ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2025, **17** (2), 472–482
doi:10.15330/cmp.17.2.472-482

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2025, Т.17, №2, С.472–482

Uniqueness of meromorphic functions and their difference or shift operators in several complex variables

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In this paper, we have studied the uniqueness problems of meromorphic functions with their difference or generalized linear shift operators in the light of partial sharing in several complex variables. By relaxing one sharing condition from CM (counting multiplicities) to IM (ignoring multiplicities), one of the results of our paper improved a result of W. Wu and T.-B. Cao [Comput. Methods Funct. Theory 2022, **22** (2), 379–399]. Our other results also extend and improve some results of the same paper.

Key words and phrases: meromorphic function, uniqueness, difference operator, several complex variables.

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1 Introduction

First, we recall some basic notions in several complex variables (see [15, 17]). For a point $z_0 \in \mathbb{C}^m$, the entire function $f(z)$ on $\mathbb{C}^m$ can be written as $f(z) = \sum_{i=0}^{\infty} Q_i(z-z_0)$, where the term $Q_i(z-z_0)$ is either identically zero or a homogeneous polynomial of degree i. For a divisor ν on $\mathbb{C}^m$, we denote the zero-multiplicity of f at z_0 by $\nu_f(z_0) = \min\{i : Q_i(z-z_0) \not\equiv 0\}$. Therefore, we can define a divisor ν_f such that $\nu_f(z_0)$ equals the zero multiplicity of ν_f at z_0 in the sense of [7, Definition 2.1] whenever z_0 is a regular point of an analytic set $|\nu_f| = \overline{\{z \in \mathbb{C}^m : \nu_f(z) \neq 0\}}$. Let $f(z)$ be a non-zero meromorphic function on $\mathbb{C}^m$. For each $z_0 \in \mathbb{C}^m$, we can choose two non-zero relatively prime holomorphic functions f_1 and f_2 (i.e. there is no common factor of f_1 and f_2) on a neighborhood U of z_0 such that $f = \frac{f_1}{f_2}$ on U and $\dim\{z \in \mathbb{C}^m : f_1(z) = f_2(z) = 0\} \leq m-2$ (see [18, p. 165]). Define $\nu_f = \nu_{f_1}$ and $\nu_{\frac{1}{f}} = \nu_{f_2}$ for any two non-zero relatively prime holomorphic functions f_1 and f_2.

Let $z = (z_1, \dots, z_m) \in \mathbb{C}^m$ and $r > 0$. We set $\|z\| = \sqrt{|z_1|^2 + \cdots + |z_m|^2}$ and

$$S_m(r) = \{z \in \mathbb{C}^m : \|z\| = r\}, \qquad B_m(r) = \{z \in \mathbb{C}^m : \|z\| < r\}.$$

Define the differential operators

$$\partial = \sum_{j=1}^{m} \frac{\partial}{\partial z_j} dz_j \qquad \text{and} \qquad \overline{\partial} = \sum_{j=1}^{m} \frac{\partial}{\partial \overline{z}_j} d\overline{z}_j.$$

УДК 517.55
2020 *Mathematics Subject Classification:* 30D35, 32H30.

We set $d = \partial + \overline{\partial}$, $d^c = \frac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial)$ and

$$\eta_m(z) := \left(dd^c\|z\|^2\right)^{m-1}, \qquad \sigma_m(z) := d^c \log \|z\|^2 \wedge \left(dd^c \log \|z\|^2\right)^{m-1}, \quad z \in \mathbb{C}^m \setminus \{0\}.$$

Set

$$n(t, \nu_f) = \begin{cases} \sum_{|z| \le t} \nu_f(z), & \text{if } m = 1, \\ \int_{|\nu| \cap B_m(t)} \nu_f(z)\eta_m(z), & \text{if } m \ge 2. \end{cases}$$

The counting functions of ν_f and the proximity function of f can be defined by

$$N(r, \nu_f) = \int_1^r \frac{n(t, \nu_f)}{t^{2m-1}}\, dt, \quad 1 < r < \infty, \qquad m(r, f) = \int_{S_m(r)} \log^+ |f(z)| \sigma_m(z),$$

respectively.

The Nevanlinna characteristic function of f is defined as $T(r, f) = m(r, f) + N(r, f)$. Then the first main theorem states (see [15, Chapter 4, A5.1]) that

$$T\left(r, \frac{1}{f - a}\right) = T(r, f) + O(1)$$

for any $a \in \mathbb{C} \cup \{\infty\}$.

Definition 1. *The defect $\delta(a, f)$ of zeros of $f - a$ is defined as*

$$\delta(a, f) = 1 - \limsup_{r \to \infty} \frac{N(r, \frac{1}{f-a})}{T(r, f)}.$$

A meromorphic function $a(z)$ is said to be a small function of a meromorphic function f if it satisfies $T(r, a) = o(T(r, f))$ as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure.

The collection of all small functions of f is denoted by S_f. Let $\hat{S}_f = S_f \cup \{\infty\}$.

Definition 2. *For a constant value a, we denote the set of all a-points (counting multiplicities or CM) of f by $E(a, f)$, and all distinct a-points (ignoring multiplicities or IM) of f by $\overline{E}(a, f)$. For any two non-constant meromorphic functions f and g, we say that f and g share the value a counting multiplicities, if $E(a, f) = E(a, g)$. On the other hand, if $\overline{E}(a, f) = \overline{E}(a, g)$ we say that f and g share the value a ignoring multiplicities. Similarly we can also define CM and IM sharing of a small function.*

We are now going to define some relaxed sharing notions, namely partial sharing, defined as follows.

Definition 3. *We say that a meromorphic function f shares $a \in \hat{S}_f$ partially CM with a meromorphic function g if $E(a, f) \subseteq E(a, g)$.*

Definition 4. *The order and hyper-order of a meromorphic function f are denoted by $\rho(f)$ and $\rho_2(f)$, respectively, and are defined by*

$$\rho(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}, \qquad \rho_2(f) = \limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r}.$$

Definition 5. *Let $c \in \mathbb{C}^m \setminus \{0\}$, $n \geq 2$ be a natural number and $f(z)$ be a meromorphic function in $\mathbb{C}^m$. The shift operator of $f(z)$ is denoted by $f(z+c)$. The difference and nth difference operators are denoted by $\Delta_c f(z)$ and $\Delta_c^n f(z)$, respectively, and are defined by*

$$\Delta_c f(z) = f(z+c) - f(z) \qquad \text{and} \qquad \Delta_c^n f(z) = \Delta_c^{n-1}(\Delta_c f(z)).$$

Now let us define $L_c^b f(z) = \sum_{j=0}^{n} b_j f(z+jc)$, where b_j's are constants such that $\sum_{j=0}^{n} b_j = b$. In particular, if $b = 1$, then we get $L_c^1 f(z)$, which generalizes $f(z+c)$. It is easy to see that $\Delta_c^2 f(z) = f(z+2c) - 2f(z+c) + f(z)$. By induction we can get

$$\Delta_c^n f(z) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{k} f(z+jc).$$

Now from the binomial formula $(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n}y^n$, substituting $x = -1$ and $y = 1$, we get $\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{k} = 0$. Thus sum of all coefficients of $f(z+jc)$ in $\Delta_c^n f(z)$ is 0. Therefore $L_c^0 f(z)$ is a generalization of $\Delta_c^n f(z)$.

2 Backgrounds and main results

The uniqueness problem of meromorphic functions plays an important role in the value distribution theory of complex analysis. In 1929, R. Nevanlinna [14] proved the famous five-value theorem that if two non-constant meromorphic functions f and g in $\mathbb{C}$ share five distinct values IM, then $f(z) \equiv g(z)$. In 1977, L.A. Rubel and C.-C. Yang [16] showed that if a non-constant entire function f and its first derivative f' share two distinct values CM, then they are identical. Later the question comes: instead of f' if one considers the difference operator $\Delta_c f$ or the shift operator $f(z+c)$, then what is the least number of values sharing are required? This uniqueness problem is very much interesting as it is very useful to study finite-order meromorphic solutions of various kind of difference equations, delay difference equations. Basics of Nevanlinna's uniqueness theory for difference operator can be found in [6,8,9]. Readers can also see [1,2,13] for most recent results related to uniqueness of meromorphic functions and its generalized shift and difference operators.

Now, it is interesting to study the uniqueness problem for a meromorphic function in several complex variables with its difference or some shift operators. Recently, in [3–5], authors established the difference analogues of second main theorem and logarithmic lemma for several complex variables. Two more important theorems, i.e. difference Picard's theorem and difference Cartan's theorem for several complex variables were proved in [10,11]. In 2018, Z.-X. Liu and Q.-C. Zhang [12] obtained a difference uniqueness result for meromorphic function f of finite order in several variables sharing a small function with $\Delta_c^n f$ and $\Delta_c^{n+1} f$. Very recently, in 2022, W. Wu and T.-B. Cao [19] considered one CM and two partial CM sharing of values between f and $\Delta_c^n f$ and obtained the following uniqueness result.

Theorem A ([19])**.** *Let f be a transcendental meromorphic function on $\mathbb{C}^m$. Suppose that $\limsup_{r\to\infty} \dfrac{\log T(r,f)}{r} = 0$ and $c \in \mathbb{C}^m \setminus \{0\}$ is such that $\Delta_c^n f \not\equiv 0$. If f and $\Delta_c^n f$ share 1 CM and $0, \infty$ partially CM, i.e. $E(1,f) = E(1,\Delta_c^n f)$, $E(0,f) \subset E(0,\Delta_c^n f)$, $E(\infty,f) \supset E(\infty,\Delta_c^n f)$, then $\Delta_c^n f \equiv f$.*

Note that in Theorem A the condition $\limsup_{r\to\infty}\frac{\log T(r,f)}{r}=0$ is stronger than the condition $\rho_2(f)<1$. Now, for $n=1$ in Theorem A, W. Wu and T.-B. Cao [19] obtained the following result.

Theorem B ([19])**.** *Let f be a transcendental meromorphic function on $\mathbb{C}^m$. Suppose that $\limsup_{r\to\infty}\frac{\log T(r,f)}{r}=0$ and $c\in\mathbb{C}^m\backslash\{0\}$ is such that $\Delta_c f\not\equiv 0$. If f and $\Delta_c f$ share 1 CM and $0,\infty$ partially CM, i.e. $E(1,f)=E(1,\Delta_c f)$, $E(0,f)\subset E(0,\Delta_c f)$, $E(\infty,f)\supset E(\infty,\Delta_c f)$, then $\Delta_c f\equiv f$.*

From Theorem B the following question comes naturally.

Question 1. *Instead of one CM and two partial CM sharing, if we consider at least one IM sharing, can we get the same result as in Theorem B?*

In our first result, by considering one IM and two partial CM sharing, we answer Question 1 affirmatively as follows.

Theorem 1. *Let f be a transcendental meromorphic function on $\mathbb{C}^m$. Let us suppose that $\limsup_{r\to\infty}\frac{\log T(r,f)}{r}=0$ and $c\in\mathbb{C}^m\backslash\{0\}$ is such that $\Delta_c f\not\equiv 0$. For $a\neq 0$, if f and $\Delta_c f$ share $0,\infty$ partially CM and a IM, i.e. $E(0,f)\subseteq E(0,\Delta_c f)$, $E(\infty,f)\supseteq E(\infty,\Delta_c f)$, $\overline{E}(a,f)=\overline{E}(a,\Delta_c f)$, then $\Delta_c f\equiv f$.*

From the definition of sharing, we know that IM sharing is weaker than CM sharing. In Theorem 1, we have considered "a IM sharing" in the place of "1 CM sharing" in Theorem B. So we relaxed the sharing condition. Thus, Theorem 1 represents a significant improvement over Theorem B.

The following example is corresponding to Theorem 1.

Example 1. *Let $f(z)=f(z_1,z_2)=e^{z_1+z_2}$ and $c=(c_1,c_2)=(\log 2,0)$. Then*

$$f(z_1+c_1,z_2+c_2)=e^{z_1+\log 2+z_2+0}=e^{z_1+z_2}e^{\log 2}=2f(z_1,z_2).$$

This implies $\Delta_c f(z)=f(z)$.

Next, we focus on uniqueness of meromorphic function f and the shift operator $f(z+c)$ over $\mathbb{C}^m$. In this perspective, W. Wu and T.-B. Cao [19] proved the following result.

Theorem C ([19])**.** *Let f be a transcendental meromorphic function on $\mathbb{C}^m$. Suppose that $\limsup_{r\to\infty}\frac{\log T(r,f)}{r}=0$ and $c\in\mathbb{C}^m\backslash\{0\}$. If f and $f(z+c)$ share 1 CM and $0,\infty$ partially CM, i.e. $E(1,f)=E(1,f(z+c))$, $E(0,f)\subset E(0,f(z+c))$, $E(\infty,f)\supset E(\infty,f(z+c))$, then $f(z+c)\equiv f(z)$.*

In our next theorem, we consider the generalized shift operator $L_c^1 f$ in place of $f(z+c)$. Also, instead of three value sharing, we take one small function and ∞ partial CM sharing.

Theorem 2. *Let f be a transcendental meromorphic function on $\mathbb{C}^m$ such that the condition $\limsup_{r\to\infty} \frac{\log T(r,f)}{r} = 0$ holds. For $c \in \mathbb{C}^m \setminus \{0\}$, let $a_1(z) \in S_f$ be any periodic function of period c and $L_c^1 f(z) \not\equiv 0$. If f and $L_c^1 f$ share $a_1(z), \infty$ partially CM, i.e. $E(a_1(z), f) \subseteq E(a_1(z), L_c^1 f)$, $E(\infty, L_c^1 f) \subseteq E(\infty, f)$ and $\delta(\alpha, f) > 0$ for some c periodic small function $\alpha(z)(\not\equiv a_1(z))$, then $L_c^1 f(z) \equiv f(z)$.*

Now, in Theorem 2, if we try to remove the condition $\delta(\alpha, f) > 0$, then we need one extra small function sharing.

Theorem 3. *Let f be a transcendental meromorphic function on $\mathbb{C}^m$ such that the condition $\limsup_{r\to\infty} \frac{\log T(r,f)}{r} = 0$ holds. For $c \in \mathbb{C}^m \setminus \{0\}$, let $a_1(z)$, $a_2(z) \in S_f$ be any two periodic functions of period c and $L_c^1 f(z) \not\equiv 0$. If f and $L_c^1 f$ share $a_1(z), a_2(z), \infty$ partially CM, i.e. $E(a_1(z), f) \subseteq E(a_1(z), L_c^1 f)$, $E(a_2(z), f) \subseteq E(a_2(z), L_c^1 f)$, $E(\infty, L_c^1 f) \subseteq E(\infty, f)$, then $L_c^1 f(z) \equiv f(z)$.*

Note that, in Theorem C, the authors considered CM sharing of 1, and partial CM sharing of 0 and ∞, but in Theorem 3, we have considered partial CM sharing of two small functions of $a_1(z), a_2(z)$ and ∞ for the generalized operator $L_c^1 f(z)$. As we know any constant is a small function, and $L_c^1 f$ is a generalization of $f(z+c)$, so Theorem 3 is an improved and extended version of Theorem C.

The next example is corresponding to Theorem 3.

Example 2. *Let $f(z) = f(z_1, z_2) = \sin(z_1 + z_2)$ and $c = (c_1, c_2) = (\frac{\pi}{2}, 0)$. Consider*

$$\begin{aligned}
L_c^1 f(z) &= \frac{2}{3} f(z+8c) - \frac{1}{4} f(z+5c) + \frac{1}{3} f(z+4c) + \frac{1}{4} f(z+c) \\
&= \frac{2}{3} \sin(z_1 + z_2 + \frac{8\pi}{2}) - \frac{1}{4} \sin(z_1 + z_2 + \frac{5\pi}{2}) \\
&\quad + \frac{1}{3} \sin(z_1 + z_2 + \frac{4\pi}{2}) + \frac{1}{4} \sin(z_1 + z_2 + \frac{\pi}{2}) \\
&= \frac{2}{3} \sin(z_1 + z_2) - \frac{1}{4} \cos(z_1 + z_2) + \frac{1}{3} \sin(z_1 + z_2) + \frac{1}{4} \cos(z_1 + z_2) \\
&= \sin(z_1 + z_2).
\end{aligned}$$

Thus, $L_c^1 f(z) = f(z)$.

3 Lemmas

Lemma 1 ([5, Theorem 2.1]). *Let f be a non-constant meromorphic function on $\mathbb{C}^m$ and $c \in \mathbb{C}^m \setminus \{0\}$. If $\limsup_{r\to\infty} \frac{\log T(r,f)}{r} = 0$, then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = o(T(r,f))$$

for all $r \notin E$ with

$$\overline{dens}\, E = \limsup_{r\to\infty} \frac{1}{r} \int_{E\cap[1,r]} dt = 0.$$

Lemma 2 ([5, Theorem 2.2]). *Let f be a non-constant meromorphic function on $\mathbb{C}^m$ and $c \in \mathbb{C}^m \setminus \{0\}$. If $\limsup_{r\to\infty} \frac{\log T(r,f)}{r} = 0$, then*

$$T(r, f(z+c)) = T(r,f) + o(T(r,f)), \qquad N(r, f(z+c)) = N(r,f) + o(N(r,f))$$

for all $r \notin E$ with $\overline{\text{dens}}\, E = 0$.

Lemma 3. *Let f be a non-constant meromorphic function on $\mathbb{C}^m$ and $c \in \mathbb{C}^m \setminus \{0\}$. If the condition $\limsup_{r\to\infty} \frac{\log T(r,f)}{r} = 0$ holds, then*

$$N\left(r, \frac{1}{f(z+c)}\right) = N\left(r, \frac{1}{f}\right) + S(r,f)$$

for all $r \notin E$ with $\overline{\text{dens}}\, E = 0$.

Proof. By the First Fundamental Theorem and Lemmas 1, 2 we get

$$\begin{aligned} m\left(r, \frac{1}{f(z+c)}\right) + N\left(r, \frac{1}{f(z+c)}\right) &= T\left(r, \frac{1}{f(z+c)}\right) = T(r, f(z+c)) + S(r,f) \\ &= T(r,f) + S(r,f) = T\left(r, \frac{1}{f}\right) + S(r,f) \\ &= m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right) + S(r,f) \\ &\le m\left(r, \frac{f(z+c)}{f}\right) + m\left(r, \frac{1}{f(z+c)}\right) \\ &\quad + N\left(r, \frac{1}{f}\right) + S(r,f) \\ &= m\left(r, \frac{1}{f(z+c)}\right) + N\left(r, \frac{1}{f}\right) + S(r,f). \end{aligned} \tag{1}$$

Hence,

$$N\left(r, \frac{1}{f(z+c)}\right) \le N\left(r, \frac{1}{f}\right) + S(r,f). \tag{2}$$

Proceeding similarly as in (1), we can obtain

$$N\left(r, \frac{1}{f}\right) \le N\left(r, \frac{1}{f(z+c)}\right) + S(r,f). \tag{3}$$

Combining (2) and (3), we obtain

$$N\left(r, \frac{1}{f(z+c)}\right) = N\left(r, \frac{1}{f}\right) + S(r,f).$$

□

From Lemma 3, we get the following remark.

Remark 1. *Let us denote* $f_c = f(z+c)$ *and* $f_{-c} = f(z-c)$. *Then under the same conditions of Lemma 3 we get*

$$N\left(r, \frac{1}{f_c}\right) = N\left(r, \frac{1}{f}\right) + S(r,f) = N\left(r, \frac{1}{f_{-c}}\right) + S(r,f).$$

Lemma 4 ([19]). *Let* f *be a non-constant meromorphic function on* $\mathbb{C}^m$ *and* $c \in \mathbb{C}^m \setminus \{0\}$. *If the condition* $\limsup\limits_{r\to\infty} \dfrac{\log T(r,f)}{r} = 0$ *holds, then for any periodic small function* $a(z)$ *of* f *with period* c *we have*

$$m\left(r, \frac{\Delta_c f(z)}{f(z) - a(z)}\right) = o(T(r,f))$$

for all $r \notin E$ *with* $\overline{\text{dens}}\, E = 0$.

Lemma 5 ([19]). *Let* f *be a non-constant meromorphic function on* $\mathbb{C}^m$ *and* $a_1, a_2, \cdots, a_q$ *be* q *distinct small functions of* f. *If* $q \geq 3$, *then we have*

$$\frac{q}{3} T(r,f) \leq \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f - a_j}\right) + o(T(r,f))$$

for all $r \notin F$, *where* F *is a Borel subset of the interval* $[0, +\infty)$ *with* $\int_F dr < +\infty$.

The difference polynomial in f is a polynomial in $f(z+c_j)$, where $c_j \in \mathbb{C}^m$, with coefficients $\alpha_j(z)$ such that $T(r, \alpha_j) = S(r,f)$.

Lemma 6. *Let* f *be a non-constant meromorphic on* $\mathbb{C}^m$ *such that*

$$\limsup_{r\to\infty} \frac{\log T(r,f)}{r} = 0,$$

and f *is a solution of* $P(z,f) = 0$, *where* $P(z,f)$ *is difference polynomial in* f. *If* $P(z,a) \not\equiv 0$ *for a small function* a, *then*

$$m\left(r, \frac{1}{f-a}\right) = S(r,f)$$

for all $r \notin E$ *with* $\overline{\text{dens}}\, E = 0$.

Proof. By substituting $f = g + a$ into $P(z,f) = 0$ we obtain

$$Q(z,g) + D(z) = 0, \tag{4}$$

where $Q(z,g) = \sum_\gamma b_\gamma(z) G_\gamma(z,f)$ is a difference polynomial in g such that all of its terms are at least of degree one, and $T(r,D) = S(r,f)$. Also $D \not\equiv 0$, since $P(z,a) \not\equiv 0$.

Next we compute $m(r, 1/g)$. Note that the integral to be evaluated vanishes, when $|g| > 1$. For $|g| \leq 1$, we have

$$\begin{aligned} \left|\frac{Q(z,g)}{g}\right| &= \frac{1}{|g|}\left|\sum_\gamma b_\gamma(z) g(z)^{l_0} g(z+c_1)^{l_1} \cdots g(z+c_\nu)^{l_\nu}\right| \\ &\leq \sum_\gamma |b_\gamma(z)| \left|\frac{g(z+c_1)}{g(z)}\right|^{l_1} \cdots \left|\frac{g(z+c_\nu)}{g(z)}\right|^{l_\nu}, \end{aligned}$$

since $\sum_{j=0}^{\nu} l_j \geq 1$ for all γ. Therefore by equation (4) and Lemma 1 we get

$$m\left(r, \frac{1}{g}\right) \leq m\left(r, \frac{D}{g}\right) + m\left(r, \frac{1}{D}\right) = m\left(r, \frac{Q(z,g)}{g}\right) + m\left(r, \frac{1}{D}\right) \leq T(r,D) + S(r,f) = S(r,f).$$

Since $g = f - a$, therefore $m\left(r, \frac{1}{f-a}\right) = S(r,f)$. □

4 Proofs of the theorems

Proof of Theorem 1. Let us assume $\Delta_c f \not\equiv f$. The given sharing conditions $E(0,f) \subseteq E(0,\Delta_c f)$, $E(\infty,f) \supseteq E(\infty,\Delta_c f)$, $\overline{E}(a,f) = \overline{E}(a,\Delta_c f)$ imply

$$N\left(r,\frac{1}{f}\right) \le N\left(r,\frac{1}{\Delta_c f}\right), \qquad N(r,\Delta_c f) \le N(r,f)$$

and

$$\overline{N}\left(r,\frac{1}{f-a}\right) = \overline{N}\left(r,\frac{1}{\Delta_c f - a}\right).$$

By first fundamental theorem and Lemma 4 we get

$$\begin{aligned} T(r,f) &= T\left(r,\frac{1}{f}\right) + O(1) = m\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f}\right) + O(1) \\ &\le m\left(r,\frac{\Delta_c f}{f}\right) + m\left(r,\frac{1}{\Delta_c f}\right) + N\left(r,\frac{1}{\Delta_c f}\right) + O(1) = T(r,\Delta_c f) + S(r,f). \end{aligned} \tag{5}$$

On the other hand, by Lemma 4 we obtain

$$\begin{aligned} T(r,\Delta_c f) &= m(r,\Delta_c f) + N(r,\Delta_c f) + O(1) \\ &\le m\left(r,\frac{\Delta_c f}{f}\right) + m(r,f) + N(r,f) + O(1) = T(r,f) + S(r,f) \end{aligned} \tag{6}$$

From (5) and (6), we have $T(r,f) = T(r,\Delta_c f) + S(r,f)$. Hence,

$$S(r,f) = S(r,\Delta_c f) := S(r).$$

Let us consider

$$\Phi = \frac{\Delta_c f}{f}. \tag{7}$$

Now, by Lemma 4, we get $m(r,\Phi) = S(r)$. From the sharing conditions, it is clear that Φ is an entire function. Hence, $N(r,\Phi) = S(r)$ and then

$$T(r,\Phi) = S(r). \tag{8}$$

Let us assume $\Phi \not\equiv 1$. From the sharing conditions and from (8) we obtain

$$\begin{aligned} \overline{N}\left(r,\frac{1}{f-a}\right) &= \overline{N}\left(r,\frac{1}{\Delta_c f - a}\right) \le \overline{N}\left(r,\frac{f}{\Delta_c f - f}\right) \\ &= \overline{N}\left(r,\frac{1}{\frac{\Delta_c f}{f} - 1}\right) \le N\left(r,\frac{1}{\Phi - 1}\right) \le T(r,\Phi) = S(r). \end{aligned} \tag{9}$$

By using (8), (9), we get

$$\begin{aligned} \overline{N}\left(r,\frac{1}{f - \frac{a}{\Phi}}\right) &= \overline{N}\left(r,\frac{1}{\frac{1}{\Phi}(\Delta_c f - a)}\right) \le \overline{N}(r,\Phi) + \overline{N}\left(r,\frac{1}{\Delta_c f - a}\right) \\ &\le T(r,\Phi) + S(r) = S(r). \end{aligned} \tag{10}$$

Note that, $(f_c)_{-c} = f$. With the help of equation (9) and applying Remark 1 for the function $f_c - a$, we obtain

$$\overline{N}\left(r, \frac{1}{f_c - a}\right) = \overline{N}\left(r, \frac{1}{(f_c - a)_{-c}}\right) + S(r) = \overline{N}\left(r, \frac{1}{f - a}\right) + S(r) \le S(r). \tag{11}$$

Now, from (7) we can write $f_c = (\Phi + 1)f$. Using this fact and (11), we get

$$\begin{aligned}\overline{N}\left(r, \frac{1}{f - \frac{a}{\Phi+1}}\right) = \overline{N}\left(r, \frac{1}{\frac{1}{\Phi+1}(f_c - a)}\right) &\le \overline{N}\left(r, \Phi + 1\right) + \overline{N}\left(r, \frac{1}{f_c - a}\right) \\ &\le T(r, \Phi) + S(r) = S(r).\end{aligned} \tag{12}$$

Next we claim that a, $\frac{a}{\Phi}$ and $\frac{a}{\Phi+1}$ are distinct. First suppose that $a \equiv \frac{a}{\Phi}$, which implies $\Phi \equiv 1$, a contradiction. Now we assume that $a \equiv \frac{a}{\Phi+1}$, which implies $\Phi + 1 \equiv 1$, i.e. $\Phi \equiv 0$. This implies $\Delta_c f \equiv 0$, a contradiction. Next we suppose that $\frac{a}{\Phi} \equiv \frac{a}{\Phi+1}$, which implies $1 \equiv 0$, a contradiction.

Now applying Lemma 5 for $q = 3$ and using equations (9), (10) and (12), we get

$$T(r, f) \le \overline{N}\left(r, \frac{1}{f - a}\right) + \overline{N}\left(r, \frac{1}{f - \frac{a}{\Phi}}\right) + \overline{N}\left(r, \frac{1}{f - \frac{a}{\Phi+1}}\right) + S(r) = S(r),$$

a contradiction. So, $\Phi \equiv 1$. Therefore $\Delta_c f \equiv f$. □

Proof of Theorem 2. Consider

$$\Psi(z) = \frac{L_c^1 f(z) - a_1(z)}{f(z) - a_1(z)}. \tag{13}$$

If $\Psi(z) \equiv 1$, then we are done. So, let us assume $\Psi(z) \not\equiv 1$. Now $E(a_1(z), f) \subseteq E(a_1(z), L_c^1 f)$ and $E(\infty, L_c^1 f) \subseteq E(\infty, f)$ implies that $\Psi(z)$ is an entire function. Thus using Lemma 1 and $\sum_{j=0}^{n} b_j = 1$, we get

$$\begin{aligned} T\left(r, \Psi(z)\right) &= m\left(r, \Psi(z)\right) \\ &\le m\left(r, \frac{\sum_{j=0}^{n} b_j \left[f(z + jc) - a_1(z)\right]}{f(z) - a_1(z)}\right) + m\left(r, \frac{\sum_{j=0}^{n} b_j a_1(z) - a_1(z)}{f(z) - a_1(z)}\right) \\ &\le \sum_{j=0}^{n} \left[m(r, b_j) + m\left(r, \frac{f(z + jc) - a_1(z)}{f(z) - a_1(z)}\right)\right] + S(r, f) = S(r, f). \end{aligned}$$

Consider $P(z, f) = \left(L_c^1 f(z) - a_1(z)\right) - \Psi(z)\left(f(z) - a_1(z)\right)$. As $\Psi(z)$ is a small function of $f(z)$, thus $P(z, f)$ is polynomial in $f(z)$ and its shifts with small functions as coefficient. So, from (13) we obtain $P(z, f) = 0$. Since $\sum_{j=0}^{n} b_j = 1$, for some periodic small function $\alpha(z)(\not\equiv a_1(z))$ we get $P(z, \alpha) = (\alpha - a_1) - \Psi(\alpha - a_1)$.

Let us assume $P(z, \alpha) \equiv 0$, which implies $\Psi(z) \equiv 1$, a contradiction. Hence $P(z, \alpha) \not\equiv 0$. Thus by Lemma 6 we have $m\left(r, \frac{1}{f - \alpha}\right) = S(r, f)$, which implies $T(r, f) = N\left(r, \frac{1}{f - \alpha}\right) + S(r, f)$. So we obtain $\delta(\alpha, f) = 0$, which contradicts our assumption $\delta(\alpha, f) > 0$. Therefore, proof of Theorem 2 follows. □

Proof of Theorem 3. Consider

$$\Psi_1(z) = \frac{L_c^1 f(z) - a_1(z)}{f(z) - a_1(z)}, \qquad \Psi_2(z) = \frac{L_c^1 f(z) - a_2(z)}{f(z) - a_2(z)}. \tag{14}$$

Now discuss the following three cases.

Case 1. If $\Psi_1(z) \equiv 1$ or $\Psi_2(z) \equiv 1$, then $L_c^1 f(z) \equiv f(z)$.

Case 2. If $\Psi_1(z) \not\equiv 1$ and $\Psi_2(z) \not\equiv 1$ but $\Psi_1(z) \equiv \Psi_2(z)$, then by simple calculations we get $L_c^1 f(z) \equiv f(z)$.

Case 3. Let $\Psi_1(z) \not\equiv 1$ and $\Psi_2(z) \not\equiv 1$ and $\Psi_1(z) \not\equiv \Psi_2(z)$. Then eliminating $L_c^1 f(z)$ from (14) we deduce

$$f(z) = \frac{a_2(z) - a_1(z) + a_1(z)\Psi_1(z) - a_2(z)\Psi_2(z)}{\Psi_1(z) - \Psi_2(z)}. \tag{15}$$

Now $E(a_1(z), f) \subseteq E(a_1(z), L_c^1 f)$, $E(a_2(z), f) \subseteq E(a_2(z), L_c^1 f)$ and $E(\infty, L_c^1 f) \subseteq E(\infty, f)$ imply that both $\Psi_1(z)$ and $\Psi_2(z)$ are entire functions. Similarly as in the proof of Theorem 2, using Lemma 1 and $\sum_{j=0}^{n} b_j = 1$, we obtain

$$T(r, \Psi_1(z)) = S(r, f), \qquad T(r, \Psi_2(z)) = S(r, f). \tag{16}$$

From (15) and (16) we obtain $T(r, f) = S(r, f)$, which is a contradiction. Therefore, from Cases 1, 2 and 3, we conclude $L_c^1 f(z) \equiv f(z)$. □

Acknowledgement

The authors wish to thank the referee for his/her valuable suggestions towards the improvement of this paper.

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Received 15.08.2024
Revised 14.12.2024

Майті С., Бісвас Т. *Єдиність мероморфних функцій та їх різницевих чи операторів зсуву кількох комплексних змінних* // Карпатські матем. публ. — 2025. — Т.17, №2. — С. 472–482.

У цій статті досліджено задачі єдиності мероморфних функцій із їх різницевими або узагальненими лінійними операторами зсуву в контексті часткового співпадіння у кількох комплексних змінних. Послабивши одну умову співпадіння з урахування кратностей до умови співпадіння без урахування кратностей, один із результатів нашої роботи покращує результат В. Ву та Т.-Б. Цао [Comput. Methods Funct. Theory 2022, **22** (2), 379–399]. Інші наші результати також розширюють і вдосконалюють деякі результати цієї ж статті.

Ключові слова і фрази: мероморфна функція, єдиність, різницевий оператор, кілька комплексних змінних.