-\alpha_2)^i \binom{\alpha}{i} \sum_{k} \mathcal{D}(n-rk-i, \beta+\alpha k) \\ &+ (b - a\alpha_1) \sum_{i=0}^{\alpha} (-\alpha_2)^i \binom{\alpha}{i} \sum_{k} \mathcal{D}(n-rk-i-1, \beta+\alpha k) \\ &+ (c - b\alpha_1 - a\alpha_2) \sum_{i=0}^{\alpha} (-\alpha_2)^i \binom{\alpha}{i} \sum_{k} \mathcal{D}(n-rk-i-1, \beta+\alpha k-1).\end{aligned}$$

By Theorem 1, we obtain

$$\begin{aligned}\sum_{i=0}^{\alpha} (-\alpha_2)^i \binom{\alpha}{i} \mathcal{H}_{n-i}^{(\alpha,\beta,r)} &= \sum_{i=0}^{\alpha} \alpha_1^{r-i} \alpha_3^i \binom{\alpha}{i} a \sum_{k} \mathcal{D}(n-rk-i, \beta+\alpha k) \\ &+ \sum_{i=0}^{\alpha} \alpha_1^{r-i} \alpha_3^i \binom{\alpha}{i} (b - a\alpha_1) \sum_{k} \mathcal{D}(n-rk-i-1, \beta+\alpha k) \\ &+ \sum_{i=0}^{\alpha} \alpha_1^{r-i} \alpha_3^i \binom{\alpha}{i} (c - b\alpha_1 - a\alpha_2) \sum_{k} \mathcal{D}(n-rk-i-1, \beta+\alpha k-1).\end{aligned}$$

Finally, again by Theorem 3, we get (13). □

4 *q*-Horadam sequence

In this section, we describe a q-analogue of the generalized Delannoy triangle using the same approach as in [1]. We then use this definition to suggest a q-deformation for the generalized Tribonacci and Horadam sequences.

Definition 2. *According to relation* (6)*, for $n \geq 2$ and $k \geq 1$ we define the coefficient of q-generalized Delannoy triangle as*

$$\mathcal{D}_q(n,k) = [\alpha_1]_q \mathcal{D}_q(n-1,k) + [\alpha_2]_q q^{n-1} \mathcal{D}_q(n-1,k-1) + [\alpha_3]_q q^{n-2} \mathcal{D}_q(n-2,k-1), \quad (14)$$

or equivalently

$$\mathcal{D}_q(n,k) = [\alpha_1]_q q^k \mathcal{D}_q(n-1,k) + [\alpha_2]_q q^{k-1} \mathcal{D}_q(n-1,k-1) + [\alpha_3]_q q^{2(k-1)} \mathcal{D}_q(n-2,k-1) \quad (15)$$

with $\mathcal{D}_q(0,0) = 1$.

Also, we use the convention that $\mathcal{D}_q(n,k)=0$ for $k<n$, $k<0$ or $n<0$.

Theorem 8. *Let $\mathbb{F}_k(x):=\sum_{n\geq 0}\mathcal{D}_q(n,k)x^n$, $k\geq 0$, be the generating function of the q-generalized Delannoy triangle coefficient. Then*

$$\mathbb{F}_k(x)=\frac{x^k q^{\binom{k}{2}}\prod_{j=0}^{k-1}([\alpha_2]_q+[\alpha_3]_q q^j x)}{\prod_{j=0}^{k}(1-[\alpha_1]_q q^j x)},\quad k\geq 0. \tag{16}$$

Proof. Using relation (14), we have

$$\begin{aligned}\mathbb{F}_k(x)&=\sum_{n\geq 0}\mathcal{D}_q(n,k)x^n\\&=[\alpha_1]_q\sum_{n\geq 0}\mathcal{D}_q(n-1,k)x^n+[\alpha_2]_q\sum_{n\geq 0}q^{n-1}\mathcal{D}_q(n-1,k-1)x^n\\&\qquad\qquad+[\alpha_3]_q\sum_{n\geq 0}q^{n-2}\mathcal{D}_q(n-2,k-1)x^n\\&=x[\alpha_1]_q\sum_{n\geq 0}\mathcal{D}_q(n,k)x^n+([\alpha_2]_q x+[\alpha_3]_q x^2)\sum_{n\geq 0}\mathcal{D}_q(n,k-1)(qx)^n.\end{aligned}$$

Then

$$\frac{(1-[\alpha_1]_q x)\mathbb{F}_k(x)}{([\alpha_2]_q x+[\alpha_3]_q x^2)}=\mathbb{F}_{k-1}(qx),$$

and, by repeating applications of this process, we get (16). □

Remark 1. *Following the same approach, one can similarly demonstrate Theorem 8, using relation* (15). *Since the relations* (14) *and* (15) *give the same generating function, then these two relations are equivalent.*

The following result establishes the explicit formula for the coefficients of q-generalized Delannoy triangle.

Theorem 9. *The coefficient $\mathcal{D}_q(n,k)$, $n\geq 0$, $k\geq 0$, satisfies*

$$\mathcal{D}_q(n,k)=\sum_j\begin{bmatrix}n-j\\k\end{bmatrix}_q^*\begin{bmatrix}k\\j\end{bmatrix}_q^*[\alpha_1]_q^{n-k-j}[\alpha_2]_q^{k-j}[\alpha_3]_q^{j},\quad n\geq 0,\ k\geq 0.$$

Proof. We have

$$\begin{aligned}\sum_{n\geq 0}\mathcal{D}_q(n,k)x^n&=\left(\frac{[\alpha_2]_{[q}}{[\alpha_1]_q}\right)^k\sum_j\begin{bmatrix}k\\j\end{bmatrix}_q^*\left(\frac{x[\alpha_3]_q}{[\alpha_2]_q}\right)^j\sum_{n\geq 0}\begin{bmatrix}n-j\\k\end{bmatrix}_q^*(x[\alpha_1]_q)^{n-j}\\&=\frac{x^k q^{\binom{k}{2}}\prod_{j=0}^{k-1}([\alpha_2]_q+[\alpha_3]_q q^j x)}{\prod_{j=0}^{k}(1-[\alpha_1]_q q^j x)}.\end{aligned}$$

The last equality comes from relations (4) and (5). □

L. Carlitz [6] and J.A. Cigler [7] proposed the q-analogue of the Fibonacci sequence. The following result establishes the recurrence relation for the q-analogue of the generalized Tribonacci sequence.

Theorem 10. *Let*

$$\mathcal{T}_{n+1,q}(x) := \sum_{k\geq 0} \mathcal{D}_q(n-k,k)x^k, \quad n \geq 0.$$

Then for $n \geq 0$ we have

$$\mathcal{T}_{n+1,q}(x) = [\alpha_1]_q \mathcal{T}_{n,q}(x) + xq^{n-2}[\alpha_2]_q \mathcal{T}_{n-1,q}(x/q) + xq^{n-3}[\alpha_3]_q \mathcal{T}_{n-2,q}(x/q),$$

or equivalently

$$\mathcal{T}_{n+1,q}(x) = [\alpha_1]_q \mathcal{T}_{n,q}(xq) + x[\alpha_2]_q \mathcal{T}_{n-1,q}(xq) + x[\alpha_3]_q \mathcal{T}_{n-2,q}(xq^2). \tag{17}$$

Proof. Using relation (14), we get

$$\begin{aligned}
\mathcal{T}_{n+1,q}(x) &= [\alpha_1]_q \sum_{k\geq 0} \mathcal{D}_q(n-k-1,k)x^k + [\alpha_2]_q \sum_{k\geq 0} \mathcal{D}_q(n-k-1,k-1)x^k q^{n-k-1} \\
&\quad + [\alpha_3]_q \sum_{k\geq 0} \mathcal{D}_q(n-k-2,k-1)x^k q^{n-k-2} \\
&= [\alpha_1]_q \sum_{k\geq 0} \mathcal{D}_q(n-k-1,k)x^k + x[\alpha_2]_q \sum_{k'\geq 0} \mathcal{D}_q(n-k'-2,k')x^{k'} q^{n-k'-2} \\
&\quad + x[\alpha_3]_q \sum_{k'\geq 0} \mathcal{D}_q(n-k'-3,k')x^{k'} q^{n-k'-3} \\
&= [\alpha_1]_q \mathcal{T}_{n,q}(x) + xq^{n-2}[\alpha_2]_q \mathcal{T}_{n-1,q}(x/q) + xq^{n-3}[\alpha_3]_q \mathcal{T}_{n-2,q}(x/q),
\end{aligned}$$

which had to be proven.

The proof is the same for the relation (17), we use relation (15). □

Definition 3. *For $n \geq 0$ we define q-generalized Horadam sequence by*

$$\begin{aligned}
\mathcal{H}_{n+1,q}(x) = \sum_k \sum_j \begin{bmatrix} k \\ j \end{bmatrix}_q^* \begin{bmatrix} n-j-k \\ k \end{bmatrix}_q^* \Bigg([a]_q + \frac{[b-a\alpha_1]_q [n-j-2k]_q}{[\alpha_1]_q [n-j-k]_q} \\
+ \frac{[c-b\alpha_1-a\alpha_2]_q [k-j]_q}{[\alpha_2]_q [n-j-k]_q} \Bigg) [\alpha_1]_q^{n-2k-j} [\alpha_2]_q^{k-j} [\alpha_3]_q^j (x)^k.
\end{aligned}$$

Theorem 11. *The relation between q-generalized Horadam sequence and q-generalized Tribonacci sequence is given by*

$$\mathcal{H}_{n+1,q}(x) = [a]_q \mathcal{T}_{n+1,q}(x) + [b-a\alpha_1]_q \mathcal{T}_{n,q}(x) + x[c-b\alpha_1-a\alpha_2]_q \mathcal{T}_{n-1,q}(qx), \quad n \geq 0.$$

Proof. By the Definition 3, we have

$$\begin{aligned}
\mathcal{H}_{n+1,q}(x) &= [a]_q \sum_{k\geq 0} \sum_j^k \begin{bmatrix} k \\ j \end{bmatrix}_q^* \begin{bmatrix} n-j-k \\ k \end{bmatrix}_q^* [\alpha_1]_q^{n-2k-j} [\alpha_2]_q^{k-j} [\alpha_3]_q^j x^k \\
&\quad + \frac{[b-a\alpha_1]_q}{[\alpha_1]_q} \sum_{k\geq 0} \sum_j^k \begin{bmatrix} k \\ j \end{bmatrix}_q^* \begin{bmatrix} n-j-k \\ k \end{bmatrix}_q^* \frac{[n-j-2k]_q}{[n-j-k]_q} [\alpha_1]_q^{n-2k-j} [\alpha_2]_q^{k-j} [\alpha_3]_q^j x^k \\
&\quad + \frac{[c-b\alpha_1-a\alpha_2]_q}{[\alpha_2]_q} \sum_{k\geq 0} \sum_j^k \begin{bmatrix} k \\ j \end{bmatrix}_q^* \begin{bmatrix} n-j-k \\ k \end{bmatrix}_q^* \frac{[k-j]_q}{[n-j-k]_q} [\alpha_1]_q^{n-2k-j} [\alpha_2]_q^{k-j} [\alpha_3]_q^j x^k
\end{aligned}$$

$$= [a]_q \sum_{k\geq 0}\sum_{j}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q^* \begin{bmatrix} n-j-k \\ k \end{bmatrix}_q^* [\alpha_1]_q^{n-2k-j}[\alpha_2]_q^{k-j}[\alpha_3]_q^j x^k$$
$$+ [b-a\alpha_1]_q \sum_{k}\sum_{j}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q^* \begin{bmatrix} n-j-k \\ k \end{bmatrix}_q^* [\alpha_1]_q^{n-2k-j}[\alpha_2]_q^{k-j}[\alpha_3]_q^j x^k$$
$$+ \frac{[c-b\alpha_1-a\alpha_2]_q}{[\alpha_2]_q} \sum_{k}\sum_{j}^{k} \begin{bmatrix} k-1 \\ j \end{bmatrix}_q^* \begin{bmatrix} n-j-k-1 \\ k-1 \end{bmatrix}_q^* [\alpha_1]_q^{n-2k-j}[\alpha_2]_q^{k-j}[\alpha_3]_q^j x^k.$$

Then

$$\mathcal{H}_{n+1,q}(x) = [a]_q \sum_{k\geq 0}\sum_{j}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q^* \begin{bmatrix} n-j-k \\ k \end{bmatrix}_q^* [\alpha_1]_q^{n-2k-j}[\alpha_2]_q^{k-j}[\alpha_3]_q^j x^k$$
$$+ [b-a\alpha_1]_q \sum_{k\geq 0}\sum_{j}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q^* \begin{bmatrix} n-j-k-1 \\ k \end{bmatrix}_q^* [\alpha_1]_q^{n-2k-j-1}[\alpha_2]_q^{k-j}[\alpha_3]_q^j x^k$$
$$+ x[c-b\alpha_1-a\alpha_2]_q \sum_{k'\geq 0}\sum_{j} \begin{bmatrix} k' \\ j \end{bmatrix}_q^* \begin{bmatrix} n-j-k'-2 \\ k' \end{bmatrix}_q^* [\alpha_1]_q^{n-2k'-j-2}[\alpha_2]_q^{k'-j}[\alpha_3]_q^j (qx)^{k'}$$
$$= [a]_q \mathcal{T}_{n+1,q}(x) + [b-a\alpha_1]_q \mathcal{T}_{n,q}(x) + x[c-b\alpha_1-a\alpha_2]_q \mathcal{T}_{n-1,q}(qx),$$

which had to be proven. □

Theorem 12. *The q-generalized Horadam sequence satisfies the following recurrence relation*

$$\mathcal{H}_{n+1,q}(x) = [\alpha_1]_q \mathcal{H}_{n,q}(qx) + [\alpha_2]_q x \mathcal{H}_{n-1,q}(qx) + [\alpha_3]_q x \mathcal{H}_{n-2,q}(q^2x), \quad n \geq 0. \tag{18}$$

Proof. By Theorem 11, we have

$$\mathcal{H}_{n+1,q}(x) = [a]_q \mathcal{T}_{n+1,q}(x) + [b-a\alpha_1]_q \mathcal{T}_{n,q}(x) + x[c-b\alpha_1-a\alpha_2]_q \mathcal{T}_{n-1,q}(qx).$$

Then, by relation (17), we get

$$\mathcal{H}_{n+1,q}(x) = [a]_q([\alpha_1]_q \mathcal{T}_{n,q}(xq) + x[\alpha_2]_q \mathcal{T}_{n-1,q}(xq) + x[\alpha_3]_q \mathcal{T}_{n-2,q}(xq^2))$$
$$+ [b-a\alpha_1]_q([\alpha_1]_q \mathcal{T}_{n-1,q}(xq) + x[\alpha_2]_q \mathcal{T}_{n-2,q}(xq) + x[\alpha_3]_q \mathcal{T}_{n-3,q}(xq^2))$$
$$+ x[c-b\alpha_1-a\alpha_2]_q + ([\alpha_1]_q \mathcal{T}_{n-2,q}(xq^2) + x[\alpha_2]_q \mathcal{T}_{n-,q}(xq^2) + x[\alpha_3]_q \mathcal{T}_{n-4,q}(xq^3)),$$

and again, by Theorem 11, we obtain (18). □

5 Conclusion

The generalization presented in this paper for the Horadam sequence of order three can be extended for orders greater than three. Recently, in [2], S. Amrouche and H. Belbachir have defined the quasi s-Pascal triangle which is an extension of Pascal and Delannoy triangles. They have established that the sum of elements lying on the diagonal rays of the quasi s-Pascal triangle gives the terms of s-Fibonacci sequence. We think that the approach outlined in this paper works for generalized Horadam sequence of order $s > 3$.

References

[1] Amrouche S., Belbachir H. *Unimodality and linear recurrences associated with rays in the Delannoy triangle*. Turkish J. Math. 2020, **44** (1), 118–130. doi:10.3906/mat-1811-109

[2] Amrouche S., Belbachir H. *Asymmetric extension of Pascal-Delannoy triangles*. Appl. Anal. Discrete Math. 2022, **16** (2), 328–349. doi:10.2298/AADM200411028A

[3] Amrouche S., Belbachir H., Ramirez J.L. *Unimodality, linear recurrences and combinatorial properties associated to rays in the generalized Delannoy matrix*. J. Differ. Equations Appl. 2019, **25** (8), 1200–1215. doi:10.1080/10236198.2019.1662413

[4] Belbachir H., Szalay L. *Unimodal rays in the ordinary and generalized Pascal triangles*. J. Integer Seq. 2008, **11** (2), article 08.2.4.

[5] Belbachir H., Komatsu T., Szalay L. *Linear recurrences associated to rays in Pascal's triangle and combinatorial identities*. Math. Slovaca 2014, **64** (2), 287–300. doi:10.2478/s12175-014-0203-0

[6] Carlitz L. *Fibonacci notes – 3: q-Fibonacci numbers.* Fibonacci Quart. 1974, **12** (4), 317–322. doi:10.1080/00150517.1974.12430696

[7] Cigler J. *A new class of q-Fibonacci polynomials*. Electron. J. Combin. 2003, **10** (1), 19. doi:10.37236/1712

[8] Fray R.D., Roselle D.P. *Weighted lattice paths*. Pacific J. Math. 1971, **37** (1), 85–96.

[9] Larcombe P.J. *Horadam sequences: a survey update and extension*. Bull. Inst. Comb. Appl. 2017, **80**, 99–118.

[10] Larcombe P.J., Bagdasar O.D., Fennessey E.J. *Horadam sequences: a survey*. Bull. Inst. Comb. Appl. 2013, **67**, 49–72.

Received 17.05.2024
Revised 25.01.2025

Амруш С., Белбахір Х., Чезарано К., Річчі П.Е. *Про послідовності Горадама третього порядку* // Карпатські матем. публ. — 2025. — Т.17, №2. — С. 406–415.

У цій статті ми визначаємо узагальнений трикутник Горадама, в якому сума елементів, розташованих уздовж діагональних ліній, відповідає членам узагальненої послідовності Горадама третього порядку. Встановлено зв'язок між узагальненим трикутником Горадама та узагальненим трикутником Делануа. Крім того, ми визначаємо q-аналог узагальненої послідовності трібоначчі та узагальненої послідовності Горадама третього порядку.

Ключові слова і фрази: послідовність Горадама, q-аналог, рекурентне співвідношення.