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Problem with integral conditions for a partial differential equation with Bessel operators

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In the rectangular domain we study a problem with integral conditions with respect to one of the variables for a partial differential equation with singular Bessel operators. A criterion for the unique solvability of this problem and sufficient conditions for the existence of its solution are established. To solve the problem of small denominators arising in some cases of the considered problem, we used the metric approach.

Key words and phrases: singular Bessel operator, Fourier-Bessel series, generalized hypergeometric function, integral condition, ill-posed problem, small denominator.

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Introduction

Classical boundary value problems for linear and some classes of quasi-linear partial differential equations and their direct generalizations are quite well studied in the theory of partial differential equations for the present. The studies of problems with nonlocal (including integral) conditions on the time variable for hyperbolic, parabolic or typeless evolutionary equations and systems of such equations are incomplete and insufficient.

Problems with integral conditions for the time variable for evolutionary equations are generally incorrect, their solvability in many cases is not stable with respect to small changes in the coefficients of the problem and domain parameters and is often associated with the problem of small denominators. A metric approach to solving the problem of small denominators turned out to be natural. In the works of B.Y. Ptashnyk [4] and his students, based on the metric approach, it is investigated the correct solvability of problems with nonlocal conditions in the time variable for linear and weakly nonlinear hyperbolic equations and for equations, which are not solved with respect to the higher time derivative, as well as for some differential-operator equations. Problems with integral conditions for linear partial differential equations with continuous coefficients using a metric approach were studied in [5]. Some direct generalizations of such problems were considered in [12].

Particularly relevant among problems with integral conditions are such problems for equations with degenerate or singular coefficients. An example of such equations are equations

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with singular Bessel operators, which appear in an investigation of various physical phenomena in cylindrical domains or with radial symmetry. A Cauchy problem for equations with singular Bessel operators (the Euler-Poisson-Darboux equation and its generalizations) was considered [13] (see also the bibliography there). In particular, boundary value problems involving integral conditions for equations with singular Bessel operators were investigated in [6–9].

In this article, we establish conditions of correct solvability of the mixed boundary problem with integral conditions in the form of arbitrary order moments of unknown function with respect to a chosen variable for the generalized Euler-Poisson-Darboux equation.

1 Preliminary statements

Basic notations. We use the following notations: $\mathbb{Z}_-$ and $\mathbb{Z}_+$ are sets of non-positive and non-negative integer numbers correspondingly; $\Gamma(x)$ is Euler gamma function, $(x)_k, k \in \mathbb{N}$, is the Pochhammer symbol of a real number x, i.e. $(x)_k = x(x+1)\cdots(x+(k-1))$; δ_{mn} is Kroneker delta of numbers n and m.

Bessel functions and Bessel differential operator. Bessel function of the first kind of index $\alpha \in \mathbb{R}$ is denoted $J_\alpha(x)$ and defined by power series of the form

$$J_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\alpha+1)} \left(\frac{x}{2}\right)^{2k+\alpha} = \left(\frac{x}{2}\right)^{\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\alpha+1)} \left(\frac{x}{2}\right)^{2k}. \tag{1}$$

The Bessel function (1) is non-periodic and has an infinite set of zeros. If $\alpha \geqslant -1$, then all zeros of the function $J_\alpha(x)$ are real and simple with the possible exception of $x = 0$. If $\alpha > -\frac{1}{2}$ and $\alpha \notin \mathbb{Z}$, then all zeros are non-negative and in the case of an integer α zeros are symmetric with respect to the point $x = 0$.

Further by $j_{\alpha,k}$ we denote the kth by number (in ascending order) the positive zero of the function $J_\alpha(x)$, $\alpha > -\frac{1}{2}$.

Bessel function $J_\alpha(x)$ satisfies the following asymptotic equalities:

$$J_\alpha(x) = \sqrt{\frac{2}{\pi x}} \left(\cos\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O(x^{-1}) \right), \quad x \to +\infty, \tag{2}$$

$$J_\alpha(x) = \frac{1}{\Gamma(\alpha+1)} \left(\frac{x}{2}\right)^{\alpha} + O(x^{\alpha+2}), \quad x \to 0. \tag{3}$$

Also if $\alpha \geqslant -\frac{1}{2}$, then for all $x > 0$ we have

$$|J_\alpha(\lambda x)| \leqslant \frac{(\lambda x)^\alpha}{2^\alpha \Gamma(\alpha+1)}. \tag{4}$$

From (2) it follows the following asymptotic estimate

$$j_{\alpha,k} = \pi k + \frac{\alpha\pi}{2} + \frac{\pi}{4} + O(k^{-1}), \quad k \to +\infty, \tag{5}$$

for the zeros of the Bessel function.

Note that Bessel functions $J_\alpha(j_{\alpha,k}x)$, $k \in \mathbb{N}$, are orthogonal on the interval $[0,1]$ with weight x, i.e.

$$\int_0^1 x J_\alpha(j_{\alpha,m}x) J_\alpha(j_{\alpha,n}x)dx = \frac{J^2_{\alpha+1}(j_{\alpha,n})}{2}\delta_{mn}. \tag{6}$$

The differential operator, given by

$$\mathcal{B}_\alpha\left(\frac{d}{dx}\right) := \frac{1}{x^{2\alpha+1}}\frac{d}{dx}\left(x^{2\alpha+1}\frac{d}{dx}\right) = \frac{d^2}{dx^2} + \frac{2\alpha+1}{x}\frac{d}{dx}, \tag{7}$$

where $\alpha \in \mathbb{R}$ is numerical parameter, is called singular Bessel operator. The general solution for the equation

$$\mathcal{B}_\alpha\left(\frac{d}{dx}\right)y(x) + \lambda^2 y(x) = 0, \quad \lambda \in \mathbb{R}, \tag{8}$$

in case of noninteger α, $\alpha \notin \mathbb{Z}$, $2\alpha+1 \geqslant 0$, is given (see [1, §9.1.52]) by the formula

$$y(x) = C_1\, x^{-\alpha} J_\alpha(\lambda x) + C_2\, x^{-\alpha} J_{-\alpha}(\lambda x), \tag{9}$$

where $J_\alpha(x)$ is a Bessel function of the first kind. In case $2\alpha+1=0$, Bessel operator (7) turns to ordinary second derivative and functions $x^{-\alpha}J_\alpha(\lambda x)$, $x^{-\alpha}J_{-\alpha}(\lambda x)$ turn into usual cosine and sine functions, namely $x^{\frac{1}{2}}J_{-\frac{1}{2}}(\lambda x) = \sqrt{\frac{2}{\pi\lambda}}\cos(\lambda x)$, $x^{\frac{1}{2}}J_{\frac{1}{2}}(\lambda x) = \sqrt{\frac{2}{\pi\lambda}}\sin(\lambda x)$.

From (3), (6) and (9) it follows that the set of functions

$$y_k(x) = \frac{J_\alpha(\lambda_k x)}{x^\alpha}, \quad \lambda_k = j_{\alpha,k}, \quad k \in \mathbb{N}, \tag{10}$$

coincides with the set of nontrivial solutions of the Sturm-Liouville problem for equation (8) with boundary conditions

$$y'(0) = 0, \quad y(1) = 0, \tag{11}$$

and $y_k(x)$, $k \in \mathbb{N}$, are orthogonal on interval $[0,1]$ with weight $x^{2\alpha+1}$.

For further information on Bessel functions, we refer the reader to [1,3].

Hypergeometric function ${}_1F_2$. Hypergeometric function ${}_1F_2(a;b,c;z)$ is defined by power series

$${}_1F_2(a;b,c;z) = \sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k(c)_k}\cdot\frac{z^k}{k!}, \quad b,c \notin \mathbb{Z}_-, \quad z \in \mathbb{C}, \tag{12}$$

and is analytical function of $z \in \mathbb{C}$ (see [2]). Some partial cases of function (12) is given below:

$${}_1F_2\left(\frac{5}{2};\frac{1}{2},\frac{3}{2};z^2\right) = \frac{2}{3}z\,\sinh(2z) + \cosh(2z), \quad {}_1F_2\left(b;b,\alpha+1;-\frac{z^2}{4}\right) = \frac{2^\alpha\Gamma(b+1)}{z^\alpha}J_\alpha(z).$$

Hypergeometric function ${}_1F_2(a;b,c;x)$, $x \in \mathbb{R}$, has the following asymptotic expansion [10], which is written here in trigonometric form

$$\begin{aligned}{}_1F_2(a;b,c;x) =& \frac{\Gamma(b)\Gamma(c)}{\Gamma(b-a)\Gamma(c-a)}(-x)^{-a}\left(1+O\left(\frac{1}{x}\right)\right)\\ &+\frac{\Gamma(b)\Gamma(c)}{\sqrt{\pi}\Gamma(a)}(-x)^{\eta}\cos\left(2\sqrt{-x}+\pi\eta\right)\left(1+O\left(\frac{1}{\sqrt{-x}}\right)\right)\\ &+\frac{K}{16}\frac{\Gamma(b)\Gamma(c)}{\sqrt{\pi}\Gamma(a)}(-x)^{\eta-\frac{1}{2}}\sin\left(2\sqrt{-x}+\pi\eta\right)\left(1+O\left(\frac{1}{\sqrt{-x}}\right)\right), \quad x\to-\infty,\end{aligned} \tag{13}$$

where

$$\eta = \frac{1}{2}\left(a-b-c+\frac{1}{2}\right), \quad K = 4(3a^2-b^2-c^2) + 8(bc+(1-a)(b+c)-a) - 3.$$

2 Statement of the problem

In the domain $\mathcal{D} := \{(t,x) : t \in (0,T), x \in (0,L)\}$, we consider the following problem:

$$\mathcal{B}_\nu\left(\frac{\partial}{\partial t}\right) u(t,x) = a^2 \mathcal{B}_\mu\left(\frac{\partial}{\partial x}\right) u(t,x), \quad (t,x) \in \mathcal{D}, \tag{14}$$

$$\int_0^T t^{r_1} u(t,x)dt = \varphi_1(x), \quad \int_0^T t^{r_2} u(t,x)dt = \varphi_2(x), \quad x \in (0,L), \tag{15}$$

$$\left.\frac{\partial u(t,x)}{\partial x}\right|_{x=0} = 0, \quad u(t,L) = 0, \quad t \in (0,T), \tag{16}$$

where $a > 0$, $\nu, \mu \geqslant -\frac{1}{2}$, $\mu, \nu \notin \mathbb{Z}$, $r_1, r_2 \in \mathbb{R}$, $r_2 > r_1$; φ_1, φ_2 are given functions, operators $\mathcal{B}_\mu, \mathcal{B}_\nu$ are Bessel operators defined by (7) for $\alpha \in \{\mu, \nu\}$.

The equation (14) is called a generalized Euler-Poisson-Darboux equation. Sometimes it is called $\mathcal{B}$-hyperbolic equation also. For $\nu = \mu = -\frac{1}{2}$ the equation (14) turns to the usual wave equation.

3 Construction of a solution to the problem

We will seek a solution to the problem (14)–(16) using the method of separation of variables. For the variable x, we obtain the following boundary value problem with respect to the function $w := w(x)$:

$$\mathcal{B}_\mu\left(\frac{\partial}{\partial x}\right) w + \lambda^2 w = 0, \quad x \in (0,L), \tag{17}$$

$$w'(0) = 0, \quad w(L) = 0. \tag{18}$$

From (6), (8)–(11) we get that the functions

$$w_{\mu,k}(x) = \frac{\sqrt{2}}{L J_{\mu+1}(j_{\mu,k})} \cdot \frac{J_\mu(\lambda_{\mu,k} x)}{x^\mu}, \quad \lambda_{\mu,k} = \frac{j_{\mu,k}}{L}, \quad k = 1, 2, \ldots, \tag{19}$$

are nontrivial solutions of the problem (17), (18) and orthonormal on interval $(0,L)$ with weight $x^{2\mu+1}$.

Solution of the original problem (14)–(16) we seek in the form of series

$$u(t,x) = \sum_{k=1}^{\infty} u_k(t) w_{\mu,k}(x), \tag{20}$$

where $w_{\mu,k}(x)$ are functions defined by the formula (19). After substituting the series (20) in (14), (15), we get the following problem for determining unknown functions $u_k(t)$, $k \in \mathbb{N}$:

$$\mathcal{B}_\nu\left(\frac{d}{dt}\right) u_k + a^2 \lambda_{\mu,k}^2 u_k = 0, \quad t \in (0,T), \tag{21}$$

$$\int_0^T t^{r_1} u_k(t)dt = \varphi_{1k}, \quad \int_0^T t^{r_2} u_k(t)dt = \varphi_{2k}, \quad r_1, r_2 \in \mathbb{R}, \tag{22}$$

where φ_{jk} are Fourier-Bessel coefficients of the functions $\varphi_1(x)$, $\varphi_2(x)$, namely

$$\varphi_{jk} = \int_0^L x^{2\mu+1} \varphi_j(x) w_{\mu,k}(x) dx, \quad j = 1, 2. \tag{23}$$

Based on (8), (9) the general solution of the equation (21) is given by the formula

$$u_k(t) = C_{1k}\, t^{-\nu} J_\nu(a\lambda_{\mu,k} t) + C_{2k}\, t^{-\nu} J_{-\nu}(a\lambda_{\mu,k} t), \tag{24}$$

where C_{1k}, C_{2k} are arbitrary constants.

Let us denote

$$I_{1k}(r,\nu,T) := \int_0^T t^{r-\nu} J_\nu(a\lambda_{\mu,k}t)dt, \quad I_{2k}(r,\nu,T) := \int_0^T t^{r-\nu} J_{-\nu}(a\lambda_{\mu,k}t)dt, \quad r \in \mathbb{R}. \tag{25}$$

If $r > \max\{-1, 2\nu - 1\}$, then from (3) it follows that each of integrals (25) converge and can be expressed in terms of a hypergeometric function ${}_1F_2$ as follows:

$$I_{1k}(r,\nu,T) = \frac{a^\nu \lambda_{\mu,k}^\nu T^{r+1}}{2^\nu (r+1)\Gamma(\nu+1)} {}_1F_2\left(\frac{r+1}{2}; \nu+1, \frac{r+3}{2}; -\left(\frac{a\lambda_{\mu,k}T}{2}\right)^2\right), \tag{26}$$

$$I_{2k}(r,\nu,T) = \frac{2^\nu a^{-\nu} \lambda_{\mu,k}^{-\nu} T^{r-2\nu+1}}{(r-2\nu+1)\Gamma(1-\nu)} {}_1F_2\left(\frac{r-2\nu+1}{2}; 1-\nu, \frac{r-2\nu+3}{2}; -\left(\frac{a\lambda_{\mu,k}T}{2}\right)^2\right). \tag{27}$$

The formulas (26) and (27) can be easily verified by considering the series for the right and the left sides of the corresponding equalities.

Under conditions

$$r_j > \max\{-1, 2\nu - 1\}, \quad j = 1, 2, \tag{28}$$

by substituting (24) in the conditions (22), we obtain the following linear system of equations for determining C_{1k}, C_{2k}:

$$C_{1k} I_{1k}(r_j, \nu, T) + C_{2k} I_{2k}(r_j, \nu, T) = \varphi_{jk}, \quad j = 1, 2. \tag{29}$$

By $\Delta(\lambda_{\mu,k}, T)$, $k \in \mathbb{N}$, we denote the determinant of the system (29), namely

$$\begin{aligned}\Delta(\lambda_{\mu,k}, T) :&= \begin{vmatrix} \int_0^T t^{r_1-\nu} J_\nu(a\lambda_{\mu,k}t)dt & \int_0^T t^{r_1-\nu} J_{-\nu}(a\lambda_{\mu,k}t)dt \\ \int_0^T t^{r_2-\nu} J_\nu(a\lambda_{\mu,k}t)dt & \int_0^T t^{r_2-\nu} J_{-\nu}(a\lambda_{\mu,k}t)dt \end{vmatrix} \\ &= I_{1k}(r_1,\nu,T) I_{2k}(r_j,\nu,T) - I_{1k}(r_2,\nu,T) I_{2k}(r_1,\nu,T).\end{aligned} \tag{30}$$

If condition

$$\forall k \in \mathbb{N} \quad \Delta(\lambda_{\mu,k}, T) \neq 0 \tag{31}$$

holds, then there exists only one solution of the system (29) given by the formulas

$$C_{1k} = \frac{I_{2k}(r_2,\nu,T)\varphi_{1k} - I_{2k}(r_1,\nu,T)\varphi_{2k}}{\Delta(\lambda_{\mu,k},T)}, \quad C_{2k} = \frac{I_{1k}(r_1,\nu,T)\varphi_{2k} - I_{1k}(r_2,\nu,T)\varphi_{1k}}{\Delta(\lambda_{\mu,k},T)}, \tag{32}$$

and the problem (21)–(22) has the unique solution (24), where C_{1k}, C_{2k} are definded by (32).

Summing up, under the conditions (28), (31), the formal solution of the problem (14)–(16) is represented as a series

$$u(t,x) = \sum_{k=1}^\infty \left(\sum_{j,q=1}^2 \frac{(-1)^{j+q+1} I_{qk}(r_{3-j},\nu,T)}{\Delta(\lambda_{\mu,k},T)} \varphi_{jk}\, t^{-\nu} J_{(-1)^q\nu}(a\lambda_{\mu,k}t) \right) w_{\mu,k}(x). \tag{33}$$

The formula (33) along with (19), (26), (27), (30) represents the exact solution of the problem (14)–(16) in the form of a series through Bessel functions of the first kind and a hypergeometric function ${}_1F_2$. If $\nu = \frac{2n+1}{2}$, n, r_1, $r_2 \in \mathbb{N}$, then under the condition (28) the determinant $\Delta(\lambda_{\mu,k}, T)$ and the solution $u(t,x)$ can be expressed in terms of elementary functions.

4 Uniqueness and existence of a solution

To establish conditions of uniqueness and existence of a solution to the problem (14)–(16), we describe corresponding functional spaces. Also further we suppose that condition (28) is always true.

By $\mathcal{P}_\mu$, $\mu \geqslant -1/2$, we denote the space of finite sums $p(x) = \sum_k p_k w_{\mu,k}(x)$, $p_k \in \mathbb{R}$, $k \in \mathbb{N}$, $x \in [0, L]$, where $w_{\mu,k}(x)$ are defined by (19). Sequence $\{p^n\}_{n=1}^\infty$ we call convergent in $\mathcal{P}_\mu$ to the element $v \in \mathcal{P}_\mu$ if:

1) $\exists n_0, N \in \mathbb{N}, \quad \forall n > n_0, \quad p^n(x) = \sum_{k \leqslant N} p_k^n \, w_{\mu,k}(x)$,

2) $\forall k \in \mathbb{N}, \quad k \leqslant N, \quad p_k^n \underset{n\to\infty}{\to} p_k$.

Let $\mathcal{P}'_\mu$, $\mu \geqslant -1/2$, be the space of formal series $p(x) = \sum_{k\in\mathbb{N}} p_k w_{\mu,k}(x)$, which coincides with the space of all linear functionals over $\mathcal{P}_\mu$. The action of a functional $p \in \mathcal{P}'_\mu$ on an element $v \in \mathcal{P}_\mu$ is given by the formula $(p, v) = \int_0^L p(x)v(x)x^{2\mu+1}dx = \sum_k p_k v_k$.

Let $\mathcal{PH}_\mu^\beta[0, L]$, $\mu \geqslant -1/2$, $\beta \in \mathbb{R}$, be the completion of the space $\mathcal{P}_\mu$ with respect to the norm

$$\|p\|_{\mathcal{PH}_\mu^\beta[0,L]} = \left(\sum_{k=1}^{\infty} |p_k|^2 \lambda_{\mu,k}^{2\beta} \right)^{\frac{1}{2}}.$$

The space $\mathcal{PH}_\mu^0[0, L]$ coincides with the $x^{2\mu+1}$-weighted $L^2[0, L]$ space of functions $f(x)$ defined on the interval $[0, L]$ such that $f(L) = 0$.

Let $C_\omega^n\big([0,T]\,;\mathcal{P}_\mu\big)$ $\big($respectively $C_\omega^n\big([0,T]\,;\mathcal{P}'_\mu\big)\big)$, $n \in \mathbb{Z}_+$, $\omega, T \in \mathbb{R}_+$, be the space of functions $u(t, x) = \sum_k u_k(t)w_{\mu,k}(x)$, where $t^{\omega+j}u_k^{(j)} \in C\,[0, T]$, $k \in \mathbb{N}$, such that for all fixed $t \in [0, T]$ derivatives $t^{\omega+j}\partial_t^j u = \sum_{k\in\mathbb{N}} t^{\omega+j}u_k^{(j)}(t)w_{\mu,k}(x)$, $j = 0, 1, \ldots, n$, belong to the space $\mathcal{P}_\mu$ $\big($respectively $\big(\mathcal{P}'_\mu\big)\big)$.

From definition of the spaces $C_\omega^n\big([0,T]\,;\mathcal{P}_\mu\big)$ and $C_\omega^n\big([0,T]\,;\mathcal{P}'_\mu\big)$ it follows that the embeddings $C_{\omega_1}^n\big([0,T]\,;\mathcal{P}_\mu\big) \subset C_{\omega_2}^n\big([0,T]\,;\mathcal{P}_\mu\big)$, $C_{\omega_1}^n\big([0,T]\,;\mathcal{P}'_\mu\big) \subset C_{\omega_2}^n\big([0,T]\,;\mathcal{P}'_\mu\big)$ hold, if $\omega_1 > \omega_2$.

By $C_\omega^n([0,T], \mathcal{PH}_\mu^\beta)$, $n \in \mathbb{Z}_+$, $T \in \mathbb{R}_+$, we denote the space of functions of view $u(t, x) = \sum_{k=1}^\infty u_k(t)w_{\mu,k}(x)$, where $t^{\omega+j}u_k^{(j)}(t) \in C[0, T]$, $j = 0, 1, \ldots, n$, such that, for every fixed $t \in [0, T]$, the derivatives $t^{\omega+j}\partial_t^j u(t, x) = \sum_{k=1}^\infty t^{\omega+j}u_k^{(j)}(t)w_{\mu,k}(x)$, $j = 0, 1, \ldots, n$, belong to the space $\mathcal{PH}_\mu^\beta$ and are continuous on $[0, T]$ with respect to t in the norm

$$\|u\|_{C_\omega^n([0,T],\mathcal{PH}_\mu^\beta)} = \sum_{j=0}^{n} \sup_{t\in[0,T]} \left\| t^{\omega+j}\partial_t^j u(t, x) \right\|_{\mathcal{PH}_\mu^\beta}.$$

Let us denote

$$u_{1k}(\nu, t) := t^{-\nu}J_{-\nu}(a\lambda_{\mu,k}t), \qquad u_{2k}(\nu, t) := t^{-\nu}J_\nu(a\lambda_{\mu,k}t). \tag{34}$$

Lemma 1. *For all $n \in \mathbb{Z}_+$ and all sufficiently large k functions $u_{qk}(\nu, t)$, $\nu > -\frac{1}{2}$, $q = 1, 2$, satisfy estimates*

$$\left|u_{2k}^{(2n+1)}(\nu, t)\right| < C_1|\lambda_{\mu,k}|^{2n+\nu+2}\, t, \quad \left|u_{2k}^{(2n)}(\nu, t)\right| < C_2|\lambda_{\mu,k}|^{2n+\nu}, \quad t \geqslant 0, \tag{35}$$

$$\left|u_{1k}^{(2n+1)}(\nu, t)\right| < C_3|\lambda_{\mu,k}|^{2n-\nu} \max\left\{ \frac{1}{t^{2\nu+1}}, \frac{1}{t^{2\nu+2n+1}} \right\}, \quad t > 0, \tag{36}$$

$$\left|u_{1k}^{(2n)}(\nu, t)\right| < C_4|\lambda_{\mu,k}|^{2n-\nu} \max\left\{ \frac{1}{t^{2\nu}}, \frac{1}{t^{2\nu+2n}} \right\}, \quad t > 0, \tag{37}$$

where C_j, $j = 1, \ldots, 4$, are constants depending only on n, α, a, ν.

Proof. Based on (1) and (8), by mathematical induction, it is easy to show the following formulas of differentiation (see also [11]):

$$\frac{d^{2n+1}}{dx^{2n+1}}\left(\frac{J_\alpha(\lambda x)}{x^\alpha}\right)=\sum_{k=0}^{n}(-1)^{n+k+1}\,C_n^k\,\lambda^{2n-k+1}(2\alpha+1)^k\,\frac{J_{\alpha+k+1}(\lambda x)}{x^{\alpha+k}}, \tag{38}$$

$$\frac{d^{2n}}{dx^{2n}}\left(\frac{J_\alpha(\lambda x)}{x^\alpha}\right)=\sum_{k=0}^{n}(-1)^{n+k}\,C_n^k\,\lambda^{2n-k}(2\alpha+1)^k\,\frac{J_{\alpha+k}(\lambda x)}{x^{\alpha+k}}, \tag{39}$$

$$\frac{d^{2n+1}}{dx^{2n+1}}\left(\frac{J_{-\alpha}(\lambda x)}{x^\alpha}\right)=(-1)^n\sum_{k=0}^{n}C_n^k\,\lambda^{2n-k+1}(2\alpha+1)^k\,\frac{J_{-\alpha-k-1}(\lambda x)}{x^{\alpha+k}}, \tag{40}$$

$$\frac{d^{2n}}{dx^{2n}}\left(\frac{J_{-\alpha}(\lambda x)}{x^\alpha}\right)=(-1)^n\sum_{k=0}^{n}C_n^k\,\lambda^{2n-k}(2\alpha+1)^k\,\frac{J_{-\alpha-k}(\lambda x)}{x^{\alpha+k}}, \tag{41}$$

where C_n^k are binomial coefficients, $\alpha \geqslant -\frac{1}{2}$, $n \in \mathbb{Z}_+$. From (4), (34) and (40) for the function $u_{1k}(\nu,t)$, $\nu > -\frac{1}{2}$, we obtain that

$$\begin{aligned}\left|u_{1k}^{(2n+1)}(\nu,t)\right| &= \left|\sum_{k=0}^{n}C_n^k\,(a\lambda_{\mu,k})^{2n-k+1}(2\nu+1)^k\,\frac{J_{-\nu-k-1}(a\lambda_{\mu,k}t)}{t^{\nu+k}}\right|\\ &\leqslant \sum_{k=0}^{n}C_n^k\,a^{2n-k+1}(2\nu+1)^k|\lambda_{\mu,k}|^{2n-k+1}\left|\frac{2^{\nu+k+1}(a\lambda_{\mu,k})^{-\nu-k-1}}{\Gamma(-\nu-k)t^{2\nu+2k+1}}\right|.\end{aligned} \tag{42}$$

By the statement of the problem (14)–(16), we have $a > 0$ and $2\nu + 1 > 0$. Since $\nu \notin \mathbb{Z}$, we obtain $\Gamma(-\nu - k) \neq 0$ for all integer k. Also taking into account that $t > 0$ and $\lambda_{\mu,k} > 1$ (which follows from (5)), for sufficiently large k from (42) it follows the estimate

$$\left|u_{1k}^{(2n+1)}(\nu,t)\right| \leqslant C_5|\lambda_{\mu,k}|^{2n-\nu}\max\left\{\frac{1}{t^{2\nu+1}},\frac{1}{t^{2\nu+2n+1}}\right\}, \tag{43}$$

where

$$C_5=\frac{(n+1)2^{2n+\alpha+1}\max\{1,a^{2n-\nu}\}\max\{1,(2\nu+1)^n\}}{\min\limits_{k\in\{0,1,\ldots,n\}}\{\Gamma(-k-\nu)\}},$$

which holds for all $t > 0$. Estimate (43) coincides with (36), if we put $C_3 = C_5$. In the same way, based on formulas (4), (38), (39) and (41), we get that estimates (35) and (37) are true. □

Based on Lemma 1 and taking into account that

$$u_{1k}(-\tfrac{1}{2},t)=\sqrt{\frac{2}{a\pi\lambda_{\mu,k}}}\sin(a\lambda_{\mu,k}t),\quad u_{2k}(-\tfrac{1}{2},t)=\sqrt{\frac{2}{a\pi\lambda_{\mu,k}}}\cos(a\lambda_{\mu,k}t),$$

it is easy to prove following result.

Corollary 1. *For any fixed* $\nu \geqslant -\frac{1}{2}$, *functions* $t^{|\nu|+\nu+j}u_{qk}^{(j)}(t)$, *where* $u_{qk}(t)$, $q = 1, 2$, *are defined by* (34), *belong to space* $C[0, T]$ *for arbitrary* $j \in \mathbb{Z}_+$.

Proposition 1. *The problem* (14)–(16) *has no more than one solution in the space* $C^n_\omega\big([0,T];\mathcal{P}_\mu\big)$ $\big(C^n_\omega\big([0,T];\mathcal{P}'_\mu\big)\big)$, $\omega \geqslant |\nu|+\nu$, *if and only if condition* (31) *holds.*

Proof. Necessity. Assume the opposite. Let the condition (31) not be fulfilled, that is, there exists $k_0 \in \mathbb{N}$ such that the determinant $\Delta(\lambda_{\mu,k_0}, T)$ is equal to zero. Then there exist nontrivial solutions $u_{k_0}(t) = \sum_{q=1}^{2} C_{qk_0} u_{q,k_0}(\nu,t)$ of the homogeneous problem (21), (22), given by (24), where the constants C_{qk_0}, $q \in 1,2$, are determined from the system $\sum_{q=1}^{2} C_{qk_0} I_{qk_0}(r_j,\nu,T) = 0$, $j \in 1,2$, whose determinant coincides with $\Delta(\lambda_{\mu,k_0}, T)$. Therefore, the homogeneous problem (21), (22) has nontrivial solutions $u_0(t,x) = u_{k_0}(\nu,t) w_{\mu,k_0}(x)$. From Corollary 1 it follows that $u_0(t,x) \in C^n_\omega\big([0,T];\mathcal{P}_\mu\big)$ $\big(u_0(t,x) \in C^n_\omega\big([0,T];\mathcal{P}'_\mu\big)\big)$ and a solution of the problem (14)–(16), if it exists, will not be unique.

Sufficiency. Let condition (31) holds. Assume that there are two solutions $u_1(t,x), u_2(t,x)$ of the problem (14)–(16) from the space $C^n_\omega\big([0,T];\mathcal{P}_\mu\big)$ $\big(C^n_\omega\big([0,T];\mathcal{P}'_\mu\big)\big)$. Then the function $\tilde{u}(t,x) = u_2(t,x) - u_1(t,x)$ will be the solution of the problem (14)–(16) from the space $C^n_\omega\big([0,T];\mathcal{P}_\mu\big)$ $\big(C^n_\omega\big([0,T];\mathcal{P}'_\mu\big)\big)$, where $\varphi_1(x) \equiv \varphi_2(x) \equiv 0$. Note that each of the coefficient $\tilde{u}_k(t)$, $k \in \mathbb{N}$, of the function $\tilde{u}(t,x)$ is the solution of the homogeneous problem (21), (22). Since $\Delta(\lambda_{\mu,k}, T) = 0$ for all $k \in \mathbb{N}$, the homogeneous problem (21), (22) has only trivial solutions for all $k \in \mathbb{N}$, and so $\tilde{u}_k(t) \equiv 0$, $k \in \mathbb{N}$. Hence, we conclude that $\tilde{u}(t,x) \equiv 0$ in the space $C^n_\omega\big([0,T];\mathcal{P}_\mu\big)$ $\big(C^n_\omega\big([0,T];\mathcal{P}'_\mu\big)\big)$, that is, $u_1(t,x) \equiv u_2(t,x)$. □

Proposition 2. *Let the condition* (31) *holds. If* $\varphi_1(x), \varphi_2(x) \in \mathcal{P}_\mu$ $\big(\mathcal{P}'_\mu\big)$, *then there exists only one solution of the problem* (14)–(16) *from the space* $C^n_\omega\big([0,T];\mathcal{P}_\mu\big)$ $\big(C^n_\omega\big([0,T];\mathcal{P}'_\mu\big)\big)$, $\omega \geqslant |\nu|+\nu$. *This solution is given by the formula* (33).

Proof. The proof follows from the construction of formula (33) and the statements of Corollary 1 and Propostion 1. □

The issue of the existence of a solution to the problem (14)–(16) in the scale of spaces $C^n_\omega\big([0,T],\mathcal{PH}^\beta_\mu\big)$, $\beta \in \mathbb{R}$, is, in many cases, closely related to the small denominator problem (see [15, Example 1.4], where the case $\nu = \mu = -\frac{1}{2}$, $r_1 = 0$, and $r_2 = 0$ is considered). Indeed, although the quantities $|\Delta(\lambda_{\mu,k}, T)|$, $k \in \mathbb{N}$, are nonzero, they may become arbitrarily small for infinitely many values of $\lambda_{\mu,k}$. As a consequence, the series (33) may diverge in the norm of the corresponding space.

Lemma 2. *Let conditions* (28) *and* (31) *hold. Then, for every* $k \in \mathbb{N}$, *the integrals* $I_{qk}(r_j,\nu,T)$, $j,q = 1,2$, *satisfy the inequalities*

$$\big|I_{qk}(r_j,\nu,T)\big| \leq C_6 \lambda_{\mu,k}^{|\nu|}, \quad j,q = 1,2,$$

where C_6 *is a positive constant independent of* k.

Proof. Based on the formula (4), we obtain the following inequalities:

$$\begin{aligned} \left|\int_0^T t^{r-\nu} J_\nu(\lambda_{\mu,k} t)\,dt\right| &\leqslant \int_0^T t^{r-\nu} \big|J_\nu(\lambda_{\mu,k} t)\big|\,dt \\ &\leqslant \frac{(\lambda_{\mu,k})^\nu}{2^\nu \Gamma(\nu+1)} \int_0^T t^r\,dt = \frac{T^{r+1}}{2^\nu (r+1)\Gamma(\nu+1)} (\lambda_{\mu,k})^\nu, \end{aligned} \tag{44}$$

$$\left|\int_0^T t^{r-\nu} J_{-\nu}(\lambda_{\mu,k} t)\,dt\right| \leqslant \frac{2^\nu T^{r-2\nu+1}}{(r-2\nu+1)\Gamma(1-\nu)} (\lambda_{\mu,k})^{-\nu}. \tag{45}$$

Let us denote $C(r,\nu) := \max\left\{\frac{T^{r+1}}{2^{\nu}(r+1)\Gamma(\nu+1)}, \frac{2^{\nu}T^{r-2\nu+1}}{(r-2\nu+1)\Gamma(1-\nu)}\right\}$. Then from the formulas (44), (45) it follows

$$|I_{qk}(r_j,\nu,T)| \leqslant \max\left\{C(r_2,\nu), C(r_1,\nu)\right\} \max\left\{(\lambda_{\mu,k})^{\nu}, (\lambda_{\mu,k})^{-\nu}\right\} = C_7(\lambda_{\mu,k})^{|\nu|}, \quad q,j=1,2,$$

where $C_7 = \max\{C(r_2,\nu), C(r_1,\nu)\}$. The lemma is proved. □

Theorem 1. *Let conditions* (28) *and* (31) *hold, and suppose that there exist real constants* $\eta > 0$ *and* $C_8 > 0$ *such that, for all but finitely many* $k \in \mathbb{N}$, *the following inequality holds*

$$|\Delta(\lambda_{\mu,k},T)| > C_8\lambda_{\mu,k}^{-\eta}. \tag{46}$$

If $\varphi_1, \varphi_2 \in \mathcal{PH}_{\mu}^{|\nu|+\nu+\eta+\beta+n+1}[0,L]$, *then there exists a unique solution* $u \in C^{n}_{|\nu|+\nu}\big([0,T], \mathcal{PH}^{\beta}_{\mu}\big)$ *to the problem* (14)–(16), *represented by the series* (33). *Moreover,*

$$\|u\|_{C^{n}_{|\nu|+\nu}([0,T],\mathcal{PH}^{\beta}_{\mu})} \leq C_9 \sum_{j=1}^{2} \|\varphi_j\|_{\mathcal{PH}_{\mu}^{|\nu|+\nu+\eta+\beta+n+1}[0,L]}.$$

Proof. First, for convenience, we introduce the notation

$$u_k(t) = \sum_{j,q=1}^{2} \frac{(-1)^{j+q+1} I_{qk}(r_{3-j},\nu,T)}{\Delta(\lambda_{\mu,k},T)} \varphi_{jk} u_{qk}(\nu,t), \quad k \in \mathbb{N}, \tag{47}$$

where the functions $u_{qk}(\nu,t)$, $q=1,2$, are given by formula (34). Now, we evaluate the norm of the obtained solution (33) in the space $C^{n}_{2\nu}\big([0,T], \mathcal{PH}^{\beta}_{\mu}\big)$, namely

$$\|u\|_{C^{n}_{|\nu|+\nu}([0,T],\mathcal{PH}^{\beta}_{\mu}[0,L])} = \sum_{j=0}^{n} \sup_{t\in[0,T]} \left\| t^{|\nu|+\nu+j} u_k^{(j)}(\nu,t)\right\|_{\mathcal{PH}^{\beta}_{\mu}[0,L]}, \tag{48}$$

where $u_k(t)$ is defined by the formula (47).

For the functions (47), based on the formulas (35)–(37), for all $j=0,1,\ldots,n$, we get the following inequalities

$$\sup_{t\in[0,T]} \left|t^{|\nu|+\nu+j} u_{qk}^{(j)}(\nu,t)\right| \leqslant C_{10}(\lambda_{\mu,k})^{\nu+n+1}, \qquad C_{10} = \max_{j=1,2,3,4}\{C_j\} \max\left\{1, T^{n+|\nu|+\nu+1}\right\}. \tag{49}$$

From the formula (47), taking into account the inequalities (46), (49) and Lemma 2, we obtain

$$\sup_{t\in[0,T]} \left|t^{|\nu|+\nu+j} u_{k}^{(j)}(\nu,t)\right| \leqslant C_{11}\big(|\varphi_{1k}|+|\varphi_{2k}|\big)(\lambda_{\mu,k})^{|\nu|+\nu+\eta+n+1}, \qquad C_{11} = \frac{4C_6C_{10}}{C_8}. \tag{50}$$

Then, based on (48), (50), we get

$$\begin{aligned}
\|u\|_{C^{n}_{|\nu|+\nu}([0,T],\mathcal{PH}^{\beta}_{\mu})} &\leqslant \sum_{j=0}^{n}\left(\sum_{k=1}^{\infty}\sup_{t\in[0,T]}\left|t^{|\nu|+\nu+j}u_{qk}^{(j)}(\nu,t)\right|^2(\lambda_{\mu,k})^{2\beta}\right)^{1/2} \\
&\leqslant (n+1)C_{11}\left(\sum_{k=1}^{\infty}(|\varphi_{1k}|+|\varphi_{2k}|)^2(\lambda_{\mu,k})^{2(|\nu|+\nu+\eta+\beta+n+1)}\right)^{1/2} \\
&= C_{12}\sum_{j=1}^{2}\|\varphi_j\|_{\mathcal{PH}_{\mu}^{|\nu|+\nu+\eta+\beta+n+1}[0,L]},
\end{aligned}$$

where $C_{12} = (n+1)C_{11}$. The resulting inequality proves the theorem. □

5 Lower bounds for denominators and the problem of small denominators

Theorem 1 was established under the assumption that inequality (46) holds. In this section, we derive sufficient conditions under which this inequality is satisfied.

5.1 Asymptotic behavior of $\Delta(\lambda_{\mu,k}, T)$ at large k

First, let us introduce the following constants:

$$A_1 := \frac{2^{r_2-\nu+\frac{1}{2}}}{\sqrt{\pi}a^{r_2-\nu+\frac{5}{2}}}\frac{\Gamma\left(\frac{r_2+1}{2}\right)}{\Gamma\left(\frac{2\nu-r_2+1}{2}\right)}, \quad A_2 := \frac{2^{r_1-\nu+\frac{1}{2}}}{\sqrt{\pi}a^{r_1-\nu+\frac{5}{2}}}\frac{\Gamma\left(\frac{r_1+1}{2}\right)}{\Gamma\left(\frac{2\nu-r_1+1}{2}\right)},$$

$$B_1 := \frac{2^{r_2-\nu+\frac{1}{2}}}{\sqrt{\pi}a^{r_2-\nu+\frac{5}{2}}}\frac{\Gamma\left(\frac{r_2-2\nu+1}{2}\right)}{\Gamma\left(\frac{1-r_2}{2}\right)}, \quad B_2 := \frac{2^{r_1-\nu+\frac{1}{2}}}{\sqrt{\pi}a^{r_1-\nu+\frac{5}{2}}}\frac{\Gamma\left(\frac{r_1-2\nu+1}{2}\right)}{\Gamma\left(\frac{1-r_1}{2}\right)},$$

$$F_1 := \frac{2^{r_2+r_1-2\nu}}{a^{r_2+r_1-2\nu+2}}\left(\frac{\Gamma\left(\frac{r_1+1}{2}\right)\Gamma\left(\frac{r_2-2\nu+1}{2}\right)}{\Gamma\left(\frac{2\nu-r_1+1}{2}\right)\Gamma\left(\frac{1-r_2}{2}\right)} - \frac{\Gamma\left(\frac{r_2+1}{2}\right)\Gamma\left(\frac{r_1-2\nu+1}{2}\right)}{\Gamma\left(\frac{2\nu-r_2+1}{2}\right)\Gamma\left(\frac{1-r_1}{2}\right)}\right),$$

$$F_2 := \frac{2(r_1-r_2)}{\pi a^4}\sin(\pi\nu), \quad \psi_\nu = \frac{2\nu+1}{4}\pi, \quad L(r) = 4\nu^2 - 8\nu + 8r - 5.$$

Note that, by the assumptions of problem (14)–(16) and condition (28), we have

$$A_1^2 + B_1^2 \neq 0, \qquad A_2^2 + B_2^2 \neq 0, \qquad F_1 \neq 0, \qquad F_2 \neq 0.$$

From the formulas (13), (26), (27) we obtain the following asymptotic expansions:

$$\begin{aligned} I_{1k}(r,\nu,T) = &\frac{2^{r-\nu}\Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{2\nu-r+1}{2}\right)(a\lambda_{\mu,k})^{r-\nu+1}}\left(1+O\left(\frac{1}{\lambda_{\mu,k}^2T^2}\right)\right) \\ &+ \frac{\sqrt{2}}{\sqrt{\pi}}\frac{T^{r-\nu-\frac{1}{2}}}{(a\lambda_{\mu,k})^{\frac{3}{2}}}\sin\left(a\lambda_{\mu,k}T-\psi_\nu\right)\left(1+O\left(\frac{1}{\lambda_{\mu,k}T}\right)\right) \\ &+ \frac{\sqrt{2}L(r)}{8\sqrt{\pi}}\frac{T^{r-\nu-\frac{3}{2}}}{(a\lambda_{\mu,k})^{\frac{5}{2}}}\cos\left(a\lambda_{\mu,k}T-\psi_\nu\right)\left(1+O\left(\frac{1}{\lambda_{\mu,k}T}\right)\right), \end{aligned} \tag{51}$$

$$\begin{aligned} I_{2k}(r,\nu,T) = &\frac{2^{r-\nu}\Gamma\left(\frac{r-2\nu+1}{2}\right)}{\Gamma\left(\frac{1-r}{2}\right)(a\lambda_{\mu,k})^{r-\nu+1}}\left(1+O\left(\frac{1}{\lambda_{\mu,k}^2T^2}\right)\right) \\ &- \frac{\sqrt{2}}{\sqrt{\pi}}\frac{T^{r-\nu-\frac{1}{2}}}{(a\lambda_{\mu,k})^{\frac{3}{2}}}\cos\left(a\lambda_{\mu,k}T+\psi_\nu\right)\left(1+O\left(\frac{1}{\lambda_{\mu,k}T}\right)\right) \\ &+ \frac{\sqrt{2}L(r))}{8\sqrt{\pi}}\frac{T^{r-\nu-\frac{3}{2}}}{(a\lambda_{\mu,k})^{\frac{5}{2}}}\sin\left(a\lambda_{\mu,k}T+\psi_\nu\right)\left(1+O\left(\frac{1}{\lambda_{\mu,k}T}\right)\right). \end{aligned} \tag{52}$$

By substituting formulas (51), (52) in (30), after direct computation we get the following asymptotic expansion

$$\begin{aligned} \Delta(\lambda_{\mu,k},T) = &\Big(A(\lambda_{\mu,k},T)\cos\left(a\lambda_{\mu,k}T+\psi_\nu\right) + B(\lambda_{\mu,k},T)\sin\left(a\lambda_{\mu,k}T-\psi_\nu\right) \\ &+ \tilde{A}(\lambda_{\mu,k},T)\cos\left(a\lambda_{\mu,k}T-\psi_\nu\right) + \tilde{B}(\lambda_{\mu,k},T)\sin\left(a\lambda_{\mu,k}T+\psi_\nu\right) \\ &+ F(\lambda_{\mu,k},T)\Big)\left(1+O\left(\frac{1}{\lambda_{\mu,k}T}\right)\right), \end{aligned} \tag{53}$$

where

$$A(\lambda_{\mu,k},T)=\frac{A_1T^{r_1-\frac{1}{2}-\nu}}{\lambda_{\mu,k}^{r_2-\nu+\frac{5}{2}}}-\frac{A_2T^{r_2-\frac{1}{2}-\nu}}{\lambda_{\mu,k}^{r_1-\nu+\frac{5}{2}}},\qquad B(\lambda_{\mu,k},T)=\frac{B_1T^{r_1-\frac{1}{2}-\nu}}{\lambda_{\mu,k}^{r_2-\nu+\frac{5}{2}}}-\frac{B_2T^{r_2-\frac{1}{2}-\nu}}{\lambda_{\mu,k}^{r_1-\nu+\frac{5}{2}}},\tag{54}$$

$$F(\lambda_{\mu,k},T)=\frac{F_1}{\lambda_{\mu,k}^{r_2+r_1-2\nu+2}}+\frac{F_2T^{r_2+r_1-2\nu-2}}{\lambda_{\mu,k}^{4}},\tag{55}$$

$$\tilde{A}(\lambda_{\mu,k},T)=\frac{A_1L(r_1)T^{r_1-\frac{3}{2}-\nu}}{\lambda_{\mu,k}^{r_2-\nu+\frac{7}{2}}}-\frac{A_2L(r_2)T^{r_2-\frac{3}{2}-\nu}}{\lambda_{\mu,k}^{r_1-\nu+\frac{7}{2}}},$$

$$\tilde{B}(\lambda_{\mu,k},T)=\frac{B_1L(r_1)T^{r_1-\frac{3}{2}-\nu}}{\lambda_{\mu,k}^{r_2-\nu+\frac{7}{2}}}-\frac{B_2L(r_2)T^{r_2-\frac{3}{2}-\nu}}{\lambda_{\mu,k}^{r_1-\nu+\frac{7}{2}}}.$$

Obtained expansion (53) we rewrite in the form

$$\Delta(\lambda_{\mu,k},T)=D(\lambda_{\mu,k},T)\left(1+O\left(\frac{1}{\lambda_{\mu,k}T}\right)\right),\quad \lambda_{\mu,k}T\to+\infty,\tag{56}$$

where

$$\begin{aligned}D(\lambda_{\mu,k},T):=&\,A(\lambda_{\mu,k},T)\cos\left(a\lambda_{\mu,k}T+\psi_\nu\right)+B(\lambda_{\mu,k},T)\sin\left(a\lambda_{\mu,k}T-\psi_\nu\right)\\&+\tilde{A}(\lambda_{\mu,k},T)\cos\left(a\lambda_{\mu,k}T-\psi_\nu\right)+\tilde{B}(\lambda_{\mu,k},T)\sin\left(a\lambda_{\mu,k}T+\psi_\nu\right)+F(\lambda_{\mu,k},T).\end{aligned}\tag{57}$$

Note that, in the cases ($\nu=\frac{1}{2}$, $r_1=1$, $r_2=2$) and ($\nu=-\frac{1}{2}$, $r_1=0$, $r_2=1$) of problem (14)–(16), formula (53) (or (56)) yields the exact expression for $\Delta(\lambda_{\mu,k},T)$, namely

$$\Delta(\lambda_{\mu,k},T)\equiv D(\lambda_{\mu,k},T)=\frac{2T}{\pi a^3\lambda_{\mu,k}^3}\sin(a\lambda_{\mu,k}T)+\frac{4}{\pi a^4\lambda_{\mu,k}^4}\cos(a\lambda_{\mu,k}T)-\frac{4}{\pi a^4\lambda_{\mu,k}^4}.\tag{58}$$

5.2 Lower bounds of $|\Delta(\lambda_{\mu,k},T)|$

To establish estimates from below to $|\Delta(\lambda_{\mu,k},T)|$ the following lemma will be useful.

Lemma 3. *For every $T\in[T_0,\infty)$, where $T_0>0$, there exists a constant $K_0=K_0(T_0)>0$ such that, for all $\lambda_{\mu,k}>K_0$, the inequality $|D(\lambda_{\mu,k},T)|>\lambda_{\mu,k}^{\eta}$ implies $|\Delta(\lambda_{\mu,k},T)|>\tilde{M}\lambda_{\mu,k}^{\eta}$, where $\tilde{M}=\tilde{M}(K_0)>0$ is a constant independent of k.*

Proof. It follows from (56) that there exists a constant $K_1>0$ such that, for all $\lambda_{\mu,k}T>K_1$, the following inequalities

$$d_1|D(\lambda_{\mu,k},T)|<|\Delta(\lambda_{\mu,k},T)|<d_2|D(\lambda_{\mu,k},T)|\tag{59}$$

hold, where d_1 and d_2 are some constants satisfying $0<d_1<d_2$, independent of k.

Let us fix some $T_0>0$. From (5) and (19), it follows that $\lambda_{\mu,k}>K_1/T_0$ for all but finitely many values of $\lambda_{\mu,k}$. Consequently, inequality (59) holds for all $T>T_0$. This completes the proof of the lemma with $\tilde{M}=d_1$ and $K_0=K_1/T_0$. □

Further, without loss of generality, we assume that $r_2>r_1$. Taking into account that $\lambda_{\mu,k}\to+\infty$ as $k\to\infty$ and using formulas (54) and (55), we rewrite formula (57) in the following form

$$D(\lambda_{\mu,k},T)=\frac{T^{r_2-\nu-\frac{1}{2}}}{\lambda_{\mu,k}^{r_1-\nu+\frac{5}{2}}}\tilde{D}(\lambda_{\mu,k},T),\tag{60}$$

where

$$\begin{aligned}\tilde{D}(\lambda_{\mu,k},T) :=& \left(\frac{A_1}{(\lambda_{\mu,k}T)^{r_2-r_1}} - A_2\right)\cos\left(a\lambda_{\mu,k}T+\psi_\nu\right)\\ &+\left(\frac{B_1}{(\lambda_{\mu,k}T)^{r_2-r_1}} - B_2\right)\sin\left(a\lambda_{\mu,k}T-\psi_\nu\right)\\ &+\left(\frac{A_1L(r_1)}{(\lambda_{\mu,k}T)^{r_2-r_1+1}} - \frac{A_2L(r_2)}{\lambda_{\mu,k}T}\right)\cos\left(a\lambda_{\mu,k}T-\psi_\nu\right)\\ &+\left(\frac{B_1L(r_1)}{(\lambda_{\mu,k}T)^{r_2-r_1+1}} - \frac{B_2L(r_2)}{\lambda_{\mu,k}T}\right)\sin\left(a\lambda_{\mu,k}T+\psi_\nu\right)\\ &+\left(\frac{F_1}{(\lambda_{\mu,k}T)^{r_2-\nu-\frac{1}{2}}}+\frac{F_2}{(\lambda_{\mu,k}T)^{-(r_1-\nu-\frac{3}{2})}}\right).\end{aligned} \tag{61}$$

Using the so-called harmonic addition theorem, we rewrite formula (61) in the form

$$\begin{aligned}\tilde{D}(\lambda_{\mu,k},T) =& \frac{H_1}{(\lambda_{\mu,k}T)^{r_2-r_1}}\sin\left(a\lambda_{\mu,k}T-\gamma_1\right) - H_2\sin\left(a\lambda_{\mu,k}T-\gamma_2\right)\\ &+\frac{H_1|L(r_1)|}{(\lambda_{\mu,k}T)^{r_2-r_1+1}}\sin(a\lambda_{\mu,k}T+\gamma_1) - \frac{H_2|L(r_2)|}{\lambda_{\mu,k}T}\sin(a\lambda_{\mu,k}T+\gamma_2)+\tilde{F}(\lambda_{\mu,k},T)\\ =& \tilde{D}_1(\lambda_{\mu,k},T)+\tilde{F}(\lambda_{\mu,k},T),\end{aligned} \tag{62}$$

where

$$H_j=\sqrt{A_j^2+B_j^2-2A_jB_j\cos(\nu\pi)},\quad \gamma_j=\arctan\left(\frac{A_j\sin(\psi_\nu)+B_j\cos(\psi_\nu)}{A_j\cos(\psi_\nu)+B_j\sin(\psi_\nu)}\right),\quad j=1,2, \tag{63}$$

$$\tilde{F}(\lambda_{\mu,k},T)=\frac{F_1}{(\lambda_{\mu,k}T)^{r_2-\nu-\frac{1}{2}}}+\frac{F_2}{(\lambda_{\mu,k}T)^{-(r_1-\nu-\frac{3}{2})}}, \tag{64}$$

and by $\tilde{D}_1(\lambda_{\mu,k},T)$ we denote "trigonometric" part of formula (62).

Theorem 2. *If $r_2>r_1>\nu+\frac{3}{2}$, then for every $T\geqslant T_0>0$ there exists a constant $K_1:=K_1(T_0)$ such that inequality* (46) *holds for all $\lambda_{\mu,k}>K_1$, where $\eta\geqslant 4$ and $C_8=F_2T_1^{r_2+r_1-2\nu-2}\tilde{M}/4$, $T_1\in(0,T_0]$, and $\tilde{M}$ is the constant from Lemma 3.*

Proof. From $r_2>r_1>\nu+\frac{3}{2}$ it follows that $r_2-\nu-\frac{1}{2}>1$, $r_1-\nu-\frac{3}{2}>0$ and $r_2+r_1-2\nu-2>0$. Thus from (64), for arbitrary fixed $T>0$, we get that inequality

$$\left|\tilde{F}(\lambda_{\mu,k},T)\right|\geqslant\left|\left|F_2(\lambda_{\mu,k}T)^{r_1-\nu-\frac{3}{2}}\right|-\left|\frac{F_1}{(\lambda_{\mu,k}T)^{r_2-\nu-\frac{1}{2}}}\right|\right|>\frac{F_2}{2}(\lambda_{\mu,k}T)^{r_1-\nu-\frac{3}{2}} \tag{65}$$

holds for all $\lambda_{\mu,k}>K_2(T):=T^{-1}(2F_1/F_2)^{1/(r_2+r_1-2\nu-2)}$. From (62), (63) we get that for any fixed $T>0$ and for all $\lambda_{\mu,k}>K_3(T)$, where $K_3(T):=T^{-1}\max\{(4H_3/(F_2))^{1/(r_1-\nu-3/2)},1\}$, $H_3=2\max\{H_1,H_2\}\max\{|L(r_1)|,|L(r_2)|\}$, the following inequality

$$\left|\tilde{D}_1(\lambda_{\mu,k},T)\right|<\frac{F_2}{4}(\lambda_{\mu,k}T)^{r_1-\nu-\frac{3}{2}} \tag{66}$$

holds. From (65), (66) it follows that for an arbitrary fixed $T>0$ the estimate

$$|\tilde{D}(\lambda_{\mu,k},T)|>\frac{F_2}{4}(\lambda_{\mu,k}T)^{r_1-\nu-\frac{3}{2}} \tag{67}$$

is valid for all $\lambda_{\mu,k} > K_4(T) := \max\{K_2(T), K_3(T)\}$. Let us fix some $T_1 > 0$. Thus from $\lambda_{\mu,k} > K_4(T_1)$ we get $\lambda_{\mu,k} > K_4(T)$ for all $T > T_1$. Therefore from (67) we obtain that for all $T > T_1$ the inequality

$$|\tilde{D}(\lambda_{\mu,k}, T)| > \frac{F_2}{4}(\lambda_{\mu,k}T_1)^{r_1-\nu-\frac{3}{2}} \tag{68}$$

holds for all $\lambda_{\mu,k} > K_4(T_1)$. From (60), (68) it follows that for all $T \in [T_1, +\infty)$ the estimate

$$|D(\lambda_{\mu,k}, T)| > \frac{F_2 T_1^{r_2+r_1-2\nu-2}}{4}\lambda_{\mu,k}^{-4} \tag{69}$$

holds for all $\lambda_{\mu,k} > K_4(T_1)$.

From (69), based on Lemma 3, we obtain that for any fixed $T_0 \geqslant T_1$ there exists such constant $K_5(T_0)$ that for any $T > T_0$ the inequality

$$|\Delta(\lambda_{\mu,k}, T)| > \tilde{M}|D(\lambda_{\mu,k}, T)| > \frac{F_2 T_1^{r_2+r_1-2\nu-2}\tilde{M}}{4}\lambda_{\mu,k}^{-4}$$

holds for all $\lambda_{\mu,k} > K_5(T_0)$. The theorem is proved. □

Theorem 3. *If $r_1 < r_2 < \nu + \frac{1}{2}$, then for every $T \geqslant T_0 > 0$ there exists a constant $K_6 = K_6(T_0)$ such that the inequality* (46) *holds for all $\lambda_{\mu,k} > K_6$, where $\eta \geqslant r_1 + r_2 - 2\nu + 2$ and $C_8 = F_1\tilde{M}/4$.*

Proof. The proof is similar to the proof of Theorem 2, taking into account that under conditions of the theorem the following inequalities

$$r_2 - \nu - \tfrac{1}{2} < 0, \qquad r_1 - \nu - \tfrac{3}{2} < 0, \qquad r_2 + r_1 - 2\nu - 2 < 0,$$

$$|\tilde{F}(\lambda_{\mu,k}, T)| \geqslant \left|\left|\frac{F_1}{(\lambda_{\mu,k}T)^{r_2-\nu-\frac{1}{2}}}\right| - \left|\frac{F_2}{(\lambda_{\mu,k}T)^{-(r_1-\nu-\frac{3}{2})}}\right|\right|$$

$$> \frac{F_1}{2(\lambda_{\mu,k}T)^{r_2-\nu-\frac{1}{2}}}, \qquad \lambda_{\mu,k} > \frac{1}{T}\left(\frac{2F_2}{F_1}\right)^{\frac{1}{2\nu+2-r_2-r_1}},$$

$$|\tilde{D}_1(\lambda_{\mu,k}, T)| < \frac{F_1}{4(\lambda_{\mu,k}T)^{r_2-\nu-\frac{1}{2}}}, \qquad \lambda_{\mu,k} > \frac{1}{T}\left(\frac{4H_3}{F_1}\right)^{\frac{2}{2\nu+1-2r_2}},$$

hold for any fixed $T > 0$, where constant H_3 is from estimate (66). □

From Theorems 2 and 3 it follows that there is no small denominator problem for (14)–(16) if either $r_1 < r_2 < \nu + \frac{1}{2}$ or $r_2 > r_1 > \nu + \frac{3}{2}$. If $\nu \geqslant \frac{5}{2}$, then $2\nu - 1 \geqslant \nu + \frac{3}{2}$ and taking into account the condition (28), we get that the conditions of the Theorem 2 always hold. Moreover, Theorem 3 applies only if $-\frac{1}{2} \leqslant \nu < \frac{3}{2}$, since in this case $\nu + \frac{1}{2} > \max\{-1, 2\nu - 1\}$.

5.3 Problem of small denominators

In the case $r_1 < \nu + \frac{3}{2}$, $r_2 > \nu + \frac{1}{2}$, it is easy to see that $\Delta(\lambda_{\mu,k}, T)$ as a function of the variable T has infinitely many zeros on $(0, +\infty)$ for all (except perhaps a finite number) values of $\lambda_{\mu,k}$. This obviously follows from (56), (60), (61) by taking into account that

$$\min\{r_2 - \nu - \tfrac{1}{2}, -r_1 + \nu + \tfrac{3}{2}\} > 0.$$

In this case, generally speaking, the solvability of the original problem is related to the problem of small denominators as a value of $|\Delta(\lambda_{\mu,k}, T)|$ may be extremely small for T, that is, close to zeros of $\Delta(\lambda_{\mu,k}, T)$ for infinitely many $\lambda_{\mu,k}$.

In the case $\nu = \frac{2n+1}{2}$, where $n, r_1, r_2 \in \mathbb{N}$, the determinant $\Delta(\lambda_{\mu,k}, T)$ can be expressed in terms of elementary functions and is, in general, a trigonometric quasipolynomial. Taking into account condition (28) together with the inequality $r_1 < \nu + \frac{3}{2}$, we see that there are only two possible values of r_1 and ν, namely $r_1 = 0$, $\nu = -\frac{1}{2}$ and $r_1 = 1$, $\nu = \frac{1}{2}$, for which $\Delta(\lambda_{\mu,k}, T)$ is a quasipolynomial in the small denominator case. The value of r_2 may, however, be arbitrarily large. To establish estimate (46) in this case, we first need the following preliminary results.

Let $\{c_k\}$, $k \in \mathbb{Z}_p$, be a sequence in $\mathbb{R}^p$ such that $d_1|k|^\sigma \leqslant |c_k| \leqslant d_2|k|^\sigma$ and $c_{-k} = -c_k$ for all $k \in \mathbb{Z}^p$, where $\sigma > 0$, $d_2 > d_1 > 0$. Let us consider the following integral problem:

$$y^{(n)}(t) + a_1(c_k)y^{(n-1)}(t) + \cdots + a_n(c_k)y(t) = 0, \quad c_k \in \mathbb{R}^p, \tag{70}$$

$$\int_0^T t^{\alpha_j} y(t)dt = 0, \quad j = 1, \ldots, n, \tag{71}$$

where $\alpha_j \in \mathbb{Z}_+$, $a_j(\eta)$, $\eta \in \mathbb{R}^p$, are polynomials with complex coefficients of degree $(n-j)$ with respect to the variable η.

Let us suppose that the characteristic equation $\lambda^n + a_1(c_k)\lambda^{n-1} + \cdots + a_n(c_k) = 0$, corresponding to (70), has only nontrivial roots λ_{lk} of multiplicity n_l, $l = 1, \ldots, m$, $m \leqslant n$, $n_1 + \cdots + n_m = n$.

The characteristic determinant of the problem (70), (71) is denoted by

$$\Delta_d(c_k, T) := \det \left\| \int_0^T t^{\alpha_j} f_{qk}(t) \right\|_{j,q=1}^n, \tag{72}$$

where functions $f_{qk}(t)$, $q = 1, 2$, are normal (at point $t = 0$) fundamental system of solutions of equation (70), thus $f_{qk}^{j-1}(0) = \delta_{j,q}$, $j, q = 1, \ldots, n$.

The following result is a partial case of [12, Theorem 5].

Theorem 4. *For almost all (in the sense of Lebesgue measure on $\mathbb{R}$) numbers $T > 0$ inequality*

$$|\Delta_d(c_k, T)| \geqslant (1 + |c_k|)^{-\eta}$$

holds for all except a finite number of c_k, $k \in \mathbb{Z}^p$, if

$$\eta > \alpha + n(n-1)/2 + (p/\sigma + 1)\left(2^n\left(1 + \alpha + \sum_{l=1}^n n_l(n_l - 1)\right) - 1\right), \quad \alpha = \alpha_1 + \cdots + \alpha_n.$$

It is well-known that functions

$$f_1(\lambda_{\mu,k}, t) = \cos(a\lambda_{\mu,k}t), \qquad f_2(\lambda_{\mu,k}, t) = \frac{\sin(a\lambda_{\mu,k}t)}{a\lambda_{\mu,k}}, \tag{73}$$

are the normal (at point $t = 0$) fundamental system of solutions of the equation

$$y''(t) + a^2\lambda_{\mu,k}^2 y(t) = 0,$$

which is a partial case of equation (70), where $a_1(\eta) \equiv 0$, $a_2(\eta) = \eta^2$,

$$c_k = a\lambda_{\mu,k}, \quad n = 2, \quad n_1 = n_2 = 1, \quad p = \sigma = 1. \tag{74}$$

In the case $\nu = \pm\frac{1}{2}$, the fundamental system of solutions (34) of equation (21) can be expressed in terms of functions (73), namely

$$u_{1k}\left(-\frac{1}{2}, t\right) = \sqrt{\frac{2a\lambda_{\mu,k}}{\pi}} f_2(\lambda_{\mu,k}, t), \qquad u_{2k}\left(-\frac{1}{2}, t\right) = \sqrt{\frac{2}{a\pi\lambda_{\mu,k}}} f_1(\lambda_{\mu,k}, t), \tag{75}$$

$$u_{1k}\left(\frac{1}{2},t\right)=\sqrt{\frac{2}{a\pi\lambda_{\mu,k}}}\frac{f_1(\lambda_{\mu,k},t)}{t},\qquad u_{2k}\left(\frac{1}{2},t\right)=\sqrt{\frac{2a\lambda_{\mu,k}}{\pi}}\frac{f_2(\lambda_{\mu,k},t)}{t}.\tag{76}$$

Taking into account (30), (72), (73), (75), (76), it is easy to show that in case $r_1=\nu+1/2$, $r_2\in\mathbb{N}$, $r_2>r_1$ and $\nu=\pm\frac{1}{2}$, we have

$$|\Delta(\lambda_{\mu,k},T)|=\frac{2}{\pi}|\Delta_d(a\lambda_{\mu,k},T)|,\tag{77}$$

where

$$\begin{cases}\alpha_1=r_1-1,\alpha_2=r_2-1, & \text{if } \nu=\frac{1}{2},\\ \alpha_1=r_1,\alpha_2=r_2, & \text{if } \nu=-\frac{1}{2}.\end{cases}\tag{78}$$

Since $\alpha_1+\alpha_2=r_1+r_2=r_2+\nu+\frac{1}{2}=r_2-\nu-\frac{1}{2}$, when $\nu=-\frac{1}{2}$, and $\alpha_1+\alpha_2=r_1+r_2-2=r_2+\nu-\frac{3}{2}=r_2-\nu-\frac{1}{2}$, when $\nu=\frac{1}{2}$, Theorem 4 together with (74), (77) and (78) yields the following result.

Theorem 5. *If $\nu=\pm\frac{1}{2}$, $r_1=\nu+1/2$, $r_2\in\mathbb{N}$, $r_2>r_1$, then for almost all (in the sense of Lebesgue measure on $\mathbb{R}$) numbers $T\in(0,+\infty)$, the estimate* (46) *holds for all (except possibly a finite number of) values $\lambda_{\mu,k}$, where $\eta\geqslant 9(r_2-\nu)+\frac{5}{2}+\varepsilon$, $\varepsilon>0$, and $C_8=1$.*

In the case $r_2=\nu+\frac{3}{2}$, statement of Theorem 5 can be strengthened.

Theorem 6. *If $\nu=\frac{1}{2}$, $r_1=1$ and $r_2=2$ or $\nu=-\frac{1}{2}$, $r_1=0$ and $r_2=1$, then for almost all (in the sense of Lebesgue measure on $\mathbb{R}$) numbers $T>0$ the estimate* (46) *holds for all but finitely many values $\lambda_{\mu,k}$, where $\eta\geqslant 13+\varepsilon$, $\varepsilon>0$, and $C_8=1$.*

Proof. In the case $\nu=\frac{1}{2}$, $r_1=1$, $r_2=2$ (or $\nu=-\frac{1}{2}$, $r_1=0$, $r_2=1$), the determinant $\Delta(\lambda_{\mu,k},T)$ is given by the formula (58), which we rewrite in following form

$$\Delta(\lambda_{\mu,k},T)=\sum_{j=1}^{3}P_j(\lambda_{\mu,k},T)e^{\sigma_{jk}T},\tag{79}$$

where

$$P_1(\lambda_{\mu,k},T)=\frac{2}{\pi a^4\lambda_{\mu,k}^4}-\frac{iT}{\pi a^3\lambda_{\mu,k}^3},\qquad P_2(\lambda_{\mu,k},T)=\frac{2}{\pi a^4\lambda_{\mu,k}^4}+\frac{iT}{\pi a^3\lambda_{\mu,k}^3},\tag{80}$$

$$P_3(\lambda_{\mu,k},T)=-\frac{4}{\pi a^4\lambda_{\mu,k}^4},\qquad \sigma_{1k}=ia\lambda_{\mu,k},\qquad \sigma_{2k}=-ia\lambda_{\mu,k},\qquad \sigma_{3k}=0.\tag{81}$$

From (79)–(81) we get the following Taylor expansion for $\Delta(\lambda_{\mu,k},T)$ at point $T=0$:

$$\Delta(\lambda_{\mu,k},T)=-\frac{1}{6\pi}T^4+\frac{a^2\lambda_{\mu,k}^2}{90\pi}T^6+o(T^6).\tag{82}$$

From (79) it follows that $\Delta(\lambda_{\mu,k},T)$ as a function of variable T is a quasi-polynomial. For the quasi-polynomial (79), we denote

$$N_\Delta:=\sum_{j=1}^{3}(\deg P_j(\lambda_{\mu,k},T)+1)=5,\tag{83}$$

$$B_\Delta(\lambda_{\mu,k}):=1+\max_{j=1,2,3}\{|\sigma_{jk}|\},\qquad M_\Delta(\lambda_{\mu,k}):=\min_{j=1,2,3}\operatorname{Re}\sigma_{jk},\tag{84}$$

$$\Psi_\Delta(\lambda_{\mu,k})=\max_{T\in[0,T_0]}e^{-M_\Delta T},\qquad G_\Delta(\lambda_{\mu,k}):=\max_{1\leqslant j\leqslant N_\Delta}\left\{\frac{|\Delta^{(j-1)}(\lambda_{\mu,k},T)|}{(B_\Delta(\lambda_{\mu,k}))^j}\right\}\Bigg|_{T=0}.\tag{85}$$

From (5), (19), (81), (84) and (85) we obtain that

$$B_\Delta(\lambda_{\mu,k}) \leqslant C_{18}\lambda_{\mu,k}, \qquad M_\Delta(\lambda_{\mu,k}) = 0, \qquad \Psi_\Delta(\lambda_{\mu,k}) = 1, \qquad C_{13} = 2\max\{1, a\}. \tag{86}$$

Taking into account (82), (83) and (85) for $G_\Delta(\lambda_{\mu,k})$ we get the following estimate

$$G_\Delta(\lambda_{\mu,k}) = \left.\frac{|\Delta^{(4)}(\lambda_{\mu,k}, T)|}{(B_\Delta(\lambda_{\mu,k}))^5}\right|_{T=0} \geqslant C_{14}\lambda_{\mu,k}^{-5}, \qquad C_{14} = (C_{13})^{-5}/(6\pi). \tag{87}$$

By $E_\Delta([0, T_0], \varepsilon)$ we denote the set of all $T \in [0, T_0]$, $T_0 > 0$, such that $|\Delta(\lambda_{\mu,k}, T)| \leqslant \varepsilon$, $\varepsilon > 0$. Let us also consider the intervals $I_q = [(q-1)T_0, qT_0]$, $q \in \mathbb{N}$. By [15, Theorem 2.1], for each $\lambda_{\mu,k}$ we obtain

$$\operatorname{mes} E_\Delta([0, T_0], \varepsilon) \leqslant C_{15} B_\Delta(\lambda_{\mu,k}) \left(\frac{4\varepsilon\Psi_\Delta(\lambda_{\mu,k})}{G_\Delta(\lambda_{\mu,k})}\right)^{1/(N_\Delta - 1)}, \qquad C_{15} = C_{15}(T_0, N_\Delta). \tag{88}$$

From (5), (19), and estimate (88), it follows that the series $\sum_{k\in\mathbb{N}} \operatorname{mes} E_\Delta([0, T_0], \lambda_{\mu,k}^{-13-\varepsilon})$ converges. Hence, by the Borel-Cantelli lemma, the set of all $T \in [0, T_0]$, belonging to infinitely many sets $E_\Delta([0, T_0], \lambda_{\mu,k}^{-13-\varepsilon})$, $k \in \mathbb{N}$, has Lebesgue measure zero. Similarly, taking into account the translation invariance of the Lebesgue measure and making the change of variables $T = \bar{T} + (q-1)T_0$, $\bar{T} \in I_1$, in the expression for $\Delta(\lambda_{\mu,k}, T)$, we conclude that, for every $q \in \mathbb{N}$, the set of all $T \in I_q$ belonging to infinitely many sets $E_\Delta(I_q, \lambda_{\mu,k}^{-13-\varepsilon})$ also has Lebesgue measure zero. Since interval $[0, \infty)$ is a union of a countable number of intervals I_q, $q \in \mathbb{N}$, it follows that the inequality $|\mathcal{D}(\mu_k, \tau)| > \lambda_{\mu,k}^{-13-\varepsilon}$ holds for almost every $T > 0$ (with respect to the Lebesgue measure) and for all but finitely many values of $\lambda_{\mu,k}$. □

Theorem 6 is a sharpened version of [15, Theorem 4.2], including the case of the wave equation ($\nu = -\frac{1}{2}$).

Conclusions

Theorems 1–6 establish the conditions for the well-posed solvability of problem (14)–(16) in the scale of spaces $C^n_{|\nu|+\nu}([0, T], \mathcal{PH}^\beta_\mu)$, where these conditions depend on the relationship between the exponents $r_1, r_2 \in \mathbb{R}$ in the integral condition (15) and the parameter ν of the Bessel operator in equation (14), with $\nu \geqslant -\frac{1}{2}$ and $\nu \notin \mathbb{Z}$.

If $r_1 < \nu + 3/2$ and $r_2 > \nu + 1/2$ or vice versa, then the solvability of the problem (14)–(16) is related to the problem of small denominators for which, in particular cases, we obtain estimates from below (see Theorems 5, 6) by using metric approach. From statements of Theorems 5 and 6 it follows the correct solvability of the problem (14)–(16) in the corresponding scale of functional spaces (Theorem 2) for almost all values of the upper limit of integration T in the condition (15). In the case $\nu = \mu = -\frac{1}{2}$, $r_1 = 0$, $r_2 = 1$ of the problem (14)–(16), Theorems 1–6 refine the results of [14, 15] for the case of wave equation. An estimate from below of denominators in the general case remains an open problem.

In the case where r_j, $j = 1, 2$, satisfy the inequalities $r_j > \max\left\{-1,\, 2\nu - 1,\, \nu + \frac{3}{2}\right\}$ (or $\max\{-1,\, 2\nu - 1\} < r_j < \nu + \frac{1}{2}$ if $\nu \in [-\frac{1}{2}, \frac{3}{2})$), the solvability of problem (14)–(16) is not affected by the small denominator problem (see Theorems 2 and 3).

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Кузь А.М. *Задача з інтегральними умовами для рівняння із частинними похідними з сингулярними операторами Бесселя* // Карпатські матем. публ. — 2026. — Т.18, №1. — С. 328–344.

У прямокутній області досліджується задача з інтегральними умовами за однією зі змінних для рівняння в частинних похідних з сингулярними операторами Бесселя. Встановлено критерій однозначної розв'язності цієї задачі та достатні умови існування її розв'язку. Для вирішення проблеми малих знаменників, що виникає в деяких випадках розглянутої задачі, використано метричний підхід.

Ключові слова і фрази: сингулярний оператор Бесселя, ряд Фур'є-Бесселя, узагальнена гіпергеометрична функція, інтегральна умова, умовно коректна задача, малий знаменник.