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Banach-Saks sets in Banach lattices

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We study the Banach-Saks sets in Banach lattices. It also investigated some of the consequences derived from this study. Among other things, we establish that each L-weakly compact set is Banach-Saks. Also, we characterize Banach lattices in which the solid hull of a Banach-Saks set is also Banach-Saks. Next, we give a necessary and sufficient condition under which an order bounded almost Dunford-Pettis operator between two Banach lattices maps Banach-Saks sets into L-weakly compact sets. In addition, we establish a new characterization of Banach lattices with order continuous norm and order weakly compact operators using Banach-Saks set and the Cesaro convergence.

Key words and phrases: Banach-Saks set, relatively weakly compact set, L-weakly compact set, order weakly compact operator, almost Dunford-Pettis operator, Banach lattice.

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Introduction

Throughout this paper, we denote by E and F (respectively, X and Y) real Banach lattices (respectively, real Banach spaces). We also denote by B_E the closed unit ball of E. We use term "operator" to refer a bounded linear mapping. For any unexplained terms from the theory of Banach lattices and positive operators, we refer the reader to [1, 15].

The Banach-Saks property was introduced by S. Banach and S. Saks in [4]. Since then, this concept has been studied by many authors, see for instance [8–12, 14]. A sequence $(x_n)_n$ in X is called *Cesàro convergent* if the sequence $\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)_n$ of its arithmetic means is norm convergent in X. A norm bounded subset A of X is called *Banach-Saks* if every sequence in A has a Cesàro convergent subsequence. A Banach space X is said to be *Banach-Saks* if the closed unit ball B_X of X is Banach-Saks. We say that X has the *weak Banach-Saks property* if every weakly convergent sequence $(x_n)_n$ in X has a Cesaro convergent subsequence. S. Banach and S. Saks proved in [4] that $L^p[0,1]$ with $1 < p < \infty$ has the Banach-Saks property. Moreover, W. Szlenik showed in [16] that $L^1[0,1]$ has the weak Banach-Saks property, but not Banach-Saks property. However, the Banach lattices $C[0,1]$ and ℓ^∞ do not have the weak Banach-Saks property (see [6]). It is clear that every relatively compact subset of X is Banach-Saks. The unit ball of $L^2[0,1]$ is a Banach-Saks set which is not relatively compact. According to [14, Proposition 2.3] every Banach-Saks set is relatively weakly compact. However, the converse is not true in general.

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Let A be a subset of a Banach lattice E. The *solid hull of* A is the smallest solid set of E containing A. Exactly,

$$Sol(A) := \bigcup_{x\in A} [-|x|, |x|].$$

A subset A of E is called *L-weakly compact* if every disjoint sequence in the solid hull of A is norm null. A sequence $(x_n)_n$ in E is called L-weakly compact if the set $\{x_n : n \in \mathbb{N}\}$ is L-weakly compact. It should be noted that every L-weakly compact set is relatively weakly compact (see [15, Proposition 3.6.5]).

The aim of this paper is to study Banach-Saks sets in Banach lattices. In Section 1, we establish that every L-weakly compact set in E is Banach-Saks (Theorem 2). As a consequence, we obtain that every L-weakly compact sequence of E has a subsequence which is Cesaro convergent (Corollary 2). Also, we present some necessary and sufficient conditions under which the solid hull of a Banach-Saks set in E is also Banach-Saks (Theorem 3 and Corollary 3). In addition, we show that for a pair of Banach lattices E and F, the order bounded almost Dunford-Pettis operators $T : E \to F$ map Banach-Saks sets into L-weakly compact sets if and only if the norm of F is order continuous. In Section 2, we characterise Banach lattices with order continuous norms using Cesaro convergence (Proposition 3). Also, we investigate the connection between order weakly compact operators from a Banach lattice E into a Banach space X and the operators that transform order bounded subsets of E into Banach-Saks sets of X (Proposition 4). As a consequence, we obtain that the Banach-Saks operators satisfy the domination problem (Corollary 5). Finally, we obtain that an operator $T : E \to X$ is order weakly compact if and only if $\left(\frac{1}{n}\sum_{k=1}^{n} Tx_k\right)_n$ is norm convergent for every increasing order bounded sequence $(x_n)_n$ in E^+ (Theorem 4).

1 Banach-Saks sets

A positive element e of a Banach lattice E is said to be a weak order unit if the band generated by e satisfies $B_e = E$ (see [15, Definition 1.2.12]). If E has order continuous norm, then it follows from [13, Proposition 1.a.9] that every separable subspace of E is included in some closed order ideal G with a weak order unit, and hence every separable Banach lattice with order continuous norm has a weak order unit. However, the Banach lattice $(\ell^\infty)'$ does not have any weak order unit. On the other hand, every closed order ideal G of an order continuous Banach lattice E is the range of a positive projection from E into G (see [13, Proposition 1.a.11]). It follows from [13, Theorem 1.b.14 and Proposition 1.a.9] that the ideal $E(X)$ generated by a separable subspace X of E can be represented by an ideal of $L_1(\Omega, \Sigma, \mu)$ for some probability measure μ such that the formal inclusion $i : E \hookrightarrow L_1(\Omega, \Sigma, \mu)$ is continuous. To continue our discussion, we need the following lemma.

Lemma 1 ([12, Lemma 2.2])**.** *Let E be a Banach lattice with an order continuous norm, and $(x_n)_n$ be a bounded sequence in E. Then $(x_n)_n$ is convergent in E if and only if it is $\|\cdot\|_1$-convergent and L-weakly compact.*

Also, we will need the following result.

Theorem 1 ([8])**.** *Every bounded sequence $(x_n)_n$ in a Banach space X has a subsequence $(x_{n_k})_k$ such that either every further subsequence of $(x_{n_k})_k$ Cesaro converges to the same limit or every further subsequence of $(x_{n_k})_k$ Cesaro diverges.*

It follows from [3, Lemma 3.2] that a subset A of a Banach space X is Banach-Saks if and only if for each $\epsilon > 0$ there exists a Banach-Saks set B of X such that $A \subset B + \epsilon B_X$. From [1, Proposition 3.6.5] it follows that every L-weakly compact subset of E is relatively weakly compact. In [14], the authors proved that each Banach-Saks set is relatively weakly compact. In the following result, we show that every L-weakly compact subset of E is Banach-Saks. Note that the converse is not necessary true in general. Indeed, the singleton $\{e\}$ of ℓ^∞ where $e = (1, 1, \dots)$ is a Banach-Saks set of ℓ^∞ but it is not L-weakly compact.

Theorem 2. *Let E be a Banach lattice. Then every L-weakly compact subset of E is Banach-Saks.*

Proof. Let A be an L-weakly compact subset of E and let $\epsilon > 0$ be fixed. So there exists $u \in (E^a)^+$ such that $A \subset [-u, u] + \epsilon B_E$. Since the norm of E^a is order continuous, it follows from [1, Theorem 4.9] that $[-u, u]$ is weakly compact. By [1, Corollary 5.54], $[-u, u]$ is L-weakly compact, and it follows from [3, Theorem 2.3] that $[-u, u]$ is Banach-Saks. Now, by [3, Lemma 3.2], A is Banach-Saks. □

Remark 1. *Let A be a subset of a Banach lattice E. Then the following implications hold:*

$$A \text{ is } L\text{-weakly compact} \;\Rightarrow\; A \text{ is Banach-Saks} \;\Rightarrow\; A \text{ is relatively weakly compact.}$$

We obtain also that all order bounded sets in a Banach lattice that has an order continuous norm are Banach-Saks.

Corollary 1. *Let E be a Banach lattice. Then the following assertions are equivalent.*

1) *The norm of E is order continuous.*

2) *Every order bounded subset of E is Banach-Saks.*

Without any condition on Banach lattice on E we obtain the following result.

Corollary 2. *Each L-weakly compact sequence of a Banach lattice E has a subsequence which is Cesaro convergent.*

It follows from [1, Theorem 4.34] that every disjoint sequence in the solid hull of a weakly relatively compact set of E converges weakly to zero. In the following result, we give a Banach-Saks version of this theorem.

Lemma 2. *Let A be a Banach-Saks subset of E^+ and let $x \in E$. Then every disjoint sequence $(f_n)_n$ in the solid hull of $A + x$ has a subsequence which Cesaro converges to 0.*

Proof. Let A be a Banach-Saks subset of E^+ and let $(f_n)_{n\in\mathbb{N}}$ be a positive disjoint sequence in the solid hull of $A + x$. For each $n \in \mathbb{N}$ pick some $g_n \in A$ such that $0 \le f_n \le g_n + |x|$. So, for all $n \in \mathbb{N}$ we have

$$0 \le \frac{1}{n}\sum_{k=1}^{n} f_k \le \frac{1}{n}\sum_{k=1}^{n} g_k + |x|.$$

Since A is Banach-Saks, it follows from Theorem 1 that there exists a subsequence of $(g_n)_n$ (which we denote by $(g_n)_n$ again) such that all of its subsequences are Cesaro convergent to the same limit $g \in E^+$. That is,

$$\left\| \frac{1}{n}\sum_{k=1}^{n} (g_k - g) \right\| \to 0.$$

This implies that

$$0 \leq \frac{1}{n}\sum_{k=1}^{n} f_k \leq \left|\frac{1}{n}\sum_{k=1}^{n}(g_k - g)\right| + g + |x|,$$

for all $n \in \mathbb{N}$.

By [1, Theorem 1.13], for each $n \in \mathbb{N}$ there exist $z_n, v_n \in E^+$ such that $\frac{1}{n}\sum_{k=1}^{n} f_k = z_n + v_n$, $0 \leq z_n \leq \left|\frac{1}{n}\sum_{k=1}^{n}(g_k - g)\right|$ and $0 \leq v_n \leq g + |x|$. So, $z_n \to 0$ and $v_n \leq \frac{1}{n}\sum_{k=1}^{n}(f_k \wedge (g + |x|))$.

On the other hand, by our hypothesis $(f_n \wedge (g + |x|))_n$ is an order bounded disjoint sequence, then it follows from [12, Proposition 3.10] that $\left\|\frac{1}{n}\sum_{k=1}^{n}(f_k \wedge (g + |x|))\right\| \to 0$, and so $\left\|\frac{1}{n}\sum_{k=1}^{n} f_k\right\| \to 0$, which concludes the proof. □

By [15, Proposition 2.5.12], the solid hull of every relatively weakly compact set $A \subset E_+$ is again relatively weakly compact if and only if the norm of E is order continuous. In the next result, we give a necessary and sufficient condition under which the solid hull of a Banach-Saks set in E^+ is also Banach-Saks.

Theorem 3. *Let E be a Banach lattice. Then the following assertions are equivalent.*

1) E has an order continuous norm.

2) The solid hull of a Banach-Saks set in E^+ is also Banach-Saks.

Proof. 1) ⇒ 2) Let A be a Banach-Saks subset of E^+ and $(x_n)_n$ be a sequence in $Sol(A)$. For each n there exists some (f_n) in A such that $|x_n| \leq f_n$. The proof will be based upon two steps.

Step 1. There is a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $\left\{\frac{1}{k}\sum_{i=1}^{k} x_{n_i} : k \in \mathbb{N}\right\}$ is L-weakly compact.

Since A is Banach-Saks, there exists a vector $g \in E$ and a subsequence (f_{n_k}) of (f_n) such that all its subsequences (g_n) satisfy $\left\|\frac{1}{n}\sum_{k=1}^{n} g_k - g\right\| \to 0$ (see Theorem 1). This implies that for all $\epsilon > 0$, there exists $u \in E^+$ such that

$$\left\{\frac{1}{k}\sum_{i=1}^{k} f_{n_i} : k \in \mathbb{N}\right\} \subseteq [-u, u] + \epsilon B_E.$$

On the other hand, by the inequality $\left|\frac{1}{k}\sum_{i=1}^{k} x_{n_i}\right| \leq \frac{1}{k}\sum_{i=1}^{k} f_{n_i}$ for all $k \in \mathbb{N}$ and [15, Proposition 3.6.2], the set $\left\{\frac{1}{k}\sum_{i=1}^{k} x_{n_i} : k \in \mathbb{N}\right\}$ is L-weakly compact.

Step 2. The solid hull of A is a Banach-Saks set of E.

Without loss of generality, we can assume that (x_n) is weakly convergent to x. Also, without loss of generality, we can replace E by the closed ideal generated by (x_n) that has a strong unit, and so isomorphic to $L^1(\mu)$. Then, $x_n \to x$ weakly in $L^1(\mu)$, and so (from weak Banach-Saks property) we can find a subsequence x_{n_k} such that each of its further subsequences Cesaro converges to x in $L^1(\mu)$. On the other hand, it follows from Step 1 that there exists a subsequence $(x_{n_{k_m}})$ which is L-weakly compact. Finally, $(x_{n_{k_m}})$ is L-weakly compact and Cesaro converges to x in $L^1(\mu)$, thus Cesaro converges to x in E according to Lemma 1. Consequently, the solid hull of A is a Banach-Saks set of E.

2) ⇒ 1) Let $x \in E^+$. Then $Sol\{x\} = [-x, x]$ is a Banach-Saks subset of E, so $[-x, x]$ is a relatively weakly compact (and is closed), then it is a weakly compact subset of E. Consequently, E has an order continuous norm (see [1, Theorem 4.9]). □

Note that if A is a subset of a Banach lattice E, then $Sol(A) = Sol(|A|)$, where $|A| := \{|x| : x \in A\}$. Following [12], we shall say that a Banach lattice has the $(BS1)$ property if for every Banach-Saks subset A of E, the set $|A| := \{|a| : a \in A\}$ is again Banach-Saks. The Banach lattice $c_0(L^2[0,1])$ is an example of a space without $(BS1)$ property (for more details, see [12]). In a Banach lattice with order continuous norm, the $(BS1)$ property characterized as follows.

Corollary 3. *Let E be a Banach lattice with order continuous norm. Then the following assertions are equivalent.*

1) E has the $(BS1)$ property.

2) The solid hull of a Banach-Saks set is also Banach-Saks.

Recall that a Banach lattice E is said to be a *Kantorovich Banach* space (or briefly a *KB-space*) whenever every increasing norm bounded sequence of E_+ is norm convergent.

Proposition 1. *Every KB-space E with the weak Banach-Saks property has the $(BS1)$ property.*

Proof. Let A be a Banach-Saks set of E. So A is relatively weakly compact. Since E is KB-space, the set $|A|$ is a relatively weakly compact set of E (see [2, Theorem 9.35]), and by our hypothesis $|A|$ is Banach-Saks, as desired. □

Recall that an operator $T : E \to X$ is called *almost Dunford-Pettis* if $\|T(x_n)\| \to 0$ for every disjoint weakly null sequence (x_n) in E^+ [18], and a Banach lattice E has the positive Schur property if $\|x_n\| \to 0$ for every positive weakly null sequence (x_n) in E (e.g., $L^1[0,1]$ has the positive Schur property). Also, it follows from [17] that the Banach lattice E has the positive Schur property if and only if the identity operator Id_E is almost Dunford-Pettis. Now, note that if E has the positive Schur property (i.e., Id_E is almost Dunford-Pettis), then every Banach-Saks set of E is L-weakly compact.

Motivated by this, we wonder if we can generalize this result by replace the identity operator with an almost Dunford-Pettis operator. The following example shows that an almost Dunford-Pettis operator between two Banach lattices does not necessarily map each Banach-Saks set into an L-weakly compact set.

Example 1. *Let $E = L^2[0,1]$ and $F = \ell^\infty$. According to [7, Example 3.1] there exists a positive almost Dunford-Pettis operator T from E into F such that $T(B_E)$ is not L-weakly compact, where B_E is a Banach-Saks set [4].*

Before giving a necessary and sufficient condition under which an order bounded almost Dunford-Pettis operator between two Banach lattices maps Banach-Saks sets into L-weakly compact sets, we need to recall the following definition.

An operator $T : X \to E$ is called *almost L-weakly compact* if T maps relatively weakly compact sets of X into L-weakly compact sets of E.

Note that the following result improves [5, Proposition 2.4].

Proposition 2. *Let E be a non zero Banach lattice and F be a Banach lattice. The following assertions are equivalent.*

1) *The norm of F is order continuous.*

2) *Every order bounded almost Dunford-Pettis operator $T : E \to F$ is almost L-weakly compact.*

3) *Every order bounded almost Dunford-Pettis operator $T : E \to F$ maps Banach-Saks sets into L-weakly compact sets.*

Proof. 1) $\Rightarrow$ 2) As the norm of F is order continuos, T is a regular almost Dunford-Pettis operator (see [1, Theorem 1.8]). Then it follows from [5, Propositions 2.1 and 2.4] that T is almost L-weakly compact.

2) $\Rightarrow$ 3) Since each Banach-Saks set of E is relatively weakly compact, every order bounded almost Dunford-Pettis operator maps Banach-Saks sets into L-weakly compact sets.

3) $\Rightarrow$ 1) Let $y \in F_+$. Fix a vector $x_0 \in E$ and $\psi \in E'$ such that $\psi(x_0) = \|x_0\| = 1$. Clearly, the operator $T : E \to F$ defined by $T(x) = \psi(x)y$ is regular (so order bounded) and it is almost Dunford-Pettis (it has rank one). On the other hand, since $A = \{x_0\}$ is a Banach-Saks set of E, $TA = \{y\}$ is L-weakly compact. Then, $\mathrm{Sol}\{y\} = [-y, y]$ is L-weakly compact (i.e., weakly compact) and hence the norm of F is order continuous (see [1, Theorem 4.9]). □

The preceding theorem is not true if T is not order bounded.

Example 2. *Let $(r_n)_n$ be the sequence of Rademacher functions on $[0, 1]$. Consider the operator $T : L^1[0, 1] \to c_0$ defined by*

$$Tf = \Big(\int_0^1 f(t)r_1(t)dt, \int_0^1 f(t)r_2(t)dt, \dots \Big),$$

for each $f \in L^1[0, 1]$. Clearly, T is a non order bounded almost Dunford-Pettis operator, but does not necessary maps Banach-Saks sets of $L^1[0, 1]$ into L-weakly compact sets of c_0. Indeed, since $L^1[0, 1]$ has the weak Banach-Saks property and the sequence $(r_n)_n$ is weakly null in $L^1[0, 1]$, $A = \{r_n : n \in \mathbb{N}\}$ is a Banach-Saks set of $L^1[0, 1]$. However, $TA = \{e_n : n \in \mathbb{N}\}$ is not an L-weakly compact set of c_0, where $e_n = (0, \dots, 0, 1_n, 0, \dots)$.

2 Banach lattices with order continuous norms

In this section, we give some characterizations of Banach lattices with order continuous norms using the Banach-Saks sets and Cesaro convergence. We start our discussion with the following useful lemma.

Lemma 3. *Let E be a Banach lattice and let $(x_n)_n$ be an increasing bounded sequence in E^+ such that $\left(\frac{1}{n}\sum_{k=1}^n x_k\right)_n$ is weakly convergent. Then $(x_n)_n$ is norm convergent.*

Proof. Let $(x_n)_n$ be an increasing order bounded sequence in E^+. According to [1, Theorem 4.9] and [15, Proposition 1.4.1], it suffices to show that $(x_n)_n$ is weakly convergent. Let $f \in (E')^+$. The sequence $(f(x_n))_n$ is a bounded and positive increasing sequence in $\mathbb{R}$, so there exists some $a \in \mathbb{R}$ such that $f(x_n) \to a$, and hence $\frac{1}{n}\sum_{k=1}^n f(x_k) \to a$. On the other hand, by our hypothesis, $\frac{1}{n}\sum_{k=1}^n f(x_k) \to f(x)$ for some $x \in E$. Then $f(x) = a$, consequently, $(x_n)_n$ is weakly convergent in E, as desired. □

In terms of the Cesaro convergence, the Banach lattices with order continuous norm are characterized as follows.

Proposition 3. *Let E be a Banach lattice. The following assertions are equivalent.*

1) *The norm of E is order continuous.*
2) *The Banach lattice E is Dedekind σ-complete, and $x_n \downarrow 0$ implies $\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)_n$ converges in norm to zero.*
3) *The sequence $\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)_n$ is norm convergent for every increasing order bounded sequence $(x_n)_n$ in E^+.*

Proof. *1) $\Rightarrow$ 2)* It follows from [1, Theorem 4.9].

2) $\Rightarrow$ 3) Let $(x_n)_n$ be an increasing order bounded sequence in E^+. As E is Dedekind σ-complete, then there exists an element $x \in E^+$ such that $\sup_n\{x_n\} = x$, i.e. $(x - x_n) \downarrow 0$, so by our hypothesis $\frac{1}{n}\sum_{k=1}^{n}(x - x_k) \to 0$. Therefore, the sequence $\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)_n$ is norm convergent.

3) $\Rightarrow$ 1) It follows from Lemma 3 and [1, Theorem 4.9]. □

Proposition 4. *Let E be a Banach lattice and X be a Banach space. The following assertions are equivalent.*

1) *An operator $T : E \to X$ is order weakly compact.*
2) *The operator T maps each order bounded subset of E into Banach-Saks sets of X.*

Proof. *2) $\Rightarrow$ 1)* It is obvious.

1) $\Rightarrow$ 2) Let A be an order bounded subset of E^+. Then there exists some $u \in E^+$ such that $T(A) \subset T([-u, u])$. On the other hand, as T is order weakly compact, it follows from [1, Theorem 5.58] that T admits a factorization through a Banach lattice F with order continuous norm, $T = S \circ Q$ such that $Q : E \to F$ is a lattice homomorphism operator and $S : F \to X$ is an operator. Since $Q[-u, u]$ is order bounded subset of F, by Corollary 1, $Q[-u, u]$ is Banach-Saks set of F, so $T(A) = S(Q(A))$ is Banach-Saks set of X. Thus, T maps each order bounded subset of E into Banach-Saks sets of X. □

Recall that a Banach lattice E is said to be an AM-space if for each $x, y \in E$ such that $\inf(x, y) = 0$ we have $\|x + y\| = \max\{\|x\|, \|y\|\}$.

Corollary 4. *Let E be an AM-space with order unit and X be a Banach space. The following assertions are equivalent.*

1) *An operator $T : E \to X$ is weakly compact.*
2) *The operator T is Banach-Saks.*

Let T, S be two operators from E into F such that $0 \leq S \leq T$ and T is Banach-Saks. The domination problem stands for finding certain conditions under which S is Banach-Saks. This problem of Banach-Saks operators has been studied by many authors (see [9, 10]). The next result ensure that Banach-Saks operators satisfy the domination problem, which is a consequence of the preceding result.

Corollary 5. *If E is an AM-space with order unit, then the Banach-Saks operators satisfy the domination problem.*

To continue our discussion, we need the following lemma.

Lemma 4. *Let $(x_n)_n$ be an increasing bounded sequence in a Banach lattice E. If there exists a subsequence $(x_{\phi(n)})_n$ of $(x_n)_n$ that converges in norm to some $x \in E$, then $(x_n)_n$ converges in norm to x.*

Proof. Let (x_n) be an increasing bounded sequence in E and let $f \in (E')^+$. So, $(f(x_n))_n$ converges to some $a \in \mathbb{R}$. By our hypothesis, $(f(x_{\phi(n)}))$ converges to $f(x)$ for some $x \in E$, then $a = f(x)$. Thus, $(x_n)_n$ converges weakly to x. The rest of the proof follows from [15, Proposition 1.4.1]. □

In terms of Cesaro convergence, the order weakly compact operators are characterized as follows.

Theorem 4. *Let T be an operator from a Banach lattice E into a Banach space X. The following assertions are equivalent.*

1) *The operator T is order weakly compact.*

2) *The sequence $\left(\frac{1}{n}\sum_{k=1}^{n} Tx_k\right)_n$ is norm convergent for every increasing order bounded sequence $(x_n)_n$ in E^+.*

Proof. 1) $\Rightarrow$ 2) It follows from [15, Theorem 3.4.4].

2) $\Rightarrow$ 1) Assume by way of contradiction that T is not order weakly compact. By [15, Corollary 3.4.5], there exists a sublattice H isomorphic to c_0 such that $T_{|H}$ is an isomorphism and B_H is order bounded in E. As H is not a KB-space, there exists an increasing positive sequence $(x_n) \in B_H$ (B_H is order bounded) such that $(x_n)_n$ does not have any subsequence convergent in norm (Lemma 4). On the other hand, by our hypothesis, any subsequence $(y_n)_n$ of $(x_n)_n$ satisfies that $\left(\frac{1}{n}\sum_{k=1}^{n} Ty_k\right)$ is norm convergent. This implies that $(T(x_n))_n$ is a Banach-Saks set of X, and so relatively weakly compact. Hence, there is a subsequence $(z_n)_n$ of $(x_n)_n$ such that $(T(z_n))_n$ is weakly convergent. Since $T_{|H}$ is an isomorphism, the sequence (z_n) is weakly convergent in H. Thus, by Dini's theorem, (z_n) is norm convergent. This is a contradiction, which completes the proof. □

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У роботі досліджуються множини Банаха-Сакса в ґратках Банаха, а також деякі наслідки, отримані з цього дослідження. Зокрема, ми встановлюємо, що кожна L-слабко компактна множина є множиною Банаха-Сакса. Крім того, наведено характеристику ґраток Банаха, у яких суцільна оболонка множини Банаха-Сакса також є множиною Банаха-Сакса. Далі ми наводимо необхідну й достатню умову, за якої обмежений за порядком оператор майже типу Данфорда-Петтіса, що діє між двома банаховими ґратками, відображає множини Банаха-Сакса в L-слабко компактні множини. Крім того, ми встановлюємо нову характеристику банахових ґраток із неперервною за порядком нормою та слабко компактних за порядком операторів, використовуючи поняття множини Банаха-Сакса та збіжності Чезаро.

Ключові слова і фрази: множина Банаха-Сакса, відносно слабко компактна множина, L-слабко компактна множина, слабко компактний за порядком оператор, оператор майже типу Данфорда-Петтіса, ґратка Банаха.