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Rings of the right (left) almost stable range 1

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We introduce a concept of rings of right (left) almost stable range 1 and we construct a theory of a canonical diagonal reduction of matrices over such rings. A description of new classes of noncommutative elementary divisor rings is done as well. In particular, for Bézout D-domain we introduced the notions of D-adequate element and D-adequate ring. We proved that every D-adequate Bézout domain has almost stable range 1. For Hermite D-ring we proved the necessary and sufficient conditions to be an elementary divisor ring. A ring R is called an L-ring if the condition $RaR = R$ for some $a \in R$ implies that a is a unit of R. We proved that every L-ring of almost stable range 1 is a ring of right almost stable range 1.

Key words and phrases: Bézout ring, almost stable range, clean ring, elementary divisor ring.

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Dedicated to 65$^{\text{th}}$ birthday of Professor Volodymyr Shchedryk

1 Introduction

The stable range of a ring is one of the most important invariants of the algebraic K-theory, which was introduced by H. Bass [2]. Rings of small stable range are of particular interest and are being actively studied. Therefore, at present, the theory of rings of stable range 1 and 2 is a rather extensive branch of the theory of rings and modules. However, it should be noted that while studies on stable range 1 rings are a very popular topic, the same is not true for stable range 2 rings (see [30, p. 2]).

Commutative rings with restrictions on their elements, resembling the definition of the stable range, were considered in [21]. Commutative Bézout rings that are of stable range 1.5, i.e. rings in which for any $a \neq 0, b, c \in R$ there exists r such that $(a + rb, c) = 1$, were introduced in [25] and actively studied in [5, 23, 26].

However, it is worth emphasizing that W. McGovern [20] is credited with the first study of commutative rings with the properties that non-trivial homomorphic images are a ring of stable range 1, i.e. rings of almost stable range 1. This ground breaking study has given impetus to further research in this area. The historical overview and current state of the topic are covered in books [24, 28] and the articles [4, 5, 8, 9, 12, 21].

In the present article, we introduce a notion of non-commutative rings with almost stable range 1. We show that every ring of stable range 1 is a ring of right (left) almost stable range 1.

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In addition, the concept of an element of right (left) almost stable range 1 is introduced. Based on the above, we introduce a notion of rings of right (left) almost stable range 1 and study their properties and connections with other ring properties.

Let R be an associative ring (not necessary commutative) with $1 \neq 0$, let $U(R)$ be the group of units of R and let $R^{n\times m}$ be the vector space of $n \times m$ matrices over R with $n, m \geq 1$. Let $\mathrm{GL_n(R)}$ be the group of units of the matrix ring $R^{n\times n}$.

The matrix $D := \mathrm{diag}(d_1, \dots, d_s) \in R^{n\times m}$ means a (possibly rectangular) matrix having $d_1, \dots, d_s$ (in which $s := \min(n, m)$) on the main diagonal and zeros elsewhere. By the main diagonal we mean the one beginning at the upper left corner.

Two matrices A and B over a ring R are called *equivalent* if there exist invertible matrices P and Q over R of suitable sizes such that $A = PBQ$ and it is denoted by $A \sim B$.

According to I. Kaplansky (see [15, p. 465]), a ring R is called an *elementary divisor ring* if for any matrix $A \in R^{n\times m}$ we have

$$A \sim D := \mathrm{diag}(d_1, \dots, d_k, 0, \dots, 0), \tag{1}$$

in which d_i is a *total divisor* of d_{i+1}, i.e. $Rd_{i+1}R \subseteq d_iR \cap Rd_i$ for each $i = 1, \dots, k-1$. In this case, we say that the matrix A has a *canonical diagonal reduction* over R.

The class of elementary divisor rings is contained in the class of Bézout rings (for example, see [15, 24, 28]), i.e. rings with nonzero unit in which every finitely generated one-sided ideal is a principal one-sided ideal. Note that elementary divisor rings are Hermite rings, i.e. rings in which each 1×2 and 2×1 matrix has a diagonal reduction, i.e. $(a, b)P = (c, 0)$ and $Q(a, b)^T = (d, 0)^T$, where $a, b, c, d \in R$ and $P, Q \in \mathrm{GL_2(R)}$ (see [15, 28]). Every right Hermite ring is a right Bézout ring.

2 Rings of almost stable range 1

Set $R^2 := R \times R = \{(a, b) : a, b \in R\}$ and $R^3 := \{(a, b, c) : a, b, c \in R\}$.

A ring R has *stable range* 1 if for each pair $(a, b) \in R^2$ the equality $aR + bR = R$ implies $(a + b\lambda)R = R$ for some $\lambda \in R$. Similarly, a ring R has *stable range* 2 if for each triple $(a, b, c) \in R^3$ the equality $aR + bR + cR = R$ implies that there exist $\lambda, \mu \in R$ such that

$$(a + c\lambda)R + (b + c\mu)R = R.$$

An element $a \in R \setminus \{0\}$ has *right (left) almost stable range* 1, if for each pair $(b, c) \in R^2$ there exists $\lambda \in R$ ($\mu \in R$) such that equality $aR + bR + cR = R$ implies $aR + (b + c\lambda)R = R$ ($Ra + Rb + Rc = R$ implies $Ra + R(b + \mu c) = R$, respectively). If each nonzero element of R is an element of right (left) almost stable range 1, then R is a ring of *right (left) almost stable range* 1.

This notion generalizes the notion of rings of almost stable range 1, which was introduced by W. McGovern for commutative rings. A right (left) Bézout ring is a ring in which every right (left) finitely generated ideal is a principal right (left) ideal. A Bézout ring is a ring that is both right and left Bézout.

Let $(a, b) \in R^2$ be such that $aR + bR = R$. The pair $(a, b) \in R^2$ is called *right diadem*, if there exists $\lambda \in R$ such that for each triple $(a + b\lambda, c, d) \in R^3$ with the property $(a + b\lambda)R + cR + dR = R$ there exists $\mu \in R$ such that $(a + b\lambda)R + (c + d\mu)R = R$.

The concept of a *left diadem* is introduced by analogy. A *diadem* is a pair that is both right and left diadem at the same time. We say that R is a ring of *right dyadic range* 1 if for each pair

$(a,b) \in R^2$ the equality $aR + bR = R$ implies that the pair (a,b) is a right diadem. Similarly we define rings of left dyadic range 1. A ring *of dyadic range* 1 is a ring that is both of right dyadic range 1 and of left dyadic range 1 at the same time (see [30, Definition 2.1 and Definition 2.3]). Note that if R has right almost stable range 1, then each pair $(a,b) \in R \setminus \{0\} \times R$ is a right diadem, because $aR + (c + b\mu)R = R$, $\lambda = 0$, $\mu \in R$.

Lemma 1. *Let R be a right Bézout ring. If R has stable range 2, then for each pair $(a,b) \in R^2$ there exists a triple $(d, a_1, b_1) \in R^3$ such that*

$$a = da_1, \qquad b = db_1 \qquad \text{and} \qquad a_1R + b_1R = R.$$

Proof. Since R is a right Bézout ring, $aR + bR = dR$ for some $d \in R$. Clearly, $a = da_0$, $b = db_0$ and $au + bv = d$ for some $d, a_0, b_0, u, v \in R$. Thus $da_0u + db_0v = d$ and $dc = 0$, where $c := 1 - a_0u - b_0v$. Consequently $a_0R + b_0R + cR = R$. Since R has stable range 2, we have $(a_0 + c\lambda)R + (b_0 + c\mu)R = R$ for some $\lambda, \mu \in R$. Finally, setting $a_1 := a_0 + c\lambda$ and $b_1 := b_0 + c\mu$, we obtain $a_1R + b_1R = R$, $a = da_1$ and $b = db_1$. □

Theorem 1. *The following conditions hold.*

(i) Each right Bézout ring of stable range 1 is a ring of right almost stable range 1.

(ii) Each ring of right (left) almost stable range 1 is a ring of right (left) dyadic range 1.

(iii) Each right (left) Bézout ring of a right (left) almost stable range 1 is a ring of stable range 2.

Proof. Let $aR + bR + cR = R$ and $bR + cR = dR$. Each ring of stable range 1 is a ring of stable range 2 (see [2, p. 14]), so $b = db_1$, $c = dc_1$ and $c_1R + b_1R = R$ for some $d, c_1, b_1 \in R$ by Lemma 1. Since R has stable range 1, from $c_1R + b_1R = R$ it follows that $b_1\lambda + c_1 = u \in U(R)$ for some $\lambda \in R$. That yields $db_1\lambda + dc_1 = du$, i.e. $b\lambda + c = du$. Since $aR + bR + cR = R$ and $bR + cR = dR$, we have

$$aR + duR = aR + bR + cR = aR + (b\lambda + c)R = R,$$

i.e. R is a ring of right almost stable range 1.

Each Bézout ring of right dyadic range 1 is a ring of stable range 2 (see [30, Theorem 2.1]). Two other statements are obvious. □

Theorem 2. *A ring R has right almost stable range 1 if and only if $R/J(R)$ has right almost stable range 1, where $J(R)$ is the Jacobson radical of R.*

Proof. Let $\overline{x} := x + J(R) \in \overline{R} := R/J(R)$, where $x \in R$. Let $(a,b,c) \in R^3$ be such that

$$\overline{a}\,\overline{R} + \overline{b}\,\overline{R} + \overline{c}\,\overline{R} = \overline{R} \quad \text{and} \quad a \notin J(R).$$

That yields $aR + bR + cR = R$ and $a \neq 0$. Since R has right almost stable range 1, we get

$$aR + (b\lambda + c)R = R, \quad \lambda \in R.$$

Thus $\overline{a}\,\overline{R} + (\overline{b}\,\overline{\lambda} + \overline{c})\overline{R} = \overline{R}$, so $\overline{R}$ has right almost stable range 1. The rest of the proof is similar. □

Theorem 3. *Let R be a right Bézout ring. If for each $(a,b) \in R^2$ with the property $aR + bR = R$ there exists $\lambda \in R$ such that $a + b\lambda$ is an element of right almost stable range 1, then R is a ring of stable range 2.*

Proof. Let $(a, b, c) \in R^3$ such that $aR + bR + cR = R$. Since R is a right Bézout ring, we have $bR + cR = dR$ for some $d \in R$, so $aR + dR = R$. By our assumption there exists $\lambda \in R$ such that the element $\nu := a + d\lambda$ has right almost stable range 1. From $bR + cR = dR$ we obtain that $a + bx + cy = \nu$ for some $x, y \in R$, so $\nu R + bR + cR = R$, because $aR + bR + cR = R$.

By the definition of an element of right almost stable range 1 we have $\nu R + (b + c\mu)R = R$ for some $\mu \in R$. Let $\nu s + (b + c\mu)t = 1$ for some $s, t \in R$. Since $Rs + Rt = R$, we have $Rs + R(xs + t)R = R$.

Let $us + v(xs + t) = ys + \mu t$ for some $u, v \in R$. Thus $(a + cu)s + (b + cv)(xs + t) = 1$, i.e. $(a + cu)R + (b + cv)R = R$, so R is a ring of stable range 2. □

3 Case of commutative rings

Let R be a commutative ring. Then in (1) each d_i is a divisor of d_{i+1} for $i = 1, \dots, k - 1$. According to W. McGovern [20, p. 393], we say that a commutative ring R has almost stable range 1, if its every proper homomorphic image has stable range 1. We have the following assertion.

Theorem 4. *For a commutative ring R the following statements are equivalent:*

(i) R is a ring of almost stable range 1,

(ii) for every $(a, b, c) \in R^3$ with the properties $a \neq 0$ and $aR + bR + cR = R$, there exists $\lambda \in R$ such that $aR + (b + c\lambda)R = R$.

Proof. The proof of the part *(i) ⇒ (ii)* can be found in [20, Theorem 3.6] putting $a \neq 0$.

Let $(a, b, c) \in R^3$ such that $aR + bR + cR = R$ and $a \neq 0$. Assume there exists $\lambda \in R$ such that $aR + (b + c\lambda)R = R$.

Set $\overline{R} := R/aR$ and $\overline{x} := x + aR$, where $x \in R$. If $(b, c) \in R^2$ such that $\overline{b}\,\overline{R} + \overline{c}\,\overline{R} = \overline{R}$, then $aR + bR + cR = R$. According to the assumption we have $aR + (b + c\lambda)R = R$ for some $\lambda \in R$. Evidently $(\overline{b} + \overline{c}\,\overline{\lambda})\overline{R} = \overline{R}$, where $\overline{b} = b + aR$, i.e. $\overline{R}$ is a ring of stable range 1. If I is a nonzero ideal of R, then for every $0 \neq a \in I$ we have the isomorphism $R/I/aR/I \simeq R/aR$. If R/aR is a ring of stable range 1, then R/I has stable range 1 as well. □

A ring is called *clean* if its every nonzero element is a sum of a unit and an idempotent [22]. A ring in which every proper homomorphic image is clean is called *neat* (see [19, p. 248]). An element $a \in R$ is called *neat* if R/aR is a clean ring.

An R-module M has *finite exchange property* if for any R-module N with the decomposition $N = M' \oplus P = \bigoplus_{i \in I} Q_i$ in which $M' \cong M$ and I is finite, there exist submodules $Q_i' \subseteq Q_i$ for each $i \in I$, such that $N = M' \oplus (\oplus Q_i')$. A ring R is called *exchange* if its regular module R_R is a finite exchange R-module.

A commutative ring R has *neat range* 1 if for each pair $(a, b) \in R^2$ with the property $aR + bR = R$ there exists a neat element $c \in R$ such that $a + bt = c$ for some $t \in R$. For the properties of the above rings and their relationship with other types of rings, see [1,16,17,19,31,32].

Theorem 5. *Let R be a commutative Bézout ring. If for each $(a, b, c) \in R^3$ with the properties $aR + bR + cR = R$ and $a \in R\backslash\{0\}$ there exists a decomposition $a = rs$ for some $r, s \in R$ such that*

$$rR + bR = R, \qquad sR + cR = R \qquad \text{and} \qquad rR + sR = R, \tag{2}$$

then R/aR is a clean ring.

Proof. Set $\overline{R} := R/aR$ and $\overline{x} := x + aR$, where $x \in R$. Since $aR + bR + cR = R$ and (2) holds, we get

$$\overline{b}\,\overline{R} + \overline{c}\,\overline{R} = \overline{R}, \qquad \overline{r}\,\overline{R} + \overline{s}\,\overline{R} = \overline{R}, \qquad \overline{b}\,\overline{R} + \overline{r}\,\overline{R} = \overline{R} \quad \text{and} \quad \overline{c}\,\overline{R} + \overline{s}\,\overline{R} = \overline{R}.$$

It follows that $\overline{r}\,\overline{u} + \overline{s}\,\overline{v} = \overline{1}$ and $\overline{r}\,\overline{s} = \overline{0}$ for some $u, v \in R$. This yields $\overline{r}^2\overline{u} = \overline{r}$ and $\overline{s}^2\overline{v} = \overline{s}$. Obviously,

$$(\overline{s}\,\overline{v})^2 = \overline{s}\,\overline{v} = \overline{e} \quad \text{and} \quad (\overline{r}\,\overline{u})^2 = \overline{r}\,\overline{u} = \overline{1} - \overline{e}.$$

Taking into account that $\overline{r}\,\overline{R} + \overline{b}\,\overline{R} = \overline{R}$, we get

$$\overline{r}\,\overline{x} + \overline{b}\,\overline{y} = \overline{1}$$

for some $\overline{x}, \overline{y} \in \overline{R}$. Thus $\overline{s}\,\overline{v}\,\overline{r}\,\overline{x} + \overline{s}\,\overline{v}\,\overline{b}\,\overline{y} = \overline{s}\,\overline{v}$ and $\overline{e}\,\overline{b}\,\overline{y} = \overline{e}$, i.e. $\overline{e} \in \overline{b}\,\overline{R}$. Using the fact that $\overline{s}\,\overline{R} + \overline{c}\,\overline{R} = \overline{R}$, we obtain $\overline{1} - \overline{e} \in \overline{c}\,\overline{R}$, so $\overline{R}$ is an exchange ring by [15, Proposition 1.1 and Theorem 2.1]. Finally, $\overline{R}$ is a clean ring by [19, p. 244]. □

In the case when R is a domain, the following result is known.

Theorem 6 ([32, Theorem 31])**.** *Let R be a commutative Bézout domain and let $a \in R \setminus \{0\}$. If R/aR is a clean ring, then for each $(b, c) \in R^2$ with the property $aR + bR + cR = R$ there exists a decomposition $a = rs$ for some $r, s \in R$ such that*

$$rR + bR = R, \qquad sR + cR = R \quad \text{and} \quad rR + sR = R.$$

Now, using Theorems 5 and 6 we have the following.

Theorem 7. *Let R be a commutative Bézout domain. The following statements hold:*

(i) R has neat range 1 if and only if for each $(a, b) \in R^2$ with the property $aR + bR = R$ there exists $\lambda \in R$ such that $R/(a + b\lambda)R$ is clean;

(ii) R is an elementary divisor ring if and only if R has neat range 1 (see [32, Theorem 33]);

(iii) if the stable range of R/aR is not 1 for each $a \in R \setminus \{0\}$, then R is not an elementary divisor ring (see [31, Theorem 5]).

Each commutative clean ring R has stable range 1 (see [19, p. 244] and [22, Proposition 1.8]). Furthermore, a neat ring is a ring of almost stable range 1 and a neat element is an element of almost stable range 1, so if R is an elementary divisor domain that is not a ring of stable range 1, then at $R \setminus U(R)$ there exists a neat element that has almost stable range 1 (see [29, Theorem 7]).

4 Rings with Dubrovin property

The *coboundary* of a one-sided ideal I of a ring R is a two-sided ideal which equals to the intersection of all two-sided ideals which contain I. Note that this definition is left-right symmetric.

A ring R in which for every $a \in R \setminus \{0\}$ there exists $b \in R$ such that

$$RaR = bR = Rb, \tag{3}$$

(in other words, the coboundary of R is a principal ideal) is called a ring with *Dubrovin property* (D-property). Simple elementary divisor rings [28, Section 4.2], quasi-duo elementary divisor rings [7, Theorem 1] and semi-local semi-primitive elementary divisor rings [11, Theorem 1], [10] are examples of elementary divisor rings with D-property.

In [3], the investigation of elementary divisor rings of stable range 1 with Dubrovin and Dubrovin-Komarnytsky property was done. In the same article, a theory of a canonical diagonal reduction of matrices over such rings was constructed as well, and new families of non-commutative rings of elementary divisor rings were presented.

A two-sided analogue of one-sided rings of almost stable range 1 is the following.

A ring R has *almost stable range* 1 if for each triple $(a, b, c) \in R^3$ with the properties

$$RaR + RbR + RcR = R \quad \text{and} \quad c \neq 0,$$

there exists $\lambda \in R$ such that $R(\lambda a + b)R + RcR = R$.

Theorem 8. *Let R be a Bézout ring. The following statements are equivalent:*

(i) R is a ring of almost stable range 1;

(ii) for each $(a, b, c) \in R^3$ with the properties $RaR + RbR + RcR = R$ and $c \neq 0$, there exist $\lambda, d \in R$ such that $(\lambda a + b)R + cR = dR$ and $RdR = R$.

Proof. Let $RaR + RbR + RcR = R$ and $(\lambda a + b)R + cR = dR$ in which $RdR = R$ for some $\lambda \in R$. Since $cR \subset RcR$ and $(\lambda a + b)R \subset R(a\lambda + mb)R$, we have

$$dR = (\lambda a + b)R + cR \subseteq R(\lambda a + b)R + RcR.$$

Hence, $RdR = R$ and $d \in R(\lambda a + b)R + RcR$, so we have $R(\lambda a + b)R + RcR = R$.

Let the condition $RaR + RbR + RcR = R$ imply that $R(\lambda a + b)R + RcR = R$ with $c \neq 0$ and some $\lambda \in R$. It is easy to see that $(\lambda a + b)R + cR = dR$ for some $d \in R$, because R is a Bézout ring. Using the fact that $(\lambda a + b)R \subseteq dR$, we obtain that $cR \subseteq dR$, so $RcR \subseteq RdR$ and $R(\lambda a + b)R \subseteq RdR$. Consequently from $R(\lambda a + b)R + RcR = R$ we obtain that $RdR = R$, which proves the equivalence of *(i)* and *(ii)*. ☐

A ring R is called an *L-ring* if the condition $RaR = R$ for some $a \in R$ implies that a is a unit of R. This definition arise from conditions of [18, Proposition 7.3(2)].

Theorem 9. *Every L-ring of almost stable range 1 is a ring of right almost stable range 1.*

Proof. Let $aR + bR + cR = R$ and $c \neq 0$. Obviously $RaR + RbR + RcR = R$ and R is a ring of almost stable range 1 by Theorem 8. Thus, $(\lambda a + b)R + cR = dR$ in which $RdR = R$. Since R is an L-ring, $(\lambda a + b)R + cR = R$, i.e. R is a ring of a right almost stable range 1. ☐

Note that every elementary divisor L-ring is a ring with D-property [28, Theorem 4.7.1]. According to Theorem 1 and Theorem 9 we have the following.

Theorem 10. *Every Bézout L-ring of almost stable range 1 has stable range 2. Moreover, each Bézout ring R of stable range 1 has almost stable range 1.*

Proof. The first part follows from Theorems 1 and 9.

Let $RaR + RbR + RcR = R$ and $Ra + Rb = Rd$. Since a ring of stable range 1 has stable range 2 (see [2, p. 14]), we have $a = a_0 d$, $b = b_0 d$ and $Ra_0 + Rb_0 = R$ by Lemma 1. The ring R has stable range 1 and $Ra_0 + Rb_0 = R$, so $a_0 + \lambda b_0 = u \in U(R)$. Then $a + \lambda b = ud$, i.e. $R(a + \lambda b) = Rud = Rd$ and $a = a_0 d$, $b = b_0 d$. Let $dR + cR = zR$. Similarly we have $dy + c = z$. Then we have $(a + \lambda b)R + cR = zR$. Since $a = a_0 d + a_0 z d_0$, we have $b = b_0 d = b_0 z d_0$ and $c = c_0 z$ for some $d_0, c_0 \in R$. It follows that $RaR \subseteq RzR$, $RbR \subseteq RzR$, and $RcR \subseteq RzR$. Since $RaR + RbR + RcR = R$, we have $R = RzR$ and R has almost stable range 1. ☐

5 Rings with D-adequacy property

The notion of the adequacy of commutative domains was introduced by O. Helmer [14]. A.I. Gatalevych [12, Definition 1, p. 116] was the first who extended this notation to non--commutative rings. Now it is known that the generalized right adequate (in the sense of Gatalevych) duo Bézout domain is an elementary divisor domain (see [12, Theorem 2, p. 117]).

In [13], it was explored the problem when a ring of matrices over either adequate rings or elementary divisor rings inherits the property of adequacy.

Let A and E be adequate and elementary divisor domains, respectively. Let $A^{2\times 2}$ and $E^{2\times 2}$ be rings of 2×2 matrices over rings A and E, respectively. In [13], it was proved that the set of full nonsingular matrices from $A^{2\times 2}$ is an adequate set in $A^{2\times 2}$ and the set of full singular matrices from $E^{2\times 2}$ is an adequate set in the set of full matrices in $E^{2\times 2}$.

In [6], another definition of adequate rings was proposed, which differs from the one proposed by A.I. Gatalevych [12, Definition 1, p. 116].

We propose a new version of the definition of adequacy, which is very close to the definitions of adequacy that were given in [6, 12, 13].

The introduction of new definitions is related to Dubrovin's condition for non-commutative rings. For a class of duo rings our definition is equivalent to the definition of A.I. Gatalevich [12, Definition 1, p. 116].

Let R be a Bézout D-domain and let $a\in R\setminus\{0\}$ be such that

$$RaR = a^*R = Ra^* \neq R.$$

The element a is called *D-adequate* if for each $b\in R$ the following conditions hold:

(i) $a = rs$ for some $r, s \in R$ such that $RrR = r^*R = Rr^*$, $RsR = s^*R = Rs^*$, and $RbR = b^*R = Rb^*$;

(ii) $r^*R + b^*R = R$;

(iii) for each non-trivial divisor s'^* of s^* we have $s'^*R + b^*R \neq R$ in which $Rs'R = s'^*R = Rs'^*$.

A ring R is called *D-adequate* if each element $0\neq a\in R$ with the property $RaR\neq R$ is D-adequate.

Theorem 11. *Every D-adequate Bézout D-domain has almost stable range* 1.

Proof. Let $(a, b, c)\in R^3$ such that $RaR + RbR + RcR = R$ and $c\neq 0$. If $RcR = R$, then the statement of our theorem is obvious. Let $RcR\neq R$. It follows that

(i) $c = rs$;

(ii) $r^*R + a^*R = R$;

(iii) for each non-trivial divisor s'^* of s^* we have $s'^*R + b^*R = R$.

Let $(a+rb)R + cR = dR$. Using the fact that $RdR = d^*R = Rd^*$, we obtain $dR\subset RdR = d^*R = Rd^*$. Hence, we obtain $(a+rb)R\subset dR\subset d^*R$. If

$$r^*R + d^*R = hR \subset h^*R = Rh^*,$$

then $aR \subset h^*R \neq R$. Clearly h^*R is a two-sided ideal, so $a^*R \subset h^*R$, which contradicts to (ii). Hence $r^*R + d^*R = R$, so $r^*u + d^*v = 1$ and $r^*us^* + d^*vs^* = s^*$ for some $u, v \in R$. Since s^* is a duo element, then $r^*s^*u' + d^*s^*v' = s^*$ for some $u', v' \in R$. It follows that

$$sR \subset s^*R \subset d^*R \qquad \text{and} \qquad d^*R + aR = h^*R \neq R.$$

It follows that $aR \subset h^*R$, $bR \subset h^*R$, $cR \subset h^*R$ and $h^*R \neq R$. Since $h^*R = Rh^*$, we get $RaR \subset h^*R$, $RbR \subset h^*R$, and $RcR \subset h^*R$, which contradicts the condition

$$RaR + RbR + RcR = R.$$

Thus, we proved that $(a + rb)R + cR = dR$ and $RdR = R$, i.e. R has almost stable range 1. □

6 Hermite *D*-rings

We denote by A_* a two-sided ideal in a ring R generated by all elements of the matrix $A = (a_{ij}) \in R^{m\times n}$. Evidently $A_* = \sum_{i=1}^{m}\sum_{j=1}^{n} Ra_{ij}R$.

We need the following well-known results.

Lemma 2. *If $A \sim B$, then $A_* = B_*$.*

Proof. Let $A = (a_{ij}), B = (b_{ij}) \in R^{m\times n}$ be such that $A \sim B$. It follows that $a_{ij} \in \sum_k \sum_s Rb_{ks}R$ and $b_{ij} \in \sum_k \sum_s Ra_{ks}R$ for each i, j, k, s. Consequently

$$\sum_i \sum_j Ra_{ij}R = \sum_k \sum_s Rb_{ks}R,$$

i.e. $A_* = B_*$. □

Lemma 3. *Each Hermite ring R with D-property is an elementary divisor ring if and only if every $A = (a_{ij}) \in R^{n\times n}$ with the property $\sum_i \sum_j Ra_{ij}R = R$ has a canonical diagonal reduction.*

Proof. Since the proof of the "if" part is obvious, we start with the proof of the "only if" part.

Let $Ra_{ij}R = a^*_{ij}R = Ra^*_{ij}R$ for each i, j (see (3)). For some duo-element $\alpha \in R$ we have

$$\sum_i \sum_j Ra_{ij}R = \sum_i \sum_j a^*_{ij}R = \sum_i \sum_j Ra^*_{ij} = \alpha R = R\alpha.$$

That yields $a_{ij} = \alpha a^0_{ij}$ and $A = \operatorname{diag}(\alpha, \ldots, \alpha)A_0$, where $A_0 = (a^0_{ij})$. Since R is Hermite, it is a Bézout ring of stable range 2 [28, p. 30, Collorarly 2.1.3, Theorem 1.2.40], so $\sum_i \sum_j Ra'_{ij}R = R$ by Lemma 1. □

Theorem 12. *Let R be a Hermite ring of almost stable range 1. If $a, b, c \in R$ with the property $RaR + RbR + RcR = R$, then there exist $P, Q \in \mathrm{GL}_2(\mathrm{R})$ such that*

$$P \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} Q = \begin{bmatrix} z & 0 \\ * & * \end{bmatrix} \qquad \textit{and} \qquad RzR = R.$$

Proof. Since every Hermite ring is Bézout [28, Corolary 2.13], then R has almost stable range 1. For each $(a, b, c) \in R^3$ with the property $RaR + RbR + RcR = R$, there exists $\lambda \in R$ such that $(\lambda a + b)R + cR = dR$ by Theorem 8. This yields that

$$\begin{bmatrix} \lambda & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} = \begin{bmatrix} \lambda a + b & c \\ a & 0 \end{bmatrix}.$$

Set $d := (\lambda a + b)x + cy$, $a_0 d := \lambda a + b$ and $c := c_0 d$ for some $a_0, c_0, x, y \in R$. Each Hermite ring is a Bézout ring of stable range 2 [28, Theorem 1.2.40], so according to Lemma 1 we have that $\lambda a + b = a_1 d$ and $c = c_1 d$, where $a_1 R + c_1 R = R$, i.e. $a_1 u + c_1 v = 1$ for some $u, v \in R$. Since $Ru + Rv = R$ and R is a Hermite ring, then the column $\begin{bmatrix} u \\ v \end{bmatrix}$ is completable to a matrix $Q = \begin{bmatrix} u & * \\ v & * \end{bmatrix} \in \mathrm{GL}_2(\mathrm{R})$ (see [28, Corolary 2.1.7]). Evidently

$$\begin{bmatrix} \lambda & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} Q = \begin{bmatrix} d & * \\ * & * \end{bmatrix},$$

where $RdR = R$. Using the fact that R is a Hermite ring, we obtain that

$$\begin{bmatrix} d & * \\ * & * \end{bmatrix} S = \begin{bmatrix} z & 0 \\ * & * \end{bmatrix}$$

for an invertible matrix S. Taking into account that $dR \subset zR$ and $RdR = R$, we get that $R = RdR \subseteq RzR$, i.e. $RzR = R$. □

For a canonical diagonal reduction of matrices over a Hermite ring R (see [15, Theorem 5.1]) it is enough to consider such reduction for matrices of the form $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$, where $a, b, c \in R$. Therefore, according to Lemma 3, for a canonical diagonal reduction of matrices over a Hermite ring R with D-property it is enough to consider such reduction for matrices $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$, where

$$RaR + RbR + RcR = R.$$

Using Theorem 12 we have the following assertion.

Theorem 13. *Let R be a Hermite ring with D-property of almost stable range 1. The ring R is an elementary divisor ring if and only if for every $a, b, c \in R$ with $RaR = R$ the matrix $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \in R^{2\times 2}$ has a canonical diagonal reduction over R.*

This result shows that the set of $a \in R$ for which $RaR = R$ is crucial for a canonical diagonal reduction of matrices over R.

Example. *Let R be a domain. An element $a \in R \setminus \{0, U(R)\}$ is called atom if a has no proper factors. An element which is either a unit or a product of atoms is called finite. Let R be a Bézout domain with D-condition of almost stable range 1. If for every $a \in R$, the condition $RaR = R$ implies that a is a finite element, then R is an elementary divisor ring. Indeed, according to Theorem 13, we can consider a matrix A of the form $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \in R^{2\times 2}$, where $RaR = R$ and a is a finite element. There exist $P, Q \in \mathrm{GL}_2(\mathrm{R})$ (see [27, Theorem 6]) such that $PAQ = \begin{bmatrix} z & 0 \\ 0 & d \end{bmatrix}$ in which $RdR \subseteq zR \cap Rz$.*

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Бовді В., Забавський Б. *Кільця правого (лівого) майже стабільного рангу* 1 // Карпатські матем. публ. — 2025. — Т.17, №2. — С. 461–471.

У роботі вводиться поняття кілець правого (лівого) майже стабільного рангу 1 та будується теорія канонічного діагонального зведення матриць над такими кільцями. Також подано опис нових класів некомутативних кілець елементарних дільників. Зокрема, для D-області Безу ми ввели поняття D-адекватного елемента та D-адекватного кільця. Ми довели, що кожна D-адекватна область Безу має майже стабільний ранг 1. Для ермітового D-кільця ми встановили необхідні й достатні умови, за яких воно є кільцем з елементарними дільниками. Кільце R називають L-кільцем, якщо з умови $RaR = R$ для деякого $a \in R$ випливає, що a є одиничним елементом у R. Ми довели, що кожне L-кільце майже стабільного рангу 1 є кільцем із правим майже стабільним рангом 1.

Ключові слова і фрази: кільце Безу, майже стабільний ранг, чисте кільце, кільце елементарних дільників.