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On the analytic extension of the Horn's hypergeometric function H_4

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The paper establishes new convergence domains of branched continued fraction expansions of Horn's hypergeometric function H_4 with real and complex parameters. These domains enabled the PC method to establish the analytical extension of analytical functions to their expansions in the studied domains of convergence. A few examples are provided at the end to illustrate this.

Key words and phrases: branched continued fraction, Horn hypergeometric function, approximation by rational functions, convergence, analytic continuation.

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Introduction

The analytic theory of branched continued fractions provides a useful means for the representation and continuation of special functions of one and several variables (see [3, 5, 6, 22–24, 30, 33]), used in mathematical physics. Many applications of branched continued fractions were made in various areas of numerical analysis and theoretical physics, chemistry and engineering [26–28, 34]. This paper continues the works [4, 18–20] and establishes new domains of the analytical continuation of Horn's hypergeometric function H_4 with certain conditions on real and complex parameters, using their representation in branched continued fractions.

Note that the Horn's hypergeometric function H_4 is defined as (see [25])

$$H_4(a,b;c,d;\mathbf{z}) = \sum_{r,s=0}^{\infty} \frac{(a)_{2r+s}(b)_s}{(c)_r(d)_s} \frac{z_1^r}{r!} \frac{z_2^s}{s!}, \quad |z_1| < p, \ |z_2| < q, \tag{1}$$

where $a,b,c,d \in \mathbb{C}$, $c,d \notin \{0,-1,-2,\dots\}$, $(\alpha)_0 = 1$, $(\alpha)_n = \alpha(\alpha+1)\dots(\alpha+n-1)$, $4p = (q-1)^2$, $q \neq 1$, $\mathbf{z} = (z_1,z_2) \in \mathbb{C}^2$.

In the paper [4], formal expansions into branched continued fractions were constructed for the following functions:

$$\frac{H_4(a,b;c,b;\mathbf{z})}{H_4(a+1,b;c+1,b;\mathbf{z})} \tag{2}$$

and

$$\frac{H_4(a,d+1;c,d;\mathbf{z})}{H_4(a+1,d+1;c,d+1;\mathbf{z})}, \quad \frac{H_4(a,d+1;c,d;\mathbf{z})}{H_4(a,d+2;c,d+1;\mathbf{z})}. \tag{3}$$

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In particular, it is proved that (2) has a formal expansion of the form

$$1-z_2-\cfrac{h_1z_1}{1-z_2-\cfrac{h_2z_1}{1-z_2-\cfrac{h_3z_1}{1-\ddots}}}, \tag{4}$$

where

$$h_k=\frac{(2c-a+k-1)(a+k)}{(c+k-1)(c+k)},\quad k\geq 1. \tag{5}$$

For h_k, $k\geq 1$, such that $0<h_k\leq h$ for all $k\geq 1$, where h is a positive number, $c\notin\{0,-1,-2,\ldots\}$, the convergence of branched continued fraction (4) to the function (2) was investigated in domain

$$\mathrm{H}_h=\{\mathbf{z}\in\mathbb{C}^2:\ z_k\notin[1/(4+4h),+\infty),\ k=1,2\}.$$

In this paper, Theorem 2 provides convergence criteria for complex h_k, $k\geq 1$, while Theorems 3 and 4 for negative and positive h_k, $k\geq 1$, respectively. Since all the studied domains of convergence contain the neighborhood of the origin, the analytical extension of the functions to these domains is established by the PC method (see [1]). Section 2 provides several examples to illustrate this.

Some other recent studies related to the function (1) can be found in [14, 29, 32, 35].

1 Branched continued fraction and analytic continuation

In this section, we establish the domains of the analytic continuation of the function (1). Here we need the following theorem, proved in [15].

Theorem 1. *Let $m_{0,k}$, $k\geq 1$, be real numbers such that $0<m_{0,k}\leq 1$ for all $k\geq 1$. Then the following is true.*

(A) The branched continued fraction

$$1-z_{1,0}-\cfrac{m_{0,1}z_{0,1}}{1-(1-m_{0,1})z_{1,1}-\cfrac{m_{0,2}(1-m_{0,1})z_{0,2}}{1-(1-m_{0,2})z_{1,2}-\cfrac{m_{0,3}(1-m_{0,2})z_{0,3}}{1-\ddots}}} \tag{6}$$

converges for $|z_{1,k}|\leq 1/2$ and $|z_{0,k}|\leq 1/2$ for all $k\geq 1$.

(B) If f_n denotes an nth approximant of (6), then $|f_n-1|\leq 1$ for all $n\geq 1$.

Note that various definitions of the convergence of branched continued fractions can be found in [15, 17].

Let us prove the following theorem.

Theorem 2. *Let a and c be complex constants such that*

$$|h_k|+\mathrm{Re}(h_k)\leq pq(1-q)\quad\textit{for all}\quad k\geq 1, \tag{7}$$

where h_k, $k\geq 1$, are defined by (5), $c\notin\{0,-1,-2,\ldots\}$, p is a positive number, and $0<q<1$. Then the following is true.

(A) *The branched continued fraction* (4) *converges uniformly on every compact subset of the domain*

$$\mathrm{H}_{p,q}^{h,l} = \mathrm{H}_{p,q} \bigcup \mathrm{H}^{h,l}, \tag{8}$$

where

$$\mathrm{H}_{p,q} = \left\{ \mathbf{z} \in \mathbb{C}^2 : \ |z_1| < \frac{1+\cos(\arg(z_1))}{2p}, \ \mathrm{Re}\left(\frac{z_2}{e^{(i/2)\arg(z_1)}}\right) < \frac{q}{2}\cos\left(\frac{\arg(z_1)}{2}\right) \right\} \tag{9}$$

and

$$\mathrm{H}^{h,l} = \left\{ \mathbf{z} \in \mathbb{C}^2 : \ |z_1| < \frac{l(1-l)}{2h}, \ |z_2| < \frac{1-l}{2} \right\}, \tag{10}$$

with

$$h = \max_{k\in\mathbb{N}}\{|h_k|\}, \quad 0 < l < 1,$$

to a function $f(\mathbf{z})$ *holomorphic in* $\mathrm{H}_{p,q}^{h,l}$.

(B) *The function* $f(\mathbf{z})$ *is an analytic continuation of* (2) *in the domain* (8).

Proof. An application of Theorem 1 (A), with $m_{0,k} = l$, $k \geq 1$, $0 < l < 1$, shows that the branched continued fraction (4) converges for all $\mathbf{z} \in \mathrm{H}^{h,l}$. Theorem 1 (B) implies that the approximants of (4) all lie in $|w-1| \leq 1$ if $\mathbf{z} \in \mathrm{H}^{h,l}$. Hence, by [2, Theorem 3], the convergence of (4) is uniform on every compact subset of (10).

Let us prove the uniform convergence of the branched continued fraction (4) on every compact subset of (9) in the same way as in [18, Theorem 2].

We set $G_n^{(n)}(\mathbf{z}) = 1$, $n \geq 1$, and, for $1 \leq k \leq n-1$, $n \geq 2$,

$$G_k^{(n)}(\mathbf{z}) = 1 - z_2 - \cfrac{h_{k+1}z_1}{1 - z_2 - \cfrac{h_{k+2}z_1}{1 - \ddots - z_2 - \cfrac{h_{n-1}z_1}{1 - z_2 - h_n z_1}}}.$$

It follows that

$$G_k^{(n)}(\mathbf{z}) = 1 - z_2 - \frac{h_{k+1}z_1}{G_{k+1}^{(n)}(\mathbf{z})}, \quad 1 \leq k \leq n-1, \quad n \geq 2, \tag{11}$$

and nth approximant of (4) we write as

$$f_n(\mathbf{z}) = 1 - z_2 - \frac{h_1 z_1}{G_1^{(k)}(\mathbf{z})}. \tag{12}$$

Let $\arg(z_1) = \varphi$, n be an arbitrary natural number, and $\mathbf{z}$ be an arbitrary fixed point in the domain (9). Let us prove that

$$\mathrm{Re}(G_k^{(n)}(\mathbf{z})e^{-i\varphi/2}) > (1-q)\cos(\varphi/2) \geq c > 0, \quad 1 \leq k \leq n, \quad n \geq 1. \tag{13}$$

From an arbitrary fixed point $\mathbf{z}$, $\mathbf{z} \in \mathrm{H}_{p,q}$, it follows that for its arbitrary neighborhood, there exists σ, $0 < \sigma \le \pi/2$, such that $|\varphi/2| \le \pi/2 - \sigma$ and, thus,

$$(1-q)\cos(\varphi/2) \ge (1-q)\cos(\varphi/2 - \sigma) = (1-q)\sin(\sigma) = c > 0.$$

If $k = n$, the first inequality in (13) is obvious. Assuming that the first inequality in (13) is true if $k = s+1 \le n$. Then, for $k = s$ from (11) we obtain

$$G_s^{(n)}(\mathbf{z})e^{-i\varphi/2} = e^{-i\varphi/2} - z_2 e^{-i\varphi/2} - \frac{h_{s+1} z_1 e^{-i\varphi}}{G_{s+1}^{(n)}(\mathbf{z})e^{-i\varphi/2}}. \tag{14}$$

Using [1, Corollary 2], (7), (9), from (14) we get

$$\begin{aligned}
\mathrm{Re}(G_s^{(n)}(\mathbf{z})e^{-i\varphi/2}) &\ge \cos(\varphi/2) - \mathrm{Re}(z_2 e^{-i\varphi/2}) - \frac{|h_{s+1}| + \mathrm{Re}(h_{s+1})}{2\,\mathrm{Re}(G_{s+1}^{(n)}(\mathbf{z})e^{-i\varphi/2})}|z_1| \\
&> \cos(\varphi/2) - \frac{q\cos(\varphi/2)}{2} - \frac{pq(1-q)}{2(1-q)\cos(\varphi/2)} \frac{1+\cos(\varphi)}{2p} \\
&= (1-q)\cos(\varphi/2).
\end{aligned}$$

Thus, $G_1^{(n)}(\mathbf{z}) \neq 0$ for all $n \ge 1$ and $\mathbf{z} \in \mathrm{H}_{p,q}$, i.e. that each approximant (12) is a holomorphic function in (9).

Now, let K be an arbitrary compact subset of (9), then there exists an open bi-disk

$$\mathrm{Q}_R = \{\mathbf{z} \in \mathbb{C}^2 : \ |z_k| < R, \ k = 1, 2\}$$

of radius R, $R > 0$, such that $\mathrm{K} \subset \mathrm{Q}_R$. Then, for any $n \ge 1$ and $\mathbf{z} \in \mathrm{H}_{p,q} \bigcap \mathrm{Q}_R$ from (12) we get

$$|f_n(\mathbf{z})| \le 1 + R + \frac{|h_1|R}{\mathrm{Re}(G_1^{(n)}(\mathbf{z})e^{-i\varphi/2})} < 1 + R + \frac{hR}{(1-q)\cos(\varphi/2)} = C(\mathrm{K}),$$

i.e. the sequence $\{f_n(\mathbf{z})\}$ is uniformly bounded on every compact subset of $\mathrm{H}_{p,q}$.

It is easy to see that for every L such that

$$0 < L < \min\left\{\frac{1-l}{2}, \frac{l(1-l)}{2h}, \frac{1}{p}, \frac{q}{2}\right\}$$

the domain

$$\mathrm{Y}_L = \{\mathbf{z} \in \mathbb{R}^2 : \ 0 < z_k < L, \ k = 1, 2\}$$

contained in $\mathrm{H}_{p,q}$, in particular $\mathrm{Y}_{L/2} \subset \mathrm{H}_{p,q}$. Using (7), for any $\mathbf{z} \in \mathrm{Y}_L$, $\mathrm{Y}_L \subset \mathrm{H}_{p,q}$, it is easy to show that $|z_2| < (1-l)/2$ and $|h_k z_1| < l(1-l)/2$, $k \ge 1$, i.e. the elements of (4) satisfy Theorem 1, with $m_{0,k} = l$, $k \ge 1$. It shows that branched continued fraction (4) converges for all $\mathbf{z} \in \mathrm{Y}_L$, $\mathrm{Y}_L \subset \mathrm{H}_{p,q}$. Hence, by [2, Theorem 3], the convergence of (4) is uniform on compact subsets of (9).

The proof of (B) is analogous to the proof of Theorem 2 (B) in [2], hence, it is omitted. □

Remark 1. *In Theorem 2, the set* $\mathrm{Re}(z_2 e^{-(i/2)\arg(z_1)}) < (q/2)\cos(\arg(z_1)/2)$ *can also be written as* $z_2 \notin [q/2, +\infty)$.

If we set $a = 0$ and replace c with $c-1$ in Theorem 2, we get the following result.

Corollary 1. *Let c be a complex constant satisfying* (7)*, where*

$$h_1 = \frac{2}{c}, \quad h_k = \frac{k(2c+k-3)}{(c+k-2)(c+k-1)} \quad \text{for all} \quad k \geq 2, \tag{15}$$

$c \notin \{0, -1, -2, \ldots\}$*, p is a positive number, and $0 < q < 1$. Then the branched continued fraction*

$$\cfrac{1}{1 - z_2 - \cfrac{h_1 z_1}{1 - z_2 - \cfrac{h_2 z_1}{1 - \ddots}}} \tag{16}$$

converges uniformly on every compact subset of (8) *to a function $f(\mathbf{z})$ holomorphic in $\mathrm{H}_{p,q}^{h,l}$, and $f(\mathbf{z})$ is an analytic continuation of $H_4(1, b; c, b; \mathbf{z})$ in the domain* (8).

An application of Theorem 2 is the following theorem.

Theorem 3. *Let a and c be real numbers such that $-h \leq h_k < 0$ for all $k \geq 1$, where h_k, $k \geq 1$, are defined by* (5)*, h is a positive number. Then the branched continued fraction* (4) *converges uniformly on every compact subset of the domain*

$$\mathrm{P}_{h,l} = \Big\{ \mathbf{z} \in \mathbb{C}^2 : \ z_1 \notin \Big(-\infty, -\frac{l(1-l)}{2h} \Big], \ z_2 \notin \Big[\frac{1}{2}, +\infty \Big) \Big\}, \tag{17}$$

where $0 < l < 1$, to function $f(\mathbf{z})$ holomorphic in $\mathrm{P}_{h,l}$, and $f(\mathbf{z})$ is an analytic continuation of (2) *in the domain* (17).

Proof. If $h_k < 0$ for all $k \geq 1$, then the condition (7) holds for all $p > 0$ and $0 < q < 1$. Let K be an arbitrary compact set contained in (17). Then $\mathrm{K} \subseteq \mathrm{H}_{p,q}^{h,l} \subseteq \mathrm{P}_{h,l}$ for some p sufficiently small and q sufficiently close to 1, whose $\mathrm{H}_{p,q}^{h,l}$ is the domain (8). Theorem 3 is, thus, an immediate consequence of Theorem 2. ☐

Corollary 2. *Let c be a real number such that $-h \leq h_k < 0$ for all $k \geq 1$, where h_k, $k \geq 1$, are defined by* (15)*, h is a positive number. Then the branched continued fraction* (16) *converges uniformly on every compact subset of* (17) *to a function $f(\mathbf{z})$ holomorphic in $\mathrm{P}_{h,l}$, and $f(\mathbf{z})$ is an analytic continuation of $H_4(1, b; c, b; \mathbf{z})$ in the domain* (17).

The following result can be proved in much the same way as Theorem 2.

Theorem 4. *Let a and c be real numbers such that $0 < h_k \leq h$ for all $k \geq 1$, where h_k, $k \geq 1$, are defined by* (5)*, h is a positive number. Then the following is true.*

(A) The branched continued fraction (4) *converges uniformly on every compact subset of the domain*

$$\mathrm{C}_{h,l} = \Big\{ \mathbf{z} \in \mathbb{C}^2 : \ z_1 \notin \Big[\frac{1}{8h}, +\infty \Big), \ z_2 \notin \Big[\frac{1-l}{2}, +\infty \Big) \Big\}, \tag{18}$$

where $0 < l < 1$, to function $f(\mathbf{z})$ holomorphic in $\mathrm{C}_{h,l}$.

(B) The function $f(\mathbf{z})$ is an analytic continuation of (2) *in the domain* (18).

Corollary 3. *Let c be a real number such that $0 < h_k \leq h$ for all $k \geq 1$, where h_k, $k \geq 1$, are defined by* (15)*, h is a positive number. Then the branched continued fraction* (16) *converges uniformly on every compact subset of* (18) *to a function $f(\mathbf{z})$ holomorphic in $\mathrm{C}_{h,l}$, and $f(\mathbf{z})$ is an analytic continuation of $H_4(1, b; c, b; \mathbf{z})$ in the domain* (18).

Remark 2. *Results similar to Theorems 2 – 4 can be obtained for the expansions of the functions* (3) *constructed in paper [4].*

2 Examples

As an example, by Corollary 3 we get

$$((1-z_2)^2-4z_1)^{-1/2} = H_4(1,b;1,b;\mathbf{z}) = \cfrac{1}{1-z_2-\cfrac{2z_1}{1-z_2-\cfrac{z_1}{1-z_2-\cfrac{z_1}{1-\ddots}}}}, \tag{19}$$

where the branched continued fraction converges and represents a single-valued branch of the analytic function on the left side of (19) in the domain (18) with $h=2$.

One more example, by Corollary 3 we obtain

$$\begin{aligned} \arctan\frac{2\sqrt{-z_1}}{1-z_2} &= 2\sqrt{-z_1}H_4(1,b;3/2,b;\mathbf{z}) \\ &= \cfrac{2\sqrt{-z_1}}{1-z_2-\cfrac{\frac{4}{3}z_1}{1-z_2-\cfrac{\frac{16}{15}z_1}{1-z_2-\ddots-\cfrac{\frac{4k^2}{4k^2-1}z_1}{1-\ddots}}}}, \end{aligned} \tag{20}$$

where the branched continued fraction converges and represents a single-valued branch of the analytic function of on the left side of (20) in the domain (18) with $h=4/3$.

3 Conclusions

We consider the representation and extension of the analytic functions by special families of functions — branched continued fractions. The main results are new domains of analytical continuation for the Horn's hypergeometric function H_4 with certain conditions on real and complex parameters, which were established using their branched continued fraction representations. Further investigations can be continued in various directions. First, we can extend the domains of convergence of the branched continued fraction expansions with real and complex coefficients in their elements using parabolic domains of convergence [11,12] and angular domains of convergence [9,10]. Other research directions are the truncation errors analysis and the computational stability of the branched continued fraction expansions. Here, we can use the results of papers [7,8,13,16,21,31].

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У статті досліджено нові області збіжності гіллястих ланцюгових дробових розвинень гіпергеометричної функції Горна H_4 з дійсними та комплексними параметрами. За допомогою PC методу встановлено аналітичні розширення деяких функцій у досліджені області збіжності. Для ілюстрації цього наприкінці наведено кілька прикладів.

Ключові слова і фрази: гіллястий ланцюговий дріб, гіпергеометрична функція Горна, апроксимація раціональними функціями, збіжність, аналітичне продовження.