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Certain characteristics of 2-variable q-truncated Tricomi functions

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This study defines certain properties of 2-variable q-truncated Tricomi functions $h_{n,q}(x,y)$, such as integral forms, generating functions and series definitions. Additionally, we present the associated 2-variable q-Laguerre polynomials, which we utilize to obtain higher order of 2-variable q-truncated Tricomi functions and examine the characteristics they possess.

Key words and phrases: quantum calculus, q-truncated exponential polynomial, q-Laguerre polynomial.

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Introduction

The powerful mathematical tool is dubbed as q-calculus. It is a generalization of ordinary calculus and has found many uses in various scientific, mathematical, and statistical domains, such as quantum mechanics, probability theory, number theory, and combinatorics. Many q-special functions have been introduced and studied recently by several researchers working in the field (see, for example, [3,4,10,18–20,23]).

Some basic quantum calculus notations and terminology are briefly reviewed from [2,12] as an introduction to the q-calculus, $0<|q|<1$. The q-analogue of complex number b has been given as $[b]_q=(1-q^b)/(1-q)$. The q-factorial number $[l]_q!$ has been provided through $[l]_q!=(q;q)_l/(1-q)^l$, $l\in\mathbb{N}$, where for $b\in\mathbb{C}$, we have

$$(b;q)_l=\begin{cases}1, & l=0,\\ \prod_{k=0}^{l-1}(1-bq^k), & l\in\mathbb{N}.\end{cases}$$

The Gauss q-binomial coefficient is offered here

$$\begin{bmatrix} l\\ k\end{bmatrix}_q=\frac{[l]_q!}{[l-k]_q![k]_q!},\quad k=0,1,\dots,l.$$

The two q-exponential functions $e_q(r)$ and $E_q(r)$ are defined as follows:

$$e_q(r)=\frac{1}{(r(1-q);q)_\infty}=\sum_{l=0}^{\infty}\frac{r^l}{[l]_q!},\quad |r|<\frac{1}{1-q},\tag{1}$$

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and

$$E_q(r) = (-r(1-q);q)_\infty = \sum_{l=0}^{\infty} q^{\binom{l}{2}} \frac{r^l}{[l]_q!}, \quad r \in \mathbb{C}. \tag{2}$$

The q-derivative operator of a function $f(r)$ with respect to r is defined by (see [13])

$$D_{q,r}f(r) = \frac{f(qr)-f(r)}{r(q-1)}, \quad r \neq 0, \quad \text{with} \quad D_{q,r}f(0) = f'(0).$$

We recall [12] some formulas for q-derivative as follows:

$$D_{q,r}(f(r)g(r)) = f(r)D_{q,r}g(r) + g(qr)D_{q,r}f(r), \tag{3}$$

$$D_{q,r}r^l = [l]_q r^{l-1}, \tag{4}$$

and $D_{q,r}e_q(\alpha r) = \alpha e_q(\alpha r)$, where α is arbitrary constant. The Heine's binomial formula is the following (see [14, p. 28, 8.1)])

$$\frac{1}{(1-t)_q^\alpha} = \sum_{k=0}^{\infty} \frac{[\alpha]_q[\alpha+1]_q \ldots [\alpha+k-1]_q}{[k]_q!} t^k = \sum_{k=0}^{\infty} \begin{bmatrix} \alpha+k-1 \\ k \end{bmatrix}_q t^k, \quad |t|<1, \quad \alpha \in \mathbb{R}. \tag{5}$$

According to [14, p. 48, 14.5], the q-derivative of the function $f_q(x) = 1/(1-t)_q^\alpha$ is given by

$$D_{q,t}\frac{1}{(1-t)_q^\alpha} = \frac{[\alpha]_q}{(1-t)_q^{\alpha+1}}. \tag{6}$$

It is worth to mentioning (see [14, p. 74, 21.6]) that

$$\int_0^{\frac{1}{1-q}} r^{\alpha-1} E_q(-qr) d_q r = \Gamma_q(\alpha), \quad \alpha > 0, \quad r \in \mathbb{C}. \tag{7}$$

Truncated exponential polynomials have proven valuable in physics and other study domains as they may be found in numerous optical and quantum mechanical problems. It has been demonstrated in the physical sciences by evaluating integrals with specified function products. Furthermore, these polynomials are important in applied mathematics because they may be described using a variety of approaches, including orthogonality requirements, generating functions, differential equations, integral transformations, recurrence relations, and operational formulas. Generalizations and extensions of these polynomials are valuable tools in applied mathematics and approximation theory, allowing for flexible function changes. The following consequence series defines the classical truncated exponential polynomials $e_l(r) = \sum_{k=0}^{l} r^k/k!$ (see [2]), which is represents $(l+1)$ elements of the Maclaurin series of the function e^r. The features and higher order of classical truncated exponential polynomials were defined by G. Dattoli et al. [8]. The below integral form (see [8], p. 15, 43) was used by G. Dattoli et al. in 2003 to present the truncated Tricomi function

$$h_l(r) = \frac{1}{l!}\int_0^\infty \exp(-\zeta) L_l(r,\zeta) d\zeta. \tag{8}$$

Many academics in mathematics and physics disciplines consider their generalizations and extensions with applications because of their numerous helpful features (e.g., [7,11,15–17,23]). N. Raza et al. [19] presented and examined q-truncated exponential polynomials recently.

The importance of q-truncated exponential polynomials $E_{l,q}(r)$, their generalizations and q-truncated families came from the various applications in mathematical and scientific disciplines. Also, in the broad applications of 2-variable q-truncated exponential polynomials in mathematical modeling, applied sciences and quantum mechanics, represent interactions between two connected quantum systems, which correspond to q-deformed algebraic structures. Their applications in signal processing and optics include studying 2-variable interdependence, such as wave functions and light propagation [4].

For q-truncated exponential polynomials, we have the following formulas as generating function and series definition [19]:

$$\frac{1}{(1-t)_q}E_q(rt) = \sum_{l=0}^{\infty} E_{l,q}(r)t^l, \qquad E_{l,q}(r) = \sum_{k=0}^{l} q^{\binom{k}{2}} \frac{r^k}{[k]_q!}.$$

The integral form produces the q-truncated exponential polynomials $E_{l,q}(r)$ [19]:

$$E_{l,q}(r) = \frac{1}{[l]_q!}\int_0^{\frac{1}{1-q}} E_q(-q\zeta)(\zeta+r)_q^l d_q\zeta.$$

The first kind q-Bessel function $J_l(r;q)$ is provided a series formulation via [5,6]:

$$J_l(r;q) = \frac{1}{(q;q)_l}\sum_{k=0}^{\infty}(-1)^k \frac{(r/2)^{l+2k}}{(q;q)_k(q^{l+1};q)_k} = \sum_{k=0}^{\infty}\frac{(-1)^k(r/2)^{l+2k}}{[l+k]_q![k]_q!}.$$

It has an absolute convergence for $|r| < 2$. The lth order q-Tricomi Bessel function of the first type $C_{l,q}(r)$ is described using the series formula [21]:

$$C_{l,q}(r) = \frac{1}{(q;q)_l}\sum_{k=0}^{\infty}(-1)^k \frac{r^k}{(q;q)_k(q^{l+1};q)_k} = \sum_{k=0}^{\infty}\frac{(-1)^k r^k}{[k]_q![l+k]_q!}.$$

For every value of r, its absolute convergence is present. For $l = 0$, aforementioned equation gives 0th order q-Bessel Tricomi function (see [3])

$$C_{0,q}(r) = \sum_{k=0}^{\infty}\frac{(-1)^k r^k}{([k]_q!)^2}. \tag{9}$$

It has an absolute convergence for all values of r. The relation between 0th order q-Bessel Tricomi function and q-exponential function can be written as $C_{0,q}(rt) = e_q(-D_{q,r}^{-1}t)\{1\}$ [3], where (see [12])

$$D_{q,r}^{-1}f(r) := \int_0^r f(\zeta)d_q\zeta = (1-q)r\sum_{l=0}^{\infty} q^l f(rq^l) \quad \text{with} \quad D_{q,r}^{-1}\{1\} = r,$$

and $(D_{q,r}^{-1})^k\{1\} = r^k/[k]_q!$, $k \in \mathbb{N}\cup\{0\}$. As stated in [3], the q-derivative of the 0th order q-Tricomi function $C_{0,q}(rt)$ yields

$$-D_{q,r}rD_{q,r}C_{0,q}(rt) = tC_{0,q}(rt). \tag{10}$$

The 2VqLP $L_{l,q}(r,s)$ are provided via the next series [3]:

$$L_{l,q}(r,s) = [l]_q!\sum_{k=0}^{l}\frac{(-1)^k r^k s^{l-k}}{([k]_q!)^2[l-k]_q!}. \tag{11}$$

According to series description, the mth order q-Laguerre polynomials of two variables are described as [3]:

$$ {}_{[m]}L_{l,q}(r,s) = [l]_q! \sum_{k=0}^{[l/m]} \frac{r^k s^{l-mk}}{([k]_q!)^2 [l-mk]_q!}. \tag{12} $$

This motivated us to generate certain features of 2-variable q-truncated Tricomi functions, as well as their higher order by introducing the associated 2-variable q-Laguerre polynomials and investigating the associated formalism. Polynomials are important in many fields, including mathematical modeling, applied sciences and quantum mechanics. They represent interactions between two connected quantum systems, resulting in q-deformed algebraic structures. Additionally, like many other special functions, it has been widely employed in q-truncated Tricomi functions to investigate overlap, which is most evident in their behavior when variables or parameters are changed. The 2-variable q-truncated Tricomi functions, their higher order and associated 2-variable q-Laguerre polynomials will be of special interest to scientists. These come from their q-analogue research and have revealed new and fascinating possibilities that could expand the theory of q-special functions, which have attracted more attention. Additionally, other q-special functions, it may be widely employed in q-truncated Tricomi functions to investigate overlap, which is most evident in how they behave when variables or parameters are changed. It is frequently possible to investigate these functions through their interrelations and recurrence qualities. Through this paper, we examine 2-variable q-truncated Tricomi functions $h_{l,q}(r,s)$ via integral forms and determine their features, including q-differential equation, generating functions and series formulations. Likewise, we evaluate the features of the associated 2-variable q-truncated Tricomi functions and construct higher order of them via the associated 2-variable q-Laguerre-type polynomials.

Recently, the q-truncated exponential polynomials [19], their generalizations and some of their families [4] were introduced and examined. The following section will examine how the scenario can be expanded to explore 2-variable q-truncated Tricomi functions and devote ourselves to exploring several of their features. Also, we examine associated 2-variable q-Laguerre polynomials and apply it to generate several features of a higher order of 2-variable q-truncated Tricomi functions.

1 Two-variable q-truncated Tricomi functions

This section discusses the 2-variable q-truncated Tricomi functions and their characteristics. Additionally, the associated 2-variable q-Laguerre polynomials are presented and used to generate higher order q-truncated Tricomi functions and their features.

We create the 2-variable q-truncated Tricomi function employing as the appropriate integral form in the context of equation (8):

$$ h_{l,q}(r,s) = \frac{1}{[l]_q!} \int_0^{\frac{1}{1-q}} E_q(-q\zeta) L_{l,q}(r,s\zeta) d_q\zeta, \tag{13} $$

where $E_q(\zeta)$ is the q-exponential function given by series (2) and $L_{l,q}(r,\zeta)$ is the 2-variable q-Laguerre polynomials given [3] by the generating function $C_{0,q}(rt)e_q(st) = \sum_{l=0}^{\infty} L_{l,q}(r,s)\frac{t^l}{[l]_q!}$ and their series is given by (11).

The series definition of the 2-variable q-truncated Tricomi function is obtained via applying equations (7) and (11), namely

$$h_{l,q}(r,s) = \sum_{k=0}^{l} \frac{(-1)^k r^k s^{l-k}}{([k]_q!)^2}, \quad r,s \in \mathbb{C}. \tag{14}$$

It has an absolute convergence for all finite values of r and s.

Theorem 1. *The two-variable truncated q-Tricomi function has the next generating formula:*

$$\sum_{l=0}^{\infty} h_{l,q}(r,s)t^l = \frac{C_{0,q}(rt)}{1-st}. \tag{15}$$

Proof. Considering equation (14), the result is

$$\sum_{l=0}^{\infty} h_{l,q}(r,s)t^l = \sum_{l=0}^{\infty}\sum_{k=0}^{l} \frac{(-1)^k r^k s^{l-k}}{([k]_q!)^2} t^l.$$

Employing the appropriate series rearrangement method, we obtain

$$\sum_{l=0}^{\infty} h_{l,q}(r,s)t^l = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k r^k s^l}{([k]_q!^2)} t^{l+k} = \sum_{l=0}^{\infty} s^l t^l \sum_{k=0}^{\infty} \frac{(-1)^k r^k}{([k]_q!)^2} t^k.$$

Applying equation (5), for $\alpha = 1$ and equation (9), we obtain assertion (15). □

Motivated by the importance of the Laguerre-type derivative $D_t\, tD_t$ [9, 22] and its applications in mathematical simulations of viscous fluid vibration phenomena, as well as in engineering problems such as oscillating chains (see [1, pp. 282–284]), we note that such operators also appear in population dynamics and in the solution of Cauchy problems related to linear dynamical systems. In this work, we employ the q-Laguerre derivative $D_{q,t}\, tD_{q,t}$ [3] to derive the q-differential equation satisfied by the 2-variable q-truncated Tricomi function $h_{l,q}(r,s)$.

Theorem 2. *The q-differential equation satisfied by $h_{l,q}(r,s)$ is*

$$[(rsD_{q,r}rD_{q,r} - (qs([l-1]_q)^2 - r + s))D_{q,r}rD_{q,r} - ([l]_q)^2]h_{l,q}(r,s) = 0. \tag{16}$$

Proof. Applying the q-derivative for formula (15) with regard to r, we get

$$\sum_{l=0}^{\infty} D_{q,r}rD_{q,r}h_{l,q}(r,s)t^l = \frac{D_{q,r}rD_{q,r}C_{0,q}(rt)}{1-st}.$$

In view of equation (10) and then again using equation (15) in the right side of resultant equation, we obtain

$$\sum_{l=0}^{\infty} D_{q,r}rD_{q,r}h_{l,q}(r,s)t^l = -\sum_{l=0}^{\infty} h_{l,q}(r,s)t^{l+1}.$$

Comparing the coefficients of equal powers of t on both sides of the preceding equation, we obtain

$$-D_{q,r}rD_{q,r}h_{l,q}(r,s) = h_{l-1,q}(r,s). \tag{17}$$

Moreover, applying formula (3) for each part of formula (15) with respect to t by taking $f(r) = C_{0,q}(rt)$ and $g(r) = 1/(1-st)$, we obtain

$$\sum_{l=0}^{\infty} h_{l,q}(r,s) D_{q,t} t D_{q,t} t^{l} = \frac{D_{q,t} t D_{q,t} C_{0,q}(rt)}{1-qst} + C_{0,q}(rt) D_{q,t} \frac{1}{1-st}, \tag{18}$$

when applied formula (4) to left part and applied formulas (10), (6) to the right part of formula (18), we obtain

$$\sum_{l=1}^{\infty} ([l]_q)^2 h_{l,q}(r,s) t^{l-1} = -r \frac{C_{0,q}(rt)}{1-qst} + C_{0,q}(rt) \frac{1}{(1-st)_q^2}. \tag{19}$$

It is worth to mention (see [14, p. 8, 3.6]) that $(1-t)_q^{l+1} = (1-t)_q^{l}(1-q^l t)$. Using this formula for $l=1$, right side of (19) yields

$$\sum_{l=1}^{\infty} ([l]_q)^2 h_{l,q}(r,s) t^{n-1} = -r \frac{C_{0,q}(rt)}{1-qst} + C_{0,q}(rt) \frac{s}{(1-st)(1-qst)}.$$

Consequently, by substituting (15) into the right-hand side of the above formula, we obtain

$$\sum_{l=1}^{\infty} ([l]_q)^2 h_{l,q}(r,s) t^{l-1} = \sum_{n=0}^{\infty} qs([l]_q)^2 h_{l,q}(r,s) t^{l} - r \sum_{n=0}^{\infty} h_{l,q}(r,s) t^{s} + rs \sum_{l=0}^{\infty} h_{l,q}(r,s) t^{l+1} + s \sum_{n=0}^{\infty} h_{l,q}(r,s) t^{l}.$$

Equating the coefficients of equal powers of t on both sides of the preceding formula, we obtain the following recurrence relation

$$h_{l+1,q}(r,s) = \frac{1}{([l+1]_q)^2} [qs([l]_q)^2 - r + s] h_{l,q}(r,s) + \frac{rs}{([l+1]_q)^2} h_{l-1,q}(r,s).$$

Substituting l by $l-1$ in the above equation, we obtain

$$h_{l,q}(r,s) = \frac{1}{([l]_q)^2} [qs([l-1]_q)^2 - r + s] h_{l-1,q}(r,s) + \frac{rs}{([l]_q)^2} h_{l-2,q}(r,s).$$

Multiplying each term of the preceding equation by $([l]_q)^2$, we obtain the equality $rs\, h_{l-2,q}(r,s) + \left[qs([l-1]_q)^2 - r + s\right] h_{l-1,q}(r,s) - ([l]_q)^2 h_{l,q}(r,s) = 0$. Then, by substituting equation (17) into the left-hand side of this expression, we arrive at the assertion (16). ☐

Example 1. *Applying formula* (16)*, we have the following:*

$$[(rsD_{q,r} r D_{q,r} - (qs([2]_q)^2 - r + s)) D_{q,r} r D_{q,r} - ([3]_q)^2] h_{3,q}(r,s) = 0$$

and

$$[(rsD_{q,r} r D_{q,r} - (qs([5]_q)^2 - r + s)) D_{q,r} r D_{q,r} - ([6]_q)^2] h_{6,q}(r,s) = 0.$$

After introducing the symbolic operator for q-Tricomi Bessel function of first type $C_{\nu,q}(r)$, we discuss the related 2-variable q-Laguerre polynomials and the associated q-truncated Tricomi function.

We define the symbolic operator $\hat{c}_{\nu,q}$ acting on the function $\psi_{r,q} := \psi_q(r) = \frac{\Gamma_q(r+1)}{\Gamma_q(\nu+1)}$ by

$$\hat{c}^k_{\nu,q}\psi_{r,q} = \psi_{r+k,q} = \frac{\Gamma_q(r+k+1)}{\Gamma_q(r+k+\nu+1)},$$

which fulfill the characteristic $\hat{c}^k_{\nu,q}\hat{c}^l_{\nu,q} = \hat{c}^{k+l}_{\nu,q}$.

In particular, for $r = 0$, we have

$$\hat{c}^k_{\nu,q}\,\psi_{0,q} = \frac{\Gamma_q(k+1)}{\Gamma_q(k+\nu+1)}, \tag{20}$$

where $\psi_{0,q}$ is called the vacuum function, because for $k = 0$ (that is, when no operator is applied), we have $\psi_{0,q} = 1/\Gamma_q(\nu+1)$, which follows directly from the definition of $\psi_{r,q}$.

The series definition of the extension of q-Tricomi Bessel function $C_{\nu,q}(r)$ can be expressed thereby in light of formulas (9) and (20) by

$$C_{\nu,q}(r) = C_{0,q}(\hat{c}_{\nu,q}r)\psi_{0,q} = \sum_{k=0}^{\infty}\frac{(-1)^k c^k_{\nu,q} r^k}{([k]_q!)^2}\psi_{0,q}.$$

Using formula (20), aforementioned equation gives

$$C_{\nu,q}(r) = \sum_{k=0}^{\infty}\frac{(-1)^k r^k}{[k]_q!\Gamma q(k+\nu+1)}, \quad \nu\in\mathbb{R}. \tag{21}$$

It should be noted that the series converges absolutely in the complex domain $|r| < 1/(1-q)^2$.

We present the associated 2-variable q-Laguerre polynomials $L^{\nu}_{l,q}(r,s)$ via the next generating function in the context of the formula (21) by

$$C_{\nu,q}(rt)e_q(st) = \sum_{l=0}^{\infty} L^{\nu}_{l,q}(r,s)\frac{t^l}{\Gamma_q(l+\nu+1)}. \tag{22}$$

By utilizing formulas (1) and (21) to expand the left part of the previously stated equation while comparing the same powers of t from each part of the consequent equation, we come to

$$L^{\nu}_{l,q}(r,s) = \Gamma_q(l+\nu+1)\sum_{k=0}^{l}\frac{(-1)^k r^k s^{l-k}}{[k]_q!\Gamma_q(k+\nu+1)[l-k]_q!}. \tag{23}$$

The integral form of the 3-parameter, 3-variable truncated q-Tricomi function $h^{(\alpha)}_{l,\nu,q}(r,s)$ is expressed as

$$h^{(\alpha)}_{l,\nu,q}(r,s) = \frac{1}{\Gamma_q(l+\nu+1)}\int_0^{\frac{1}{1-q}}\zeta^{\alpha}E_q(-q\zeta)L^{\nu}_{l,q}(r,s\zeta)d_q\zeta, \tag{24}$$

where $r,s,\zeta\in\mathbb{C}$, $\alpha,\nu\in\mathbb{R}$, $l\in\mathbb{N}\cup\{0\}$, which on using equations (23) and (7), gives the following series definition of 3-parameter 2-variable truncated q-Tricomi function $h^{(\alpha)}_{l,\nu,q}(r,s)$:

$$h^{(\alpha)}_{l,\nu,q}(r,s) = \sum_{k=0}^{l}\frac{(-1)^k r^k s^{l-k}\Gamma_q(l-k+\alpha+1)}{[k]_q!\Gamma_q(k+\nu+1)[l-k]_q!}. \tag{25}$$

We offer a theorem to characterize the generating function for the 3-parameter 2-variable q-truncated Tricomi function $h^{(\alpha)}_{l,\nu,q}(r,s)$.

Theorem 3. *The 3-parameter 2-variable q-truncated Tricomi function $h^{(\alpha)}_{l,\nu,q}(r,s)$ fulfills a specific generating function*

$$\sum_{l=0}^{\infty} h^{(\alpha)}_{l,\nu,q}(r,s)t^l = \frac{\Gamma_q(\alpha+1)}{(1-st)_q^{\alpha+1}} C_{\nu,q}(rt), \quad |r| < \frac{1}{(1-q)^2}, \quad |s| < \frac{1}{1-q}, \quad \alpha \in \mathbb{R}, \tag{26}$$

where $C_{\nu,q}(rt)$ is the associated q-Tricomi function, which defined in equation (21).

Proof. In view of (25), we have

$$\sum_{l=0}^{\infty} h^{(\alpha)}_{l,\nu,q}(r,s)t^l = \sum_{r=0}^{\infty} \frac{(-1)^k r^k s^{l-k} \Gamma_q(l-k+\alpha+1)}{[k]_q! \Gamma_q(k+\nu+1)[l-k]_q!} t^l.$$

Using the appropriate series rearrangement approach, we obtain

$$\begin{aligned}\sum_{l=0}^{\infty} h^{(\alpha)}_{l,\nu,q}(r,s)t^l &= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k \Gamma_q(l+\alpha+1) r^k s^l}{[k]_q! \Gamma_q(k+\nu+1)[l]_q!} t^{l+k} \\ &= \Gamma_q(\alpha+1) \sum_{l=0}^{\infty} \begin{bmatrix} l+\alpha \\ l \end{bmatrix}_q s^l t^l \sum_{k=0}^{\infty} \frac{(-1)^k r^k}{[k]_q! \Gamma_q(k+\nu+1)} t^k,\end{aligned}$$

which yields the assertion (26) when employing equations (5) and (21). □

Remark 1. *We derive the integral form, series definition and generating function of the 2-variable q-truncated associated Tricomi function at $\alpha = 0$ in formulas* (24), (25) *and* (26) *in the following forms*

$$h_{l,\nu,q}(r,s) = \frac{1}{\Gamma_q(l+\nu+1)} \int_0^{\frac{1}{1-q}} E_q(-q\zeta) L^{\nu}_{l,q}(r,s\zeta) d_q\zeta,$$

$$h_{l,\nu,q}(r,s) = \sum_{k=0}^{l} \frac{(-1)^k r^k s^{l-k}}{[k]_q! \Gamma_q(k+\nu+1)}, \qquad \sum_{l=0}^{\infty} h_{l,\nu,q}(r,s)t^l = \frac{C_{\nu,q}(rt)}{1-st},$$

respectively.

Remark 2. *By inserting $s = 1$ at equations* (13), (14), (15) *and* (16), *we derive the next integral form, series definition, generating function and q-differential equation for q-truncated Tricomi function $h_{l,q}(x)$*

$$h_{l,q}(r) = \frac{1}{[l]_q!} \int_0^{\frac{1}{1-q}} E_q(-q\zeta) L_{l,q}(r,\zeta) d_q\zeta, \qquad h_{l,q}(r) = \sum_{k=0}^{l} \frac{(-1)^k r^k}{([k]_q!)^2}, \qquad \sum_{n=0}^{\infty} h_{l,q}(r)t^l = \frac{C_{0,q}(rt)}{1-t},$$

and

$$[(rD_{q,r}rD_{q,r} - (q([l-1]_q)^2 - r + 1))D_{q,r}rD_{q,r} - ([l]_q)^2]h_{l,q}(r) = 0,$$

respectively.

Also, for $s = 1$ in equations (24), (26) *and* (25), *we get the following integral form, series definition and generating function of the 3-parameter truncated q-Tricomi function $h^{(\alpha)}_{l,\nu,q}(r)$:*

$$h^{(\alpha)}_{l,\nu,q}(r) = \frac{1}{\Gamma_q(l+\nu+1)} \int_0^{\frac{1}{1-q}} \zeta^{\alpha} E_q(-q\zeta) L^{\nu}_{l,q}(r,\zeta) d_q\zeta,$$

$$h^{(\alpha)}_{l,\nu,q}(r) = \sum_{k=0}^{l} \frac{(-1)^k r^k \Gamma_q(l-k+\alpha+1)}{[k]_q! \Gamma_q(k+\nu+1)[l-k]_q!}, \qquad \sum_{l=0}^{\infty} h^{(\alpha)}_{l,\nu,q}(r)t^l = \frac{\Gamma_q(\alpha+1)}{(1-t)_q^{\alpha+1}} C_{\nu,q}(rt),$$

where $|r| < 1/(1-q)^2$, $\alpha \in \mathbb{R}$, $C_{\nu,q}(rt)$ *is the associated* q*-Tricomi function, defined by* (21)*.*

Finally, substituting $s = 1$ *into formulas* (22) *and* (23)*, respectively, we obtain the following generating function and series expansion of the associated* q*-Laguerre polynomials* $L^{\nu}_{l,q}(r)$*:*

$$C_{\nu,q}(rt)e_q(t) = \sum_{n=0}^{\infty} L^{\nu}_{l,q}(r)\frac{t^l}{\Gamma_q(l+\nu+1)}, \qquad L^{\nu}_{l,q}(r) = \Gamma_q(l+\nu+1)\sum_{k=0}^{l}\frac{(-1)^k r^k}{[k]_q!\Gamma_q(k+\nu+1)[l-k]_q!}.$$

2 Higher order 2-variable q-truncated Tricomi functions

In this part, we explore the features of mth order q-Laguerre-type polynomials and present higher order of 2-variable q-truncated Tricomi functions.

The mth order 2-variable q-truncated Tricomi function ${}_{[m]}h_{l,q}(r,s)$ can be expressed using the following integral representation

$$ {}_{[m]}h_{l,q}(r,s) = \frac{1}{[l]_q!}\int_0^{\frac{1}{1-q}} E_q(-q\zeta)\,{}_{[m]}L_{l,q}(r,s\zeta)d_q\zeta, \quad r,s,\zeta \in \mathbb{C}, \quad m \in \mathbb{N}, \quad l \in \mathbb{N}\cup\{0\}, \tag{27}$$

which provides the next series expansion for the mth order 2-variable q-truncated Tricomi function when applied to equation (12) and (7):

$$ {}_{[m]}h_{l,q}(r,s) = \sum_{k=0}^{[l/m]} \frac{r^k s^{l-mk}}{([k]_q!)^2}. \tag{28}$$

Theorem 4. *The* m*th order 2-variable* q*-truncated Tricomi function satisfies the next generating function*

$$\sum_{l=0}^{\infty} {}_{[m]}h_{l,q}(r,s)t^l = \frac{C_{0,q}(-rt^m)}{1-st}. \tag{29}$$

Proof. In view of equation (28), we have

$$\sum_{l=0}^{\infty} {}_{[m]}h_{l,q}(r,s)t^l = \sum_{l=0}^{\infty}\sum_{k=0}^{[l/m]} \frac{r^k s^{l-mk}}{([k]_q!)^2}t^l.$$

Using the following appropriate series rearrangement approach [2]

$$\sum_{l=0}^{\infty}\sum_{k=0}^{\infty} A(k,l) = \sum_{l=0}^{\infty}\sum_{k=0}^{[l/m]} A(k,l-mk), \tag{30}$$

we obtain

$$\sum_{l=0}^{\infty} {}_{[m]}h_{l,q}(r,s)t^l = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{r^k s^l}{([k]_q!)^2}t^{l+mk} = \sum_{n=0}^{\infty} s^l t^l \sum_{k=0}^{\infty}\frac{r^k}{([k]_q!)^2}t^{mk},$$

which on using equations (5) and (9), gives claim (29). □

Considering (21), we possess

$$C_{\nu,q}(-r) = \sum_{k=0}^{\infty}\frac{r^k}{[k]_q!\Gamma_q(k+\nu+1)}.$$

We present the corresponding mth order 2-variable q-Laguerre polynomials ${}_{[m]}L^{\nu}_{l,q}(r,s)$ from the previously mentioned equation via the generating function

$$C_{\nu,q}(-rt^m)e_q(st) = \sum_{l=0}^{\infty} {}_{[m]}L^{\nu}_{l,q}(r,s)\frac{t^l}{\Gamma_q(l+\nu+1)}. \tag{31}$$

By extending the previously mentioned equation via (1) and (21) and contrasting the same powers of t on both parts of the resulting equation, we arrive at

$${}_{[m]}L^{\nu}_{l,q}(r,s) = \Gamma_q(l+\nu+1)\sum_{k=0}^{[l/m]} \frac{r^k s^{l-mk}}{[k]_q!\Gamma_q(k+\nu+1)[l-mk]_q!}. \tag{32}$$

We thus obtain a generating function and a series representation for ${}_{[m]}L^{\nu}_{l,q}(r)$ by setting $s=1$ in formulas (31) and (32):

$$C_{\nu,q}(-rt^m)\,e_q(t) = \sum_{l=0}^{\infty} {}_{[m]}L^{\nu}_{l,q}(r)\frac{t^l}{\Gamma_q(l+\nu+1)},$$

and

$${}_{[m]}L^{\nu}_{l,q}(r) = \Gamma_q(l+\nu+1)\sum_{k=0}^{\lfloor l/m\rfloor} \frac{r^k}{[k]_q!\,\Gamma_q(k+\nu+1)\,[l-mk]_q!},$$

respectively. Currently, we define the 3-parameter mth order 2-variable q-truncated Tricomi function ${}_{[m]}h^{(\alpha)}_{l,\nu,q}(r,s)$ via the following integral representation:

$${}_{[m]}h^{(\alpha)}_{l,\nu,q}(r,s) = \frac{1}{\Gamma_q(l+\nu+1)}\int_0^{\frac{1}{1-q}} \zeta^{\alpha}\, E_q(-q\zeta)\, {}_{[m]}L^{\nu}_{l,q}(r,s\zeta)\, d_q\zeta, \tag{33}$$

which leads to the following series representation of the 3-parameter mth order 2-variable q-truncated Tricomi function ${}_{[m]}h^{(\alpha)}_{l,\nu,q}(r,s)$, obtained by applying formulas (32) and (7):

$${}_{[m]}h^{(\alpha)}_{l,\nu,q}(r,s) = \sum_{k=0}^{\lfloor l/m\rfloor} \frac{r^k s^{l-mk}\,\Gamma_q(l-mk+\alpha+1)}{[k]_q!\,\Gamma_q(k+\nu+1)\,[l-mk]_q!}. \tag{34}$$

Theorem 5. *The 3-parameter mth order 2-variable truncated q-Tricomi function possesses the following generating function:*

$$\sum_{n=0}^{\infty} {}_{[m]}h^{(\alpha)}_{l,\nu,q}(r,s)t^l = \frac{\Gamma_q(\alpha+1)}{(1-st)_q^{\alpha+1}}C_{\nu,q}(-rt^m), \tag{35}$$

where $C_{\nu,q}(-rt^m)$ is the associated mth order q-Tricomi function defined by equation (21).

Proof. Considering equation (34), we get

$$\sum_{n=0}^{\infty} {}_{[m]}h^{(\alpha)}_{l,\nu,q}(r,s)t^l = \sum_{l=0}^{\infty}\sum_{k=0}^{[l/m]} \frac{\Gamma_q(l-mk+\alpha+1)r^k s^{l-mk}}{[k]_q!\Gamma_q(k+\nu+1)[l-mk]_q!}t^l,$$

which on using equation (30), gives

$$\begin{aligned}\sum_{l=0}^{\infty} {}_{[m]}h^{(\alpha)}_{l,\nu,q}(r,s)t^l &= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k\Gamma_q(l+\alpha+1)r^k s^l}{[k]_q!\Gamma_q(k+\nu+1)[l]_q!}t^{l+mk} \\ &= \Gamma_q(\alpha+1)\sum_{l=0}^{\infty}\begin{bmatrix} l+\alpha \\ l\end{bmatrix}_q s^l t^s \sum_{k=0}^{\infty}\frac{r^k}{[k]_q!\Gamma_q(k+\nu+1)}t^{mk},\end{aligned}$$

which yields the statement (35) when employing formulas (5) and (21). □

Remark 3. *Note that in view of* (12), *for* $m = 2$, *the* m*th order 2-variable* q*-Laguerre polynomials* ${}_{[m]}L_{l,q}(r,\zeta)$ *reduce to the second order 2-variable* q*-Laguerre polynomials* ${}_{[2]}L_{l,q}(r,\zeta)$. *Therefore, for* $m = 2$ *in equations* (27), (28) *and* (29), *the* m*th order 2-variable* q*-truncated Tricomi function reduces towards the next integral form, series expansion and generating function of second order 2-variable* q*-truncated Tricomi function:*

$$ {}_{[2]}h_{l,q}(r,s) = \frac{1}{[l]_q!}\int_0^{\frac{1}{1-q}} E_q(-q\zeta)\,{}_{[2]}L_{l,q}(r,s\zeta)d_q\zeta, $$

$$ {}_{[2]}h_{l,q}(r,s) = \sum_{k=0}^{[l/2]} \frac{r^k s^{l-2k}}{([k]_q!)^2}, \qquad \sum_{n=0}^{\infty} {}_{[2]}h_{l,q}(r,s)t^l = \frac{C_{l,q}(-rt^2)}{1-st}, $$

respectively. The 1-parameters m*th order 2-variable* q*-truncated Tricomi function can be defined via the next integral form, series definition and generating function:*

$$ {}_{[m]}h_{l,\nu,q}(r,s) = \frac{1}{\Gamma_q(l+\nu+1)}\int_0^{\frac{1}{1-q}} E_q(-q\zeta)\,{}_{[m]}L^{\nu}_{l,q}(r,s\zeta)d_q\zeta, \tag{36} $$

$$ {}_{[m]}h_{l,\nu,q}(r,s) = \sum_{k=0}^{[l/m]} \frac{r^k s^{l-mk}}{[k]_q!\Gamma_q(k+\nu+1)}, \qquad \sum_{n=0}^{\infty} {}_{[m]}h_{l,\nu,q}(r,s)t^l = \frac{C_{\nu,q}(-rt^m)}{1-st}, \tag{37} $$

by taking $\alpha = 0$ *at formulas* (33), (34) *and* (35), *respectively.*

Similarly, by taking $m = 2$ *in equations* (36) *and* (37), *we derive the following integral form, series definition and generating function of the 1-parameters second order 2-variable* q*-truncated Tricomi function:*

$$ {}_{[2]}h_{l,\nu,q}(r,s) = \frac{1}{\Gamma_q(l+\nu+1)}\int_0^{\frac{1}{1-q}} E_q(-q\zeta)\,{}_{[2]}L^{\nu}_{l,q}(r,s\zeta)d_q\zeta, $$

$$ {}_{[2]}h_{l,\nu,q}(r,s) = \sum_{k=0}^{[l/2]} \frac{r^k s^{l-2k}}{[k]_q!\Gamma_q(k+\nu+1)}, \qquad \sum_{l=0}^{\infty} {}_{[2]}h_{l,\nu,q}(r,s)t^l = \frac{C_{\nu,q}(-rt^2)}{1-st}, $$

respectively.

Remark 4. *For* $s = 1$ *at formulas* (27), (28) *and* (29), *we obtain the next integral form, series definition, generating function for* m*th order* q*-truncated Tricomi function* ${}_{[m]}h_{l,q}(r)$:

$$ {}_{[m]}h_{l,q}(r) = \frac{1}{[l]_q!}\int_0^{\frac{1}{1-q}} E_q(-q\zeta)\,{}_{[m]}L_{l,q}(r,\zeta)d_q\zeta, $$

$$ {}_{[m]}h_{l,q}(r) = \sum_{k=0}^{[l/m]} \frac{r^k}{([k]_q!)^2}, \qquad \sum_{l=0}^{\infty} {}_{[m]}h_{l,q}(r)t^l = \frac{C_{0,q}(-rt^m)}{1-t}, $$

respectively.

Also, for $s = 1$ *at formulas* (33), (34) *and* (35), *we get next integral form, series definition and generating function of the 2-parameters* q*-truncated Tricomi function* $h^{(\alpha)}_{l,\nu,q}(r)$:

$$ {}_{[m]}h^{(\alpha)}_{l,\nu,q}(r) = \frac{1}{\Gamma_q(l+\nu+1)}\int_0^{\frac{1}{1-q}} \zeta^{\alpha}E_q(-q\zeta)\,{}_{[m]}L^{\nu}_{l,q}(r,\zeta)d_q\zeta, $$

$$ {}_{[m]}h^{(\alpha)}_{l,\nu,q}(r) = \sum_{k=0}^{[l/m]} \frac{r^k\Gamma_q(l-mk+\alpha+1)}{[k]_q!\Gamma_q(k+\nu+1)[l-mk]_q!}, \qquad \sum_{l=0}^{\infty} {}_{[m]}h^{(\alpha)}_{l,\nu,q}(r)t^n = \frac{\Gamma_q(\alpha+1)}{(1-t)_q^{\alpha+1}}C_{\nu,q}(-rt^m), $$

respectively.

3 Conclusions

The q-truncated exponential polynomials $E_{l,q}(r)$ [19] and their generalizations, q-truncated families of various types are of great importance and have a variety of applications. The 2-variable q-truncated exponential polynomials have been used in quantum mechanics, applied sciences, and mathematical modeling to represent interactions between two connected quantum systems, which correspond to q-deformed algebraic structures. Their applications in signal processing and optics include examining 2-variable interdependence, such as wave functions and light propagation [4]. In this explanation, we present a weaving of new aspects related to the 2-variable q-Tricomi functions, 2-variable q-Laguerre polynomials $L^{\nu}_{l,q}(r,s)$, 2-parameter 2-variable truncated q-Tricomi function $h^{(\alpha)}_{l,\nu,q}(r,s)$, and their related formalism. Particularly, we have introduced 2-variable truncated q-Tricomi functions $h_{l,q}(r,s)$. Also, we have introduced the associated 2-variable q-Laguerre polynomials $L^{\nu}_{l,q}(r,s)$ and we have used it to create the 2-parameter 2-variable truncated q-Tricomi function $h^{(\alpha)}_{l,\nu,q}(r,s)$ and obtained their integral forms, series definition and generating function. Further, we have introduced the mth order 2-variable truncated q-Tricomi function ${}_{[m]}h_{l,q}(r,s)$ by means of integral forms and established their characteristics such as series definition and generating function. In addition, we have introduced the associated mth order 2-variable q-Laguerre polynomials ${}_{[m]}L^{\nu}_{l,q}(r,s)$ and we have used it to derive the 2-parameters mth order 2-variable truncated q-Tricomi function ${}_{[m]}h^{(\alpha)}_{l,\nu,q}(r,s)$ and studied their features. Then we have showed for $s=1$, all previous results reduced to the corresponding results for truncated q-Tricomi functions $h_{l,q}(r)$, 2-parameter truncated q-Tricomi function $h^{(\alpha)}_{l,\nu,q}(r)$ and mth order truncated q-Tricomi function ${}_{[m]}h^{(\alpha)}_{l,\nu,q}(r)$. The examination of these functions has uncovered important claims and methods as well as new avenues for expanding the theory of q-special functions. We want to learn more about the new q-special functions and q-polynomials and find out how they can be used in mathematics, applied science, as well as engineering.

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Фадель М., Раза Н., Чезарано К., Агарвал П. *Деякі характеристики q-усічених функцій Трікомі від двох змінних* // Карпатські матем. публ. — 2026. — Т.18, №1. — С. 36–48.

У цій роботі встановлено певні властивості q-усічених функцій Трікомі від двох змінних, такі як інтегральні представлення, генеруючі функції та означення у вигляді рядів. Крім того, ми вводимо пов'язані q-поліноми Лагерра від двох змінних, які використовуються для отримання q-усічених функцій Трікомі вищого порядку від двох змінних, і досліджуємо їхні характеристики.

Ключові слова і фрази: квантове числення, q-усічений експоненціальний поліном, q-поліном Лагерра.