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(p, θ, q, η)-Nuclear Bloch maps

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In this paper, new developments in the theory of ideals of Bloch maps are utilized to introduce and analyze the properties of (p, θ, q, η)-nuclear Bloch maps from the open unit disk $\mathbb{D}$ to a complex Banach space X, where $1 \leq p, q < \infty$ and $0 \leq \theta, \eta < 1$ satisfy $(1-\theta)/p + (1-\eta)/q = 1$. The main emphasis is placed on defining these maps, establishing their Banach space properties, and investigating fundamental characteristics such as Pietsch domination, Bloch compactness and Möbius invariance. Finally, we conclude the paper by presenting a Bloch reasonable crossnorm and illustrating the isometric isomorphism between the defined space and its dual space.

Key words and phrases: summing operator, vector-valued Bloch map, compact Bloch map, Pietsch domination, Kwapień's factorization.

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1 Introduction

The class of (p, θ)-absolutely continuous maps was initially defined by U. Matter in [12] using the interpolative construction. Following that, J.A. López Molina and E.A. Sánchez Pérez examined the factorization properties and the trace duality of these operators across a series of papers [10, 11, 14]. On the other hand, the notion of strongly (p, θ)-continuous maps was introduced by D. Achour et. al. in [1] for scrutinizing the cohort of operators whose adjoint mappings are (p^*, θ)-absolutely continuous, aiming to analyze the duality properties of this significant operator ideal. E. Dahia et. al. [7] introduced the concept of (p, θ, q, η)-nuclear maps, where $1 \leq p,\ q < \infty$ and $0 \leq \theta,\ \eta < 1$ satisfying $(1-\theta)/p + (1-\eta)/q = 1$. They also provided some characterization properties in this context, moreover, when $\theta = \eta = 0$, we come across the class of p-nuclear operators, originally introduced by J.S. Cohen in [6]. Additionally, these maps were introduced in the context of multilinear maps in [2].

Based on the recent works of some authors investigating certain operators in the context of Bloch maps, such as the study of (p, σ)-absolutely continuous Bloch maps in [4] and strongly (p, σ)-continuous Bloch maps in [3], we will also delve into this study in this paper. Our main objective is to introduce and establish the most notable properties of a notion of (p, θ, q, η)-nuclear Bloch maps on the open unit disk $\mathbb{D}$ into a complex Banach space X. Our paper will be divided into several sections after the introduction. In the second section, we will provide the most important notations and basic definitions used throughout this paper. In the third section, we will provide the fundamental definition of zero-preserving (p, θ, q, η)-nuclear Bloch maps from $\mathbb{D}$ into X, denoted the set consisting of such maps by $\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D}, X)$, and

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proved that they form a Banach space with their natural corresponding norms $\|\cdot\|_{\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}}$. Furthermore, we will present the Pietsch domination theorem of these maps. In the fourth section, we will establish some of the most important properties, such as Bloch compactness, Kwapień's factorization theorem, and the Banach-Bloch ideal property of these maps. Also we show that the space $\left(\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D},X);\|\cdot\|_{\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}}\right)$ is invariant by Möbius transformations of $\mathbb{D}$. Finally, in the fifth section, we conclude the paper by introducing a Bloch reasonable crossnorm $\mathcal{R}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}$ on the tensor product space $\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}X^*$ and demonstrating that the space $\left(\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D},X);\|\cdot\|_{\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}}\right)$ is isometrically isomorphic to the dual space $\left(\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_{\mathcal{R}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}}X^*\right)^*$.

2 Preliminaries

For Banach spaces X and Y, we denote the closed unit ball of X by B_X and $\mathcal{L}(X,Y)$ represents the space of all continuous linear maps from X to Y equipped with the usual norm. When Y is the scalar field $\mathbb{K}$ we simplify $\mathcal{L}(X,\mathbb{K})$ as X^*. For any $1<p<\infty$, p^* denotes the Hölder conjugate of p given by $1/p+1/p^*=1$. Recall from [12] that a mapping $T\in\mathcal{L}(X,Y)$ is considered (p,θ)-absolutely continuous, where $p\in[1,\infty)$ and $\theta\in[0,1)$, if there exists a constant $C>0$ such that

$$\left(\sum_{i=1}^{n}\left\|T\left(x_i\right)\right\|^{\frac{p}{1-\theta}}\right)^{\frac{1-\theta}{p}}\leq C\sup_{x^*\in B_{X^*}}\left(\sum_{i=1}^{n}\left(\left|x^*\left(x_i\right)\right|^{1-\theta}\|x_i\|^{\theta}\right)^{\frac{p}{1-\theta}}\right)^{\frac{1-\theta}{p}}$$

for any $n\in\mathbb{N}$ and $x_1,\ldots,x_n\in X$. The infimum of such constants C is denoted by $\pi_{p,\theta}(T)$, and the Banach space of all (p,θ)-summing maps of X to Y under the norm $\pi_{p,\theta}$ is denoted by $\Pi_{p,\theta}(X,Y)$. If $\theta=0$, then $(\Pi_{p,\theta};\pi_{p,\theta})=(\Pi_p;\pi_p)$, the Banach space of all p-summing maps. Furthermore, in [1], a mapping $T\in\mathcal{L}(X,Y)$ is termed strongly (p,η)-continuous, with $p,r\in[1,\infty)$ and $\eta\in[0,1)$ satisfying $1/r+(1-\eta)/p^*=1$, if there exists a constant $C>0$ such that

$$\sum_{i=1}^{n}\left|\left\langle T\left(x_i\right);y_i^*\right\rangle\right|\leq C\left(\sum_{i=1}^{n}\|x_i\|^{r}\right)^{\frac{1}{r}}\sup_{y^{**}\in B_{Y^{**}}}\left(\sum_{i=1}^{n}\left(\left|y^{**}\left(y_i^*\right)\right|^{1-\eta}\|y_i^*\|^{\eta}\right)^{\frac{p^*}{1-\eta}}\right)^{\frac{1-\eta}{p^*}}$$

for any $x_1,\ldots,x_n\in X$ and any $y_1^*,\ldots,y_n^*\in Y^*$. The collection of all strongly (p,η)-continuous maps from X into Y is denoted by $\mathcal{D}_{p,\eta}(X,Y)$, which is readily seen to be a subspace of $\mathcal{L}(X,Y)$. The least C for which the inequality holds will be written as $d_{p,\eta}$, serving as a norm for the space $\mathcal{D}_{p,\eta}(X,Y)$.

Let X be a complex Banach space, a map $f\in\mathcal{H}(\mathbb{D},X)$ is termed Bloch if its Bloch seminorm, defined as $\rho_{\mathcal{B}}(f):=\sup\left\{(1-|z|^2)\|f'(z)\|:z\in\mathbb{D}\right\}$, is finite. The space of all holomorphic maps from $\mathbb{D}$ into X satisfying this property is denoted by $\mathcal{B}(\mathbb{D},X)$. The normalized Bloch space $\widehat{\mathcal{B}}(\mathbb{D},X)$ is the closed subspace of $\mathcal{B}(\mathbb{D},X)$ consisting of all maps f for which $f(0)=0$, under the Bloch norm $\rho_{\mathcal{B}}$. For simplicity, we denote $\widehat{\mathcal{B}}(\mathbb{D})$ instead of $\widehat{\mathcal{B}}(\mathbb{D},\mathbb{C})$. Also, $\widehat{\mathcal{H}}(\mathbb{D},\mathbb{D})$ will denote the set of all holomorphic functions h from $\mathbb{D}$ into itself for which $h(0)=0$.

In a recent paper [4], the concept of (p,θ)-absolutely continuous maps was adapted to address the (p,θ)-absolute continuity property within the framework of Bloch maps, as described below.

For any $p \in [1,\infty)$ and $\theta \in [0,1)$, we define a map $f \in \mathcal{H}(\mathbb{D},X)$ as (p,θ)-absolutely continuous Bloch if there exists $C>0$ such that for any n in $\mathbb{N}$, $\beta_1,\dots,\beta_n$ in $\mathbb{C}$ and $z_1,\dots,z_n$ in $\mathbb{D}$, the following inequality holds

$$\left(\sum_{i=1}^{n}|\beta_i|^{\frac{p}{1-\theta}}\left\|f'(z_i)\right\|^{\frac{p}{1-\theta}}\right)^{\frac{1-\theta}{p}} \le C\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^{n}\left(|\beta_i|\left(\frac{1}{1-|z_i|^2}\right)^{\theta}\left|h'(z_i)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}}\right)^{\frac{1-\theta}{p}}.$$

The infimum of the constants C, denoted by $\pi^{\mathcal{B}}_{p,\theta}$, defines a seminorm on space $\Pi^{\mathcal{B}}_{p,\theta}(\mathbb{D},X)$ of all p-absolutely continuous Bloch maps from $\mathbb{D}$ into X. Furthermore, this seminorm becomes a norm on the subspace $\Pi^{\widehat{\mathcal{B}}}_{p,\theta}(\mathbb{D},X)$ consisting of all those maps $f\in\Pi^{\mathcal{B}}_{p,\theta}(\mathbb{D},X)$ so that $f(0)=0$. Additionally, in [1], a map $f\in\mathcal{H}(\mathbb{D},X)$ is said to be strongly (p,η)-absolutely continuous Bloch with $p,r\in[1,\infty)$ and $\eta\in[0,1)$ such that $1/r+(1-\eta)/p^*=1$, if there exists a constant $C>0$, such that

$$\sum_{i=1}^{n}|\beta_i|\left|x_i^*\left(f'(z_i)\right)\right| \le C\left(\sum_{i=1}^{n}\left(\frac{|\beta_i|}{1-|z_i|^2}\right)^{r}\right)^{\frac{1}{r}}\sup_{x^{**}\in B_{X^{**}}}\left(\sum_{i=1}^{n}\left(\left|x^{**}(x_i^*)\right|^{1-\eta}\|x_i^*\|^{\eta}\right)^{\frac{p^*}{1-\eta}}\right)^{\frac{1-\eta}{p^*}}$$

for all $n\in\mathbb{N}$, $\beta_1,\dots,\beta_n\in\mathbb{C}$, $z_1,\dots,z_n\in\mathbb{D}$ and $x_1^*,\dots,x_n^*\in X^*$. The linear space of all strongly (p,η)-absolutely continuous Bloch maps from $\mathbb{D}$ to X is denoted by $\mathcal{D}^{\mathcal{B}}_{p,\eta}(\mathbb{D},X)$, and its subspace consisting of all those mappings f so that $f(0)=0$ by $\mathcal{D}^{\widehat{\mathcal{B}}}_{p,\eta}(\mathbb{D},X)$.

Now let us consider the space

$$\operatorname{lin}\big(\Gamma(\mathbb{D})\big)\otimes X^* := \operatorname{lin}\big(\{\gamma_z\otimes x^*\colon z\in\mathbb{D},\ x^*\in X^*\}\big)\subseteq\widehat{\mathcal{B}}(\mathbb{D},X)^*,$$

where $\gamma_z\otimes x^*\colon\widehat{\mathcal{B}}(\mathbb{D},X)\to\mathbb{C}$ is the functional defined by $(\gamma_z\otimes x^*)(f)=x^*(f'(z))$ for all $f\in\widehat{\mathcal{B}}(\mathbb{D},X)$.

Each element $\gamma\in\operatorname{lin}\big(\Gamma(\mathbb{D})\big)\otimes X^*$ is of the form $\gamma=\sum_{i=1}^n\beta_i\gamma_{z_i}\otimes x_i^*$ for some $n\in\mathbb{N}$, $\beta_i\in\mathbb{C}$, $z_i\in\mathbb{D}$ and $x_i^*\in X^*$ for $i=1,\dots,n$. Its action is given as $\gamma(f)=\sum_{i=1}^n\beta_i x_i^*(f'(z_i))$ for all $f\in\widehat{\mathcal{B}}(\mathbb{D},X)$,.

We will also require some results from the paper [9] regarding the Bloch-free Banach space over $\mathbb{D}$.

For each $z\in\mathbb{D}$, a Bloch atom of $\mathbb{D}$ is the function $\gamma_z\colon\widehat{\mathcal{B}}(\mathbb{D})\to\mathbb{C}$ defined by $\gamma_z(f)=f'(z)$ for all $f\in\hat{\mathcal{B}}(\mathbb{D})$. It is worth noting that $\gamma_z\in\widehat{\mathcal{B}}(\mathbb{D})^*$ with $\|\gamma_z\|=1/\left(1-|z|^2\right)$. The elements of the linear space $\operatorname{lin}\big(\{\gamma_z\colon z\in\mathbb{D}\}\big)\subseteq\widehat{\mathcal{B}}(\mathbb{D})^*$ are termed Bloch molecules of $\mathbb{D}$. The Bloch-free Banach space over $\mathbb{D}$ is defined as $\mathcal{G}(\mathbb{D}):=\overline{\operatorname{lin}}\big(\{\gamma_z\colon z\in\mathbb{D}\}\big)\subseteq\widehat{\mathcal{B}}(\mathbb{D})^*$.

The property outlined in [9], summarizes following several important properties of $\mathcal{G}(\mathbb{D})$:

1) the mapping $\Gamma\colon\mathbb{D}\to\mathcal{G}(\mathbb{D})$, defined by $\Gamma(z)=\gamma_z$ for all $z\in\mathbb{D}$, is holomorphic;

2) the space $\widehat{\mathcal{B}}(\mathbb{D})$ is isometrically isomorphic to $\mathcal{G}(\mathbb{D})^*$ under the map $\Lambda\colon\widehat{\mathcal{B}}(\mathbb{D})\to\mathcal{G}(\mathbb{D})^*$ defined as $\Lambda(f)(\gamma)=\sum_{k=1}^n\lambda_k f'(z_k)$ for $f\in\widehat{\mathcal{B}}(\mathbb{D})$, where $\gamma=\sum_{k=1}^n\lambda_k\gamma_{z_k}\in\operatorname{lin}\big(\Gamma(\mathbb{D})\big)$;

3) for each function $h\in\widehat{\mathcal{B}}(\mathbb{D},\mathbb{D})$, there exists a unique operator $\widehat{h}\in\mathcal{L}\big(\mathcal{G}(\mathbb{D}),\mathcal{G}(\mathbb{D})\big)$ such that $\widehat{h}\circ\Gamma=h'\cdot(\Gamma\circ h)$, furthermore, $\|\widehat{h}\|\le 1$;

4) for each $z\in\mathbb{D}$, the function $f_z:\mathbb{D}\to\mathbb{C}$ defined by

$$f_z(w)=\frac{(1-|z|^2)w}{1-\overline{z}w},\quad w\in\mathbb{D},$$

belongs to $\widehat{\mathcal{B}}(\mathbb{D})$ with $\rho_{\mathcal{B}}(f_z)=1=\left(1-|z|^2\right)f_z'(z)$.

3 Definition and Banach structure

From now on, unless otherwise stated, X and Y will denote Banach complex spaces and we will suppose that $1 \leq p,q < \infty$ and $0 \leq \theta,\eta < 1$ satisfy $(1-\theta)/p + (1-\eta)/q = 1$. Following [7, Theorem 4], a map $T \in \mathcal{L}(X,Y)$ is said to be (p,θ,q,η)-nuclear if there exist Banach spaces G, H, maps $S \in \Pi_p(X,G), R \in \Pi_q(Y^*,H)$ and a constant $C > 0$ such that

$$\left|y^*(T(x))\right| \leq C\|x\|^\theta \left\|S(x)\right\|^{1-\theta} \|y^*\|^\eta \left\|R(y^*)\right\|^{1-\eta} \tag{1}$$

for all $x \in X$ and $y^* \in Y^*$. In such case, we put

$$\|T\|_{\mathcal{N}_{(p,\theta,q,\eta)}} = \inf\left\{C\pi_p(S)^{1-\theta}\pi_q(R)^{1-\eta}\right\},$$

taking the infimum over all $S \in \Pi_p(X,G), R \in \Pi_q(Y^*,H)$ and $C > 0$ such that (1) holds. We denote by $\left[\mathcal{N}_{(p,\theta,q,\eta)}; \|\cdot\|_{\mathcal{N}_{(p,\theta,q,\eta)}}\right]$ the Banach ideal of (p,θ,q,η)-nuclear maps.

Now, we introduce the Bloch analogue of the notion of (p,θ,q,η)-nuclear operators.

Definition 1. *A map $f \in \mathcal{H}(\mathbb{D},X)$ is said to be (p,θ,q,η)-nuclear Bloch if there exist complex Banach spaces G and H, a Bloch map $g \in \Pi_p^{\mathcal{B}}(\mathbb{D},G)$, a map $T \in \Pi_q(X^*,H)$ and a positive constant C such that*

$$\left|x^*\left(f'(z)\right)\right| \leq C\left(\frac{1}{1-|z|^2}\right)^\theta \left\|g'(z)\right\|^{1-\theta} \|x^*\|^\eta \left\|T(x^*)\right\|^{1-\eta} \tag{2}$$

for all $z \in \mathbb{D}$ and $x^* \in X^*$. The linear space of all (p,θ,q,η)-nuclear Bloch maps from $\mathbb{D}$ to X is denoted by $\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}(\mathbb{D},X)$, and its subspace consisting of all those mappings f so that $f(0) = 0$ by $\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D},X)$. We denote by $\|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}$ the infimum of all values $C\pi_p^{\mathcal{B}}(g)^{1-\theta}\pi_q(T)^{1-\eta}$, where the infimum is taken over all Bloch maps g, maps T, and constants C admitted in inequality above.

Our next result is a reformulation for (p,θ,q,η)-nuclear Bloch maps of Pietsch domination theorem for (p,θ,q,η)-nuclear maps. This is a particular case of the general characterization of (p,θ,q,η)-dominated maps [14, Theorem 2.4].

Let us remind that $\widehat{\mathcal{B}}(\mathbb{D})$ represents a dual Banach space. Therefore, $\mathcal{P}(B_{\widehat{\mathcal{B}}(\mathbb{D})})$ and $\mathcal{P}(B_{X^{**}})$ denote the sets of all Borel regular probability measures μ on $B_{\widehat{\mathcal{B}}(\mathbb{D})}$ and ν on $B_{X^{**}}$, respectively, equipped with the weak-* topology. Given $\mu \in \mathcal{P}(B_{\widehat{\mathcal{B}}(\mathbb{D})})$, $\nu \in \mathcal{P}(B_{X^{**}})$, $p \in [1,\infty)$ and $\sigma \in [0,1)$, let us consider the following inclusion operators

$$I_{\infty,p/(1-\sigma)}\colon L_\infty(\mu) \to L_{p/(1-\sigma)}(\mu), \qquad j_{\infty,\mathbb{D}} : C\left(B_{\widehat{\mathcal{B}}(\mathbb{D})}\right) \to L_\infty(\mu)$$

and

$$j_{\infty,X} : C\left(B_{X^{**}}\right) \to L_\infty(\nu).$$

We also define the map $\iota_{\mathbb{D}}\colon \mathbb{D} \to C(B_{\widehat{\mathcal{B}}(\mathbb{D})})$ by

$$\iota_{\mathbb{D}}(z)(g) = g'(z)$$

for $g \in B_{\widehat{\mathcal{B}}(\mathbb{D})}$ and $z \in \mathbb{D}$, and the isometric linear embedding $\iota_X\colon X \to \ell_\infty(B_{X^*})$ given by

$$\iota_X(x)(x^*) = x^*(x)$$

for $x^* \in B_{X^*}$ and $x \in X$.

Theorem 1 (Pietsch domination)**.** *Given $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$, the following assertions are equivalent:*

1) $f \in \mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D}, X)$;

2) there is a constant $C > 0$ and measures $\mu \in \mathcal{P}\big(B_{\widehat{\mathcal{B}}(\mathbb{D})}\big)$ and $\nu \in \mathcal{P}\big(B_{X^{**}}\big)$ such that the following inequality

$$\left|x^*(f'(z))\right| \le C\left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\frac{1}{(1-|z|^2)^\theta}\left|h'(z)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}} d\mu\right)^{\frac{1-\theta}{p}} \times \left(\int_{B_{X^{**}}} \left(|x^{**}(x^*)|^{1-\eta}\,\|x^*\|^\eta\right)^{\frac{q}{1-\eta}} d\nu\right)^{\frac{1-\eta}{q}}$$

holds for every $z \in \mathbb{D}$ and $x^* \in X^*$;

3) there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ and any sequences $(z_i)_{i=1}^n$ in $\mathbb{D}$, $(\beta_i)_{i=1}^n$ in $\mathbb{C}$ and $\left(x_i^*\right)_{i=1}^n$ in X^* we have

$$\sum_{i=1}^n |\beta_i|\left|x_i^*\left(f'(z_i)\right)\right| \le C \sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^n \left(|\beta_i|\frac{1}{(1-|z_i|^2)^\theta}\left|h'(z_i)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}}\right)^{\frac{1-\theta}{p}} \times \sup_{x^{**}\in B_{X^{**}}} \left(\sum_{i=1}^n \left(|x^{**}(x_i^*)|^{1-\eta}\,\|x_i^*\|^\eta\right)^{\frac{q}{1-\eta}}\right)^{\frac{1-\eta}{q}}.$$

Moreover, the infimum of the constants $C > 0$ in 2) and 3) is $\|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}$.

Proof. 1) $\Rightarrow$ 2) If $f \in \mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D}, X)$, then there exist a constant $C_1 > 0$, complex Banach spaces G and H, a Bloch map $g \in \Pi^{\widehat{\mathcal{B}}}_p(\mathbb{D}, G)$ and a map $T \in \Pi_q(X^*, H)$ such that

$$\left|x^*(f'(z))\right| \le C_1\left(\frac{1}{1-|z|^2}\right)^\theta \left\|g'(z)\right\|^{1-\theta}\|x^*\|^\eta\,\|T(x^*)\|^{1-\eta}$$

for all $z \in \mathbb{D}$ and $x^* \in X^*$. Applying [5, Theorem 1.4] to g and [8, Theorem 2.12] to T, we obtain measures μ on $B_{\widehat{\mathcal{B}}(\mathbb{D})}$ and ν on $B_{X^{**}}$ such that

$$\left\|g'(z)\right\| \le \pi_p^{\mathcal{B}}(g)\left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left|h'(z)\right|^p d\mu\right)^{\frac{1}{p}}, \qquad \left\|T(x^*)\right\| \le \pi_q(T)\left(\int_{B_{X^{**}}} |x^{**}(x^*)|^q\, d\nu\right)^{\frac{1}{q}}$$

for all $(z, x^*) \in \mathbb{D}\times X^*$. Taking $C = C_1\pi_p^{\mathcal{B}}(g)^{1-\theta}\pi_q(T)^{1-\eta}$, we get

$$\left|x^*(f'(z))\right| \le C\left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\frac{1}{(1-|z|^2)^\theta}\left|h'(z)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}} d\mu\right)^{\frac{1-\theta}{p}} \times \left(\int_{B_{X^{**}}} \left(|x^{**}(x^*)|^{1-\eta}\,\|x^*\|^\eta\right)^{\frac{q}{1-\eta}} d\nu\right)^{\frac{1-\eta}{q}}$$

for all $z \in \mathbb{D}$ and $x^* \in X^*$.

Moreover $\|f\|_{\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}} \leq C$ and so $\inf\{C > 0 \text{ satisfying } 2)\} \leq \|f\|_{\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}}$.

2) $\Rightarrow$ *3)* Consider the sequences $(z_i)_{i=1}^n$ in $\mathbb{D}$, $(\beta_i)_{i=1}^n$ in $\mathbb{C}$ and $(x_i^*)_{i=1}^n$ in X^*. Hölder's inequality gives

$$\begin{aligned}\sum_{i=1}^n |\beta_i||x^*(f'(z_i))| &\leq C\left(\sum_{i=1}^n \int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(|\beta_i|\frac{1}{(1-|z_i|^2)^\theta}\left|h'(z_i)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}} d\mu\right)^{\frac{1-\theta}{p}} \\ &\quad\times\left(\sum_{i=1}^n \int_{B_{X^{**}}} \left(|x^{**}(x_i^*)|^{1-\eta}\|x_i^*\|^\eta\right)^{\frac{q}{1-\eta}} d\nu\right)^{\frac{1-\eta}{q}} \\ &\leq C\left(\sum_{i=1}^n \int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}} \sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(|\beta_i|\frac{1}{(1-|z_i|^2)^\theta}\left|h'(z_i)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}} d\mu\right)^{\frac{1-\theta}{p}} \\ &\quad\times\left(\sum_{i=1}^n \int_{B_{X^{**}}} \sup_{x^{**}\in B_{X^{**}}} \left(|x^{**}(x_i^*)|^{1-\eta}\|x_i^*\|^\eta\right)^{\frac{q}{1-\eta}} d\nu\right)^{\frac{1-\eta}{q}} \\ &\leq C \sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^n \left(|\beta_i|\frac{1}{(1-|z_i|^2)^\theta}\left|h'(z_i)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}}\right)^{\frac{1-\theta}{p}} \\ &\quad\times\sup_{x^{**}\in B_{X^{**}}}\left(\sum_{i=1}^n \left(|x^{**}(x_i^*)|^{1-\eta}\|x_i^*\|^\eta\right)^{\frac{q}{1-\eta}}\right)^{\frac{1-\eta}{q}},\end{aligned}$$

and this proves *3)*.

3) $\Rightarrow$ *2)* Define the functions $R_1 : B_{\widehat{\mathcal{B}}(\mathbb{D})} \times \mathbb{D} \times \mathbb{R} \to [0,\infty)$, $R_2 : B_{X^{**}} \times \mathbb{D} \times X^* \to [0,\infty)$, $S : \widehat{\mathcal{B}}(\mathbb{D}, X) \times \mathbb{D} \times \mathbb{R} \times X^* \to [0,\infty)$ by

$$R_1(h,z,\beta) = |\beta|\frac{1}{(1-|z|^2)^\theta}\left|h'(z)\right|^{1-\theta}, \qquad R_2(x^{**},z,x^*) = |x^{**}(x^*)|^{1-\eta}\|x^*\|^\eta,$$
$$S(f,z,\beta,x^*) = |\beta|\left|x^*\left(f'(z)\right)\right|,$$

respectively.

We will apply a general Pietsch domination theorem (see [13, Theorem 4.6]), notice that R_1, R_2 and S satisfy the conditions of [13, Definition 4.4], namely

(C1) for each $z \in \mathbb{D}$, $\beta \in \mathbb{R}$ and $x^* \in X^*$, the mappings $(R_1)_{z,\beta} : B_{\widehat{\mathcal{B}}(\mathbb{D})} \to [0,\infty)$ and $(R_2)_{z,x^*} : B_{X^{**}} \to [0,\infty)$, defined by

$$(R_1)_{z,\beta}(h) = R_1(h,z,\beta), \qquad (R_2)_{z,x^*}(x^{**}) = R_2(x^{**},z,x^*)$$

are continuous;

(C2) the equalities

$$R_1(h,z,\beta_1\beta) = \beta_1 R_1(h,z,\beta), \qquad R_2(x^{**},z,\beta_2 x^*) = \beta_2 R_2(x^{**},z,x^*)$$
$$S(f,z,\beta_1\beta,\beta_2 x^*) = \beta_1\beta_2 S(f,z,\beta,x^*)$$

hold for all $h \in B_{\widehat{\mathcal{B}}(\mathbb{D})}$, $x^{**} \in B_{X^{**}}$, $z \in \mathbb{D}$, $\beta \in \mathbb{R}$, $x^* \in X^*$ and $\beta_1, \beta_2 \in [0,1]$.

We now prove that the map f is R_1-, R_2-, S-abstract $(p/(1-\theta), q/(1-\eta))$-summing. Indeed, let $n \in \mathbb{N}$, $\beta_1, \ldots, \beta_n \in \mathbb{C}$, $z_1, \ldots, z_n \in \mathbb{D}$ and $x_1^*, \ldots, x_n^* \in X^*$. By *3)*, we have a constant $C > 0$ such that

$$\sum_{i=1}^{n} |\beta_i| |x_i^*(f'(z_i))| \leq C \sup_{h \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^{n} \left(|\beta_i| \frac{1}{(1-|z_i|^2)^\theta} |h'(z_i)|^{1-\theta} \right)^{\frac{p}{1-\theta}} \right)^{\frac{1-\theta}{p}} \times \sup_{x^{**} \in B_{X^{**}}} \left(\sum_{i=1}^{n} \left(|x^{**}(x_i^*)|^{1-\eta} \|x_i^*\|^\eta \right)^{\frac{q}{1-\eta}} \right)^{\frac{1-\eta}{q}},$$

and so we find

$$\begin{aligned} \sum_{i=1}^{n} S(f, z_i, \beta_i, x_i^*) &= \sum_{i=1}^{n} |\beta_i| |x_i^*(f'(z_i))| \\ &\leq C \sup_{h \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^{n} \left(|\beta_i| \frac{1}{(1-|z_i|^2)^\theta} |h'(z_i)|^{1-\theta} \right)^{\frac{p}{1-\theta}} \right)^{\frac{1-\theta}{p}} \\ &\quad \times \sup_{x^{**} \in B_{X^{**}}} \left(\sum_{i=1}^{n} \left(|x^{**}(x_i^*)|^{1-\eta} \|x_i^*\|^\eta \right)^{\frac{q}{1-\eta}} \right)^{\frac{1-\eta}{q}} \\ &= C \sup_{h \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^{n} R_1(h, z_i, \beta_i)^{\frac{p}{1-\theta}} \right)^{\frac{1-\theta}{p}} \\ &\quad \times \sup_{x^{**} \in B_{X^{**}}} \left(\sum_{i=1}^{n} R_2(x^{**}, z_i, x_i^*)^{\frac{q}{1-\eta}} \right)^{\frac{1-\eta}{q}}. \end{aligned}$$

By [13, Theorem 4.6], there are measures $\mu \in \mathcal{P}(B_{\widehat{\mathcal{B}}(\mathbb{D})})$ and $\nu \in \mathcal{P}(B_{X^{**}})$ such that for all $(z, \beta, x^*) \in \mathbb{D} \times \mathbb{R} \times X^*$ we get

$$S(f, z, \beta, x^*) \leq C \left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}} R_1(x^{**}, z, \beta)^{\frac{p}{1-\theta}} d\mu \right)^{\frac{1-\theta}{p}} \left(\int_{B_{X^{**}}} R_2(x^{**}, z, x^*)^{\frac{p^*}{1-\sigma}} d\nu \right)^{\frac{1-\sigma}{p^*}}.$$

It follows that

$$|x^*(f'(z))| \leq C \left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\frac{1}{(1-|z|^2)^\theta} |h'(z)|^{1-\theta} \right)^{\frac{p}{1-\theta}} d\mu \right)^{\frac{1-\theta}{p}} \times \left(\int_{B_{X^{**}}} \left(|x^{**}(x^*)|^{1-\eta} \|x^*\|^\eta \right)^{\frac{q}{1-\eta}} d\nu \right)^{\frac{1-\eta}{q}}.$$

2) $\Rightarrow$ *1)* By [5, Lemma 1.5], there exists a map $k \in \widehat{\mathcal{B}}(\mathbb{D}, L_\infty(\mu))$ with $\rho_{\mathcal{B}}(k) = 1$ such that $k' = j_\infty \circ \iota_{\mathbb{D}}$. So, $k \in \Pi_p^{\widehat{\mathcal{B}}}(\mathbb{D}, L_\infty(\mu))$ with $\pi_p^{\mathcal{B}}(k) = 1$ and by [8, 2.4 and 2.9], we can take the map $T = I_{\infty,q} \circ j_{\infty,X} \circ \iota_X$, so $T \in \Pi_q(X^*, L_q(\nu))$ with $\pi_q(T) \leq 1$. Since *2)* holds, for every $z \in \mathbb{D}$ and $x^* \in X^*$ we can write

$$
\begin{aligned}
\left|x^*\left(f'(z)\right)\right| &\le C\left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\frac{1}{(1-|z|^2)^\theta}\left|h'(z)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}}d\mu\right)^{\frac{1-\theta}{p}}\\
&\quad\times\left(\int_{B_{X^{**}}}\left(\left|x^{**}\left(x^*\right)\right|^{1-\eta}\|x^*\|^\eta\right)^{\frac{q}{1-\eta}}d\nu\right)^{\frac{1-\eta}{q}}\\
&= C\frac{1}{(1-|z|^2)^\theta}\left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left|\left(I_{\infty,p}\circ j_{\infty,\mathbb{D}}\circ\iota_{\mathbb{D}}\right)(z)\,(h)\right|^p d\mu\right)^{\frac{1-\theta}{p}}\\
&\quad\times\|x^*\|^\eta\left(\int_{B_{X^{**}}}\left|\left(I_{\infty,q}\circ j_{\infty,X}\circ\iota_X\right)(x^*)\,(x^{**})\right|^q d\nu\right)^{\frac{1-\eta}{q}}\\
&= C\frac{1}{(1-|z|^2)^\theta}\left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left|\left(I_{\infty,p}\circ k\right)'(z)\,(h)\right|^p d\mu\right)^{\frac{1-\theta}{p}}\\
&\quad\times\|x^*\|^\eta\left(\int_{B_{X^{**}}}\left|T(x^*)\,(x^{**})\right|^q d\nu\right)^{\frac{1-\eta}{q}}\\
&= C\frac{1}{(1-|z|^2)^\theta}\left\|g'(z)\right\|_{L_p(\mu)}^{1-\theta}\|x^*\|^\eta\left\|T(x^*)\right\|_{L_q(\nu)}^{1-\eta},
\end{aligned}
$$

where $g=I_{\infty,p}\circ k\in\Pi_p^{\widehat{\mathcal{B}}}\left(\mathbb{D},L_p(\mu)\right)$, we conclude that $f\in\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D},X)$. □

First, we establish the fact that the introduced functions are indeed Bloch functions. For two semi-normed spaces $(X;\rho_X)$ and $(Y;\rho_Y)$ the inequality $(X;\rho_X)\le(Y;\rho_Y)$ means that $X\subseteq Y$ and $\rho_Y(x)\le\rho_X(x)$ for all $x\in X$.

Proposition 1. *We encounter the following inequalities:*

1) $\left(\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}(\mathbb{D},X);\|\cdot\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}\right)\le\left(\Pi^{\mathcal{B}}_{p,\theta}(\mathbb{D},X);\pi^{\mathcal{B}}_{p,\theta}\right)\le\left(\mathcal{B}(\mathbb{D},X);\rho_{\mathcal{B}}\right)$,

2) $\left(\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}(\mathbb{D},X);\|\cdot\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}\right)\le\left(\mathcal{D}^{\mathcal{B}}_{q^*,\eta}(\mathbb{D},X);d^{\mathcal{B}}_{q^*,\eta}\right)\le\left(\mathcal{B}(\mathbb{D},X);\rho_{\mathcal{B}}\right)$.

Proof. 1) Let $f\in\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D},X)$, Theorem 1 provides us with

$$
\begin{aligned}
\|f'(z)\| = \sup_{x^*\in B_{X^*}}\left|x^*\left(f'(z)\right)\right| &\le \|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}\left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\frac{1}{(1-|z|^2)^\theta}\left|h'(z)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}}d\mu\right)^{\frac{1-\theta}{p}}\\
&\quad\times\sup_{x^*\in B_{X^*}}\left(\int_{B_{X^{**}}}\left(\left|x^{**}(x^*)\right|^{1-\eta}\|x^*\|^\eta\right)^{\frac{q}{1-\eta}}d\nu\right)^{\frac{1-\eta}{q}}\\
&\le \|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}\left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\frac{1}{(1-|z|^2)^\theta}\left|h'(z)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}}d\mu\right)^{\frac{1-\theta}{p}}\\
&\quad\times\sup_{x^*\in B_{X^*}}\sup_{x^{**}\in B_{X^{**}}}\left|x^{**}(x^*)\right|^{1-\eta}\|x^*\|^\eta\\
&\le \|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}\left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\frac{1}{(1-|z|^2)^\theta}\left|h'(z)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}}d\mu\right)^{\frac{1-\theta}{p}}
\end{aligned}
$$

for all $z \in \mathbb{D}$ and $x^* \in X^*$. Thus, according to Pietsch's domination theorem for (p,θ)-absolutely continuous Bloch maps [4, Theorem 2.1.], we have $f \in \Pi^{\mathcal{B}}_{p,\theta}(\mathbb{D}, X)$ and $\pi^{\mathcal{B}}_{p,\theta}(f) \le \|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}$. For the second inequality, we employ [4, Proposition 1.1], thus $f \in \mathcal{B}(\mathbb{D}, X)$ and $\rho_{\mathcal{B}}(f) \le \pi^{\mathcal{B}}_{p,\theta}(f)$.

2) If $f \in \mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}(\mathbb{D}, X)$, according to Theorem 1, we have

$$
\begin{aligned}
\left|x^*\left(f'(z)\right)\right| &\le \|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}} \left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\frac{1}{(1-|z|^2)^\theta}\left|h'(z)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}} d\mu\right)^{\frac{1-\theta}{p}} \\
&\quad \times \left(\int_{B_{X^{**}}} \left(\left|x^{**}(x^*)\right|^{1-\eta}\|x^*\|^\eta\right)^{\frac{q}{1-\eta}} d\nu\right)^{\frac{1-\eta}{q}} \\
&\le \|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}} \frac{1}{(1-|z|^2)^\theta}\left(\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left|h'(z)\right|\right)^{1-\theta} \\
&\quad \times \left(\int_{B_{X^{**}}} \left(\left|x^{**}(x^*)\right|^{1-\eta}\|x^*\|^\eta\right)^{\frac{q}{1-\eta}} d\nu\right)^{\frac{1-\eta}{q}} \\
&\le \|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}} \frac{1}{(1-|z|^2)^\theta}\frac{1}{(1-|z|^2)^{1-\theta}} \\
&\quad \times \left(\int_{B_{X^{**}}} \left(\left|x^{**}(x^*)\right|^{1-\eta}\|x^*\|^\eta\right)^{\frac{q}{1-\eta}} d\nu\right)^{\frac{1-\eta}{q}} \\
&\le \|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}} \frac{1}{1-|z|^2}\left(\int_{B_{X^{**}}} \left(\left|x^{**}(x^*)\right|^{1-\eta}\|x^*\|^\eta\right)^{\frac{q}{1-\eta}} d\nu\right)^{\frac{1-\eta}{q}}
\end{aligned}
$$

for all $z \in \mathbb{D}$ and $x^* \in X^*$. Therefore, according to Pietsch's domination theorem of maps $\mathcal{D}^{\mathcal{B}}_{q^*,\eta}$ [3, Theorem 2.2], we conclude that $f \in \mathcal{D}^{\mathcal{B}}_{q^*,\eta}(\mathbb{D}, X)$ and $d^{\mathcal{B}}_{q^*,\eta}(f) \le \|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}$. The second inequality follows from [3, Proposition 2.1]. □

In the next proposition, we will be able to prove that the linear space of all (p,θ,q,η)-nuclear Bloch maps, along with its norm, forms a Banach space.

Proposition 2. *The pair* $\left(\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D}, X); \|\cdot\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}\right)$ *constitutes a Banach space.*

Proof. If $f \in \mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}(\mathbb{D}, X)$ and $\|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}} = 0$, then $\rho_{\mathcal{B}}(f) = 0$. Therefore, according to Proposition 1, we get $f = 0$. Let us establish the triangle inequality. Let $f_i \in \mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D}, X)$ for $i = 1, 2$. For each $\varepsilon > 0$ there exist complex Banach spaces G_i and H_i, Bloch maps $g_i \in \Pi^{\widehat{\mathcal{B}}}_p(\mathbb{D}, G_i)$, bounded maps $T_i \in \Pi_q(X^*, H_i)$ and positive constants C_i such that

$$
\left|x^*\left(f_i'(z)\right)\right| \le C_i \frac{1}{(1-|z|^2)^\theta}\left\|g_i'(z)\right\|^{1-\theta}\|x^*\|^\eta\left\|T_i(x^*)\right\|^{1-\eta}
$$

and

$$
C_i \pi^{\mathcal{B}}_p(g_i)^{1-\theta}\pi_q(T_i)^{1-\eta} \le \|f_i\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}} + \varepsilon. \tag{3}
$$

Given $(z,x^*)\in\mathbb{D}\times X^*$, we have

$$\left|x^*\left(f_i'(z)\right)\right|\leq M_i\frac{1}{(1-|z|^2)^{\theta}}\left\|S_i'(z)\right\|^{1-\theta}\|x^*\|^{\eta}\left\|R_i\left(x^*\right)\right\|^{1-\eta},$$

where

$$M_i=C_i\pi_p^{\mathcal{B}}(g_i)^{1-\theta}\pi_q(T_i)^{1-\eta},\qquad S_i=C_i^{\frac{1}{p}}\pi_p^{\mathcal{B}}(g_i)^{\frac{1-\theta}{p}}\pi_q(T_i)^{\frac{1-\eta}{p}}\frac{g_i}{\pi_p^{\mathcal{B}}(g_i)},$$

$$R_i=C_i^{\frac{1}{q}}\pi_p^{\mathcal{B}}(g_i)^{\frac{1-\theta}{q}}\pi_q(T_i)^{\frac{1-\eta}{q}}\frac{T_i}{\pi_q(T_i)}.$$

Therefore, we find $S_i\in\Pi_p^{\widehat{\mathcal{B}}}(\mathbb{D},G_i)$ and $R_i\in\Pi_q(X^*,H_i)$ for $i=1,2$. From (3) we have

$$M_i\leq\|f_i\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}+\varepsilon,\qquad\pi_p^{\mathcal{B}}(g_i)\leq\left(\|f_i\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}+\varepsilon\right)^{1/p},\qquad\pi_q(T_i)\leq\left(\|f_i\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}+\varepsilon\right)^{1/q}.$$

Let G be a complex Banach space obtained as a direct ℓ_p-sum of G_1 and G_2 and let H be a complex Banach space obtained as a direct ℓ_q-sum of H_1 and H_2. Let g be a Bloch map from $\mathbb{D}$ into G defined as $g(z)=\left(g_i(z)\right)_{i=1}^2$ for $z\in\mathbb{D}$ and T be a bounded map from X^* into H given by $T(x^*)=(T_i(x^*))_{i=1}^2$ for $x^*\in X^*$. For every $m\in\mathbb{N}$ and any sequence $(z_j)_{j=1}^m$ in $\mathbb{D}$ we establish

$$\begin{aligned}\left\|\left(g(z_j)\right)_{j=1}^m\right\|_{\ell_p(G)}=\left(\sum_{j=1}^m\|g(z_j)\|^p\right)^{1/p}&\leq\left(\sum_{j=1}^m\sum_{i=1}^2\|g_i(z_j)\|^p\right)^{1/p}\\&\leq\left(\sum_{i=1}^2\pi_p^{\mathcal{B}}(g_i)^p\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\sum_{j=1}^m\left|h'(z_j)\right|^p\right)^{1/p}\\&\leq\left(\sum_{i=1}^2\pi_p^{\mathcal{B}}(g_i)^p\right)^{1/p}\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{j=1}^m\left|h'(z_j)\right|^p\right)^{1/p}.\end{aligned}$$

Consequently, the map g is Bloch p-summing with norm

$$\pi_p^{\mathcal{B}}(g)\leq\left(\sum_{i=1}^2\pi_p^{\mathcal{B}}(g_i)^p\right)^{1/p}\leq\left(\|f_1\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}+\|f_2\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}+2\varepsilon\right)^{1/p}.$$

On the other hand, for every sequence $(x_j^*)_{j=1}^m$ in X^* we also have

$$\begin{aligned}\left\|\left(T(x_j^*)\right)_{j=1}^m\right\|_{\ell_q(H)}=\left(\sum_{j=1}^m\|T(x_j^*)\|^q\right)^{1/q}&\leq\left(\sum_{j=1}^m\sum_{i=1}^2\|T_i(x_j^*)\|^q\right)^{1/q}\\&\leq\left(\sum_{i=1}^2\pi_q(T_i)^q\sup_{x^{**}\in B_{X^{**}}}\sum_{j=1}^m\left|x^{**}(x_j^*)\right|^q\right)^{1/q}\\&\leq\left(\sum_{i=1}^2\pi_q(T_i)^q\right)^{1/q}\sup_{x^{**}\in B_{X^{**}}}\left(\sum_{j=1}^m\left|x^{**}(x_j^*)\right|^q\right)^{1/q}.\end{aligned}$$

So, we can observe that the map T is q-summing with norm

$$\pi_q(T)\leq\left(\sum_{i=1}^2\pi_q(T_i)^q\right)^{1/q}\leq\left(\|f_1\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}+\|f_2\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}+2\varepsilon\right)^{1/q}.$$

Finally, since

$$\begin{aligned}\left|x^*\left((f_1'+f_2')(z)\right)\right| &\leq \sum_{i=1}^{2} C_i \frac{1}{(1-|z|^2)^\theta} \left\|g_i'(z)\right\|^{1-\theta} \|x^*\|^\eta \left\|T_i\left(x^*\right)\right\|^{1-\eta} \\ &\leq \frac{1}{(1-|z|^2)^\theta} \|x^*\|^\eta \left(\sum_{i=1}^{2} C_i\right) \left(\sum_{i=1}^{2} \left\|g_i'(z)\right\|^p\right)^{(1-\theta)/p} \left(\sum_{i=1}^{2} \left\|T_i(x^*)\right\|^q\right)^{(1-\eta)/q} \\ &\leq \left(\sum_{i=1}^{2} C_i\right) \frac{1}{(1-|z|^2)^\theta} \|x^*\|^\eta \left\|g'(z)\right\|^{1-\theta} \left\|T\left(x^*\right)\right\|^{1-\eta}\end{aligned}$$

and

$$\begin{aligned}\|f_1+f_2\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}} &\leq \left(\sum_{i=1}^{2} C_i\right) \pi_p^{\mathcal{B}}(g)^{1-\theta} \pi_q(T)^{1-\eta} \\ &\leq \left(\|f_1\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}} + \|f_2\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}} + 2\varepsilon\right)^{(1-\theta)/p+(1-\eta)/q},\end{aligned}$$

we deduce that $f_1+f_2 \in \mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D},X)$ and $\|f_1+f_2\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}} \leq \|f_1\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}} + \|f_2\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}$.

Let $\lambda \in \mathbb{C}$ and $f \in \mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D},X)$, then there exist a constant $C>0$, complex Banach spaces G and H, a Bloch map $g \in \Pi_p^{\widehat{\mathcal{B}}}(\mathbb{D},G)$ and a map $T \in \Pi_q\left(X^*,H\right)$ such that

$$\left|x^*(f'(z))\right| \leq C\frac{1}{(1-|z|^2)^\theta} \left\|g'(z)\right\|^{1-\theta} \|x^*\|^\eta \left\|T\left(x^*\right)\right\|^{1-\eta}$$

for all $(z,x^*) \in \mathbb{D}\times X^*$. Hence,

$$\begin{aligned}\left|x^*\left((\lambda f)'(z)\right)\right| &\leq C\left|\lambda\right| \frac{1}{(1-|z|^2)^\theta} \left\|g'(z)\right\|^{1-\theta} \|x^*\|^\eta \left\|T\left(x^*\right)\right\|^{1-\eta} \\ &= C\frac{1}{(1-|z|^2)^\theta} \left\|\left(\lambda^{1/(1-\theta)}g\right)'(z)\right\|^{1-\theta} \|x^*\|^\eta \left\|T\left(x^*\right)\right\|^{1-\eta}.\end{aligned}$$

We find that $\lambda^{1/(1-\theta)}g \in \Pi_p^{\widehat{\mathcal{B}}}(\mathbb{D},G)$, it follows that $\lambda f \in \mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D},X)$ with

$$\|\lambda f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}} \leq C\pi_p^{\mathcal{B}}\left(\lambda^{1/(1-\theta)}g\right)^{1-\theta}\pi_q(T)^{1-\eta} = |\lambda|\, C\pi_p^{\mathcal{B}}(g)^{1-\theta}\pi_q(T)^{1-\eta}.$$

If $\lambda=0$, then we obtain $\|\lambda f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}} = 0 = |\lambda|\,\|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}$. For $\lambda \neq 0$, we find that $\|\lambda f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}} \leq |\lambda|\,\|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}$. Therefore, $\|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}} \leq (1/\left|\lambda\right|)\,\|\lambda f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}$. This implies that $|\lambda|\,\|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}} \leq \|\lambda f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}$, and thus $\|\lambda f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}} = |\lambda|\,\|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}$. Consequently, we deduce that $\left(\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D},X)\,;\|\cdot\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}\right)$ is a complex normed space.

Let us show the completeness of $\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D},X)$. Consider an arbitrary Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D},X)$. We will prove the convergence of $(f_n)_{n\in\mathbb{N}}$ to $f \in \mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D},X)$. For every $\varepsilon>0$ there exists an n_0 such that for all $m,n \geq n_0$ the sequence $(f_n)_{n\in\mathbb{N}}$ being Cauchy implies $\|f_m-f_n\|_{\mathcal{N}_{(p,\theta,q,\eta)}^{\mathcal{B}}} \leq \varepsilon$. According to Proposition 1, we have the inequality $\rho_{\mathcal{B}}\left(f_m-f_n\right) \leq \|f_m-f_n\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}$. Therefore, $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space

$\widehat{\mathcal{B}}(\mathbb{D},X)$. So, there is a Bloch map f with $\lim\limits_{n\to 0}\rho_{\mathcal{B}}(f-f_n)=0$. Theorem 1 implies that there exist regular Borel probability measures μ_{nm} on $B_{\widehat{\mathcal{B}}(\mathbb{D})}$ and ν_{nm} on $B_{X^{**}}$ such that for any z in $\mathbb{D}$ and x^* in X^* we have

$$\begin{aligned}\left|x^*\left((f_m-f_n)'(z)\right)\right| &\le \varepsilon\left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\frac{1}{(1-|z|^2)^{\theta}}\left|h'(z)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}}d\mu_{nm}\right)^{\frac{1-\theta}{p}}\\ &\quad\times\left(\int_{B_{X^{**}}}\left(|x^{**}(x^*)|^{1-\eta}\,\|x^*\|^{\eta}\right)^{\frac{q}{1-\eta}}d\nu_{nm}\right)^{\frac{1-\eta}{q}}.\end{aligned}$$

On $\mathcal{P}(B_{\widehat{\mathcal{B}}(\mathbb{D})})\times\mathcal{P}(B_{X^{**}})$ we define a subnet $\mathcal{A}$, and its values form a subsequence of the measures in $\mathcal{P}(B_{\widehat{\mathcal{B}}(\mathbb{D})})\times\mathcal{P}(B_{X^{**}})$. For a fixed $n\ge n_0$, the weak compactness of $\mathcal{P}(B_{\widehat{\mathcal{B}}(\mathbb{D})})$ and $\mathcal{P}(B_{X^{**}})$ implies that there is a subnet $(\mu_{nm}(\alpha),\nu_{nm}(\alpha))_{\alpha\in\mathcal{A}}$ converging to $(\mu_n,\nu_n)\in\mathcal{P}(B_{\widehat{\mathcal{B}}(\mathbb{D})})\times\mathcal{P}(B_{X^{**}})$ equipped with the weak-* topology. Thus, there exists $\alpha_0\in\mathcal{A}$ such that for any $z\in\mathbb{D}$ and $x^*\in X^*$ and for any $\alpha\in\mathcal{A}$ with $\alpha\ge\alpha_0$, we have

$$\begin{aligned}\left|x^*\left((f_{m(\alpha)}-f_n)'(z)\right)\right| &\le \varepsilon\Bigg(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\frac{1}{(1-|z|^2)^{\theta}}\left|h'(z)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}}d\left(\mu_{nm(\alpha)}-\mu_n\right)\\ &\quad+\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\frac{1}{(1-|z|^2)^{\theta}}\left|h'(z)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}}d\mu_n\Bigg)^{\frac{1-\theta}{p}}\\ &\times\left(\int_{B_{X^{**}}}\left(|x^{**}(x^*)|^{1-\eta}\,\|x^*\|^{\eta}\right)^{\frac{q}{1-\eta}}d\left(\nu_{nm}-\nu_n\right)+\int_{B_{X^{**}}}\left(|x^{**}(x^*)|^{1-\eta}\,\|x^*\|^{\eta}\right)^{\frac{q}{1-\eta}}d\nu_n\right)^{\frac{1-\eta}{q}},\end{aligned}$$

and by taking limits as $\alpha\in\mathcal{A}$, we obtain

$$\begin{aligned}\left|x^*\left((f-f_n)'(z)\right)\right| &\le \varepsilon\left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\frac{1}{(1-|z|^2)^{\theta}}\left|h'(z)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}}d\mu_n\right)^{\frac{1-\theta}{p}}\\ &\quad\times\left(\int_{B_{X^{**}}}\left(|x^{**}(x^*)|^{1-\eta}\,\|x^*\|^{\eta}\right)^{\frac{q}{1-\eta}}d\nu_n\right)^{\frac{1-\eta}{q}}\end{aligned}$$

for all $z\in\mathbb{D}$ and $x^*\in X^*$. It follows that $f-f_n\in\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D},X)$, and therefore that $f\in\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D},X)$. If $n\ge n_0$, then from the last inequality it follows $\|f_m-f_n\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}\le\varepsilon$, hence $\left(\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D},X);\|\cdot\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}\right)$ is a Banach space. □

4 Properties and Möbius invariance

In the domain of complex analysis, the concept of Bloch compactness stands as a significant result illuminating the behavior of holomorphic functions on the unit disk $\mathbb{D}$. This theorem states that, under specific conditions, holomorphic mappings belonging to certain Bloch-type spaces exhibit a remarkable property: compactness.

Let us recall that the *Bloch range* of a function $f\in\mathcal{H}(\mathbb{D},X)$, denoted by $\mathrm{rang}_{\mathcal{B}}(f)$, is the set

$$\left\{(1-|z|^2)f'(z)\in X : z\in\mathbb{D}\right\}.$$

A map $f\in\mathcal{H}(\mathbb{D},X)$ is called *(weakly) compact Bloch* if $\mathrm{rang}_{\mathcal{B}}(f)$ is a (weakly) compact set in X, respectively.

Corollary 1 (Bloch compactness). *If X is a reflexive complex Banach space, then every function $f \in \mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D}, X)$ is a compact Bloch map.*

Proof. Let $f \in \mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D}, X)$. Then, by Proposition 1, $f \in \Pi^{\widehat{\mathcal{B}}}_{p,\theta}(\mathbb{D}, X)$. Consequently, f is a compact Bloch map according to [4, Proposition 4.2]. □

The next result states Kwapień's factorization theorem for the space $\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}$.

Theorem 2 (Kwapień's factorization). *A map $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$ belongs to $\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D}, X)$ if and only if there exist a Banach space Z, a closed subspace $Y \subseteq Z$, a map $h \in \Pi^{\widehat{\mathcal{B}}}_{p,\theta}(\mathbb{D}, Z)$ with $h'(\mathbb{D}) \subseteq Y$ and a map $T \in \mathcal{D}_{q^*,\eta}(Y, X)$ such that $f' = T \circ h'$.*

Proof. Necessity. Let $f \in \mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D}, X)$. According to Pietsch's factorization theorem (see [4, Theorem 3.1]), there exists a map $h \in \widehat{\mathcal{B}}(\mathbb{D}, L_{p/(1-\theta)}(\mu))$ such that $h = I_{\infty,p/(1-\theta)} \circ k'$, where $k' = j_\infty \circ \iota_{\mathbb{D}}$ with $k \in \widehat{\mathcal{B}}(\mathbb{D}, \ell_\infty)$ and $\rho_{\mathcal{B}}(k) = 1$. By [5, Lemma 1.5], we have that $k \in \Pi^{\widehat{\mathcal{B}}}_{p,\theta}(\mathbb{D}, \ell_\infty)$ and $\rho_{\mathcal{B}}(k) = 1$. Moreover, $h \in \Pi^{\widehat{\mathcal{B}}}_{p,\theta}(\mathbb{D}, L_{p/(1-\theta)}(\mu))$ and $\pi^{\mathcal{B}}_{p,\theta}(h) \leq 1$.

Consider the linear subspace $Y = \overline{\operatorname{lin}}(h'(\mathbb{D})) \subseteq L_{p/(1-\theta)}(\mu)$ and the map $T \in \mathcal{L}(Y, X)$ defined by $T(h'(z)) = f'(z)$ for all $z \in \mathbb{D}$. By Theorem 1, we have

$$\begin{aligned} \left\|T^*(x^*)\right\| &= \sup\left\{\left|T^*(x^*)(h'(z))\right| : z \in \mathbb{D}, \|h'(z)\| \leq 1\right\} \\ &= \sup\left\{\left|x^*\left(T\left(h'(z)\right)\right)\right| : z \in \mathbb{D}, \|h'(z)\| \leq 1\right\} \\ &= \sup\left\{\left|x^*\left(f'(z)\right)\right| : z \in \mathbb{D}, \|h'(z)\| \leq 1\right\}, \end{aligned}$$

where the Pietsch domination theorem (see [4, Theorem 2.1]) gives us

$$\left\|h'(z)\right\| \leq \pi^{\mathcal{B}}_{p,\theta}(h)\left(\left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\frac{1}{(1-|z|^2)^\theta}\left|h'(z)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}} d\mu\right)^{\frac{1-\theta}{p}}\right).$$

So we find

$$\left\|T^*(x^*)\right\| \leq C\left(\int_{B_{X^{**}}}\left(|x^{**}(x^*)|^{1-\eta}\,\|x^*\|^\eta\right)^{\frac{q}{1-\eta}} d\nu\right)^{\frac{1-\eta}{q}}$$

for all $x^* \in X^*$. Thus $T^* \in \Pi_{q,\eta}(X^*, Y^*)$. According to [1, Remark 3.3], $T \in \mathcal{D}_{q^*,\eta}(Y, X)$ with $d_{q^*,\eta}(T) \leq C$, thus $\pi^{\mathcal{B}}_{p,\theta}(h)\, d_{q^*,\eta}(T) \leq C$.

Sufficiency. Let us assume that there is a Banach space Z, a closed subspace $Y \subseteq Z$, a map $h \in \Pi^{\mathcal{B}}_{p,\theta}(\mathbb{D}, Z)$ with $h'(\mathbb{D}) \subseteq Y$ and a map $T \in \mathcal{D}_{q^*,\eta}(Y, X)$ such that $f' = T \circ h'$. We find $f \in \mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D}, X)$ by employing the domination theorem for h and the domination theorem for T^*, where T^* is a (q,η)-summing map and $C = \inf\left\{\pi^{\mathcal{B}}_{p,\theta}(h) d_{q^*,\eta}(T) : f' = T \circ h'\right\}$. □

The concept of a Banach normalized Bloch ideal on $\mathbb{D}$ was initially presented in [9, Definition 5.11]. Now, we show that $\left(\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D}, X); \|\cdot\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}\right)$ exhibits the same property. Let us recall that for any complex Banach space X the inequality

$$\sup_{x^* \in B_{X^*}}\left(\sum_{i=1}^{\infty}|x^*(x_i)|^{\frac{p}{1-\theta}}\right)^{\frac{1-\theta}{p}} \leq \sup_{x^* \in B_{X^*}}\left(\sum_{i=1}^{n}\left(|x^*(x_i)|^{1-\theta}\,\|x_i\|^\theta\right)^{\frac{p}{1-\theta}}\right)^{\frac{1-\theta}{p}} \leq \left(\sum_{i=1}^{n}\|x_i\|^{\frac{p}{1-\theta}}\right)^{\frac{1-\theta}{p}} \tag{4}$$

holds for every $1 \leq p < \infty$, $0 \leq \theta < 1$, $(x_i)_{i=1}^n$ in X such that $\left(\sum_{i=1}^{n}\|x_i\|^{\frac{p}{1-\theta}}\right)^{\frac{1-\theta}{p}} < \infty$.

Proposition 3. *The space* $\left(\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D},X);\|\cdot\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}\right)$ *forms a Banach normalized Bloch ideal.*

Proof. The space $\left(\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D},X);\|\cdot\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}\right)$ is Banach space according to Proposition 2, where $\rho_{\mathcal{B}}(f)\le\|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}$ holds for all $f\in\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D},X)$.

According to [9, Proposition 5.3], if $g\in\widehat{\mathcal{B}}(\mathbb{D})$ and $x\in X$, then $\rho_{\mathcal{B}}(g\cdot x)=\rho_{\mathcal{B}}(g)\|x\|$. Assume $g\neq0$ and $x\neq0$. By (4) and Hölder's inequality, for all $(\beta_i)_{i=1}^n$ in $\mathbb{C}$, $(z_i)_{i=1}^n$ in $\mathbb{D}$ and $(x_i^*)_{i=1}^n$ in X^*, the following inequalities

$$\begin{aligned}
\sum_{i=1}^n|\beta_i||x_i^*(g\cdot x)'(z_i)| &= \rho_{\mathcal{B}}(g)\|x\|\sum_{i=1}^n|\beta_i|\left|\left(\frac{g}{\rho_{\mathcal{B}}(g)}\right)'(z_i)x_i^*\left(\frac{x}{\|x\|}\right)\right| \\
&= \rho_{\mathcal{B}}(g)\|x\|\sum_{i=1}^n|\beta_i|\left|\left(\frac{g}{\rho_{\mathcal{B}}(g)}\right)'(z_i)J_X\left(\frac{x}{\|x\|}\right)(x_i^*)\right| \\
&\le \rho_{\mathcal{B}}(g)\|x\|\left(\sum_{i=1}^n\left|\beta_i\left(\frac{g}{\rho_{\mathcal{B}}(g)}\right)'(z_i)\right|^{\frac{p}{1-\theta}}\right)^{\frac{1-\theta}{p}} \\
&\quad\times\left(\sum_{i=1}^n\left\|J_X\left(\frac{x}{\|x\|}\right)(x_i^*)\right\|^{\frac{q}{1-\eta}}\right)^{\frac{1-\eta}{q}} \\
&\le \rho_{\mathcal{B}}(g)\|x\|\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^n\left(|\beta_i|\frac{1}{(1-|z_i|^2)^\theta}\left|h'(z_i)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}}\right)^{\frac{1-\theta}{p}} \\
&\quad\times\sup_{x^{**}\in B_{X^{**}}}\left(\sum_{i=1}^n\|x^{**}(x_i^*)\|^{\frac{q}{1-\eta}}\right)^{\frac{1-\eta}{q}} \\
&\le \rho_{\mathcal{B}}(g)\|x\|\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^n\left(|\beta_i|\frac{1}{(1-|z_i|^2)^\theta}\left|h'(z_i)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}}\right)^{\frac{1-\theta}{p}} \\
&\quad\times\sup_{x^{**}\in B_{X^{**}}}\left(\sum_{i=1}^n\left(|x^{**}(x_i^*)|^{1-\eta}\|x_i^*\|^\eta\right)^{\frac{q}{1-\eta}}\right)^{\frac{1-\eta}{q}}
\end{aligned}$$

hold, where J_X represents the canonical injection of X into X^{**}. As per Theorem 1, we ascertain that $g\cdot x\in\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D},X)$ with $\|g\cdot x\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}\le\rho_{\mathcal{B}}(g)\|x\|$. Since

$$\rho_{\mathcal{B}}(g)\|x\|=\rho_{\mathcal{B}}(g\cdot x)\le\|g\cdot x\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}},$$

it follows that $\|g\cdot x\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}=\rho_{\mathcal{B}}(g)\|x\|$.

Consider $f\in\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\mathbb{D},X)$, $T\in\mathcal{L}(X,Y)$. Let $g:\mathbb{D}\to\mathbb{D}$ be a holomorphic map with $g(0)=0$. It is immediate that

$$(T\circ f\circ g)'=T\circ(f\circ g)'=T\circ g'\cdot(f'\circ g).$$

For all $(\beta_i)_{i=1}^n$ in $\mathbb{C}$, $(z_i)_{i=1}^n$ in $\mathbb{D}$ and $\left(y_i^*\right)_{i=1}^n$ in Y^*, the following inequalities

$$\begin{aligned}
\sum_{i=1}^n |\beta_i||y_i^*((T\circ f\circ g)'(z_i))| &= \sum_{i=1}^n |\beta_i||y_i^*\left[g'(z_i)T(f'(g(z_i)))\right]| \\
&\le \|T\| \left(\sum_{i=1}^n \left(|\beta_i| \left\|f'\left(g(z_i)\right)\right\| |g'(z_i)|\right)^{\frac{p}{1-\theta}}\right)^{\frac{1-\theta}{p}} \sup_{y^{**}\in B_{Y^{**}}} \left(\sum_{i=1}^\infty |y^{**}\left(y_i^*\right)|^{\frac{q}{1-\eta}}\right)^{\frac{1-\eta}{q}} \\
&\le \|T\| \|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}} \sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^n \left(|\beta_i| \frac{1}{(1-|g(z_i)|^2)^\theta} \left|g'\left(z_i\right)\right| \left|h'\left(g(z_i)\right)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}}\right)^{\frac{1-\theta}{p}} \\
&\quad\times \sup_{y^{**}\in B_{Y^{**}}} \left(\sum_{i=1}^n \left(|y^{**}\left(y_i^*\right)|^{1-\eta} \|y_i^*\|^\eta\right)^{\frac{q}{1-\eta}}\right)^{\frac{1-\eta}{q}} \\
&\le \|T\| \|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}} \sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^n \left(|\beta_i| \left(\frac{g'(z_i)}{1-|g(z_i)|^2}\right)^\theta \left|g'(z_i)h'\left(g(z_i)\right)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}}\right)^{\frac{1-\theta}{p}} \\
&\quad\times \sup_{y^{**}\in B_{Y^{**}}} \left(\sum_{i=1}^n \left(|y^{**}(y_i^*)|^{1-\eta} \|y_i^*\|^\eta\right)^{\frac{q}{1-\eta}}\right)^{\frac{1-\eta}{q}}
\end{aligned}$$

hold, derived from (4) and Hölder's inequality.

The Pick-Schwarz Lemma states that $\dfrac{|g'(z)|}{1-|g(z)|^2} \le \dfrac{1}{1-|z|^2}$ for all z in $\mathbb{D}$. So, we get

$$\begin{aligned}
&\sum_{i=1}^n |\beta_i||y_i^*((T\circ f\circ g)'(z_i))| \\
&\qquad\le \|T\| \|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}} \sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^n \left(|\beta_i| \frac{1}{(1-|z_i|^2)^\theta} \left|(h\circ g)'\left(z_i\right)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}}\right)^{\frac{1-\theta}{p}} \\
&\qquad\quad\times \sup_{y^{**}\in B_{Y^{**}}} \left(\sum_{i=1}^n \left(|y^{**}(y_i^*)|^{1-\eta} \|y_i^*\|^\eta\right)^{\frac{q}{1-\eta}}\right)^{\frac{1-\eta}{q}} \\
&\qquad\le \|T\| \|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}} \sup_{k\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^n \left(|\beta_i| \frac{1}{(1-|z_i|^2)^\theta} \left|k'\left(z_i\right)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}}\right)^{\frac{1-\theta}{p}} \\
&\qquad\quad\times \sup_{y^{**}\in B_{Y^{**}}} \left(\sum_{i=1}^n \left(|y^{**}(y_i^*)|^{1-\eta} \|y_i^*\|^\eta\right)^{\frac{q}{1-\eta}}\right)^{\frac{1-\eta}{q}}.
\end{aligned}$$

Note that $\rho_{\mathcal{B}}\left(h\circ g\right) \le \rho_{\mathcal{B}}\left(g\right)$. Therefore, $T\circ f\circ g \in \mathcal{N}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}\left(\mathbb{D}, X\right)$ with

$$\|T\circ f\circ g\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}} \le \|T\| \|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}.$$

□

The Möbius group of $\mathbb{D}$, denoted as Aut($\mathbb{D}$), comprises all biholomorphic bijections from $\mathbb{D}$ onto itself.

Recall that a linear space $\mathcal{A}(\mathbb{D},X)\subseteq\mathcal{B}(\mathbb{D},X)$, equipped with a seminorm $\rho_{\mathcal{A}}$, is *Möbius-invariant* if:

(i) there is $C>0$ such that $\rho_{\mathcal{B}}(f)\leq C\rho_{\mathcal{A}}(f)$ for all $f\in\mathcal{A}(\mathbb{D},X)$,

(ii) $f\circ\phi\in\mathcal{A}(\mathbb{D},X)$ with $\rho_{\mathcal{A}}(f\circ\phi)=\rho_{\mathcal{A}}(f)$ for all $\phi\in\mathrm{Aut}(\mathbb{D})$ and $f\in\mathcal{A}(\mathbb{D},X)$.

The invariance of (p,θ,q,η)-nuclear Bloch maps under Möbius transformations over $\mathbb{D}$ can now be derived.

Proposition 4 (Möbius invariance)**.** *Space* $\left(\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}(\mathbb{D},X);\|\cdot\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}\right)$ *is Möbius-invariant.*

Proof. Let us show the conditions of above definition.

(i) Proposition 1 yields $\left(\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}(\mathbb{D},X);\|\cdot\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}\right)\leq\left(\mathcal{B}(\mathbb{D},X);\rho_{\mathcal{B}}\right)$.

(ii) A review of the proof of Proposition 3 reveals that $f\circ\phi\in\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}(\mathbb{D},X)$ with $\|f\circ\phi\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}\leq\|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}$ if $f\in\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}(\mathbb{D},X)$ and $\phi\in\mathrm{Aut}(\mathbb{D})$. Moreover, from this we also deduce that $\|f\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}=\left\|(f\circ\phi)\circ\phi^{-1}\right\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}\leq\|f\circ\phi\|_{\mathcal{N}^{\mathcal{B}}_{(p,\theta,q,\eta)}}$. □

5 Crossnorms and duality

We are now ready to study the duality of the space of (p,θ,q,η)-nuclear Bloch maps from $\mathbb{D}$ into a complex Banach space X.

Recall that a norm α on $\mathrm{lin}(\Gamma(\mathbb{D}))\otimes X$ is a *Bloch reasonable crossnorm* if the following conditions hold:

(1) $\alpha(\gamma_z\otimes x^*)\leq\|\gamma_z\|\,\|x^*\|$ for all $z\in\mathbb{D}$ and $x^*\in X^*$,

(2) for every $g\in\widehat{\mathcal{B}}(\mathbb{D})$ and $x^{**}\in X^{**}$, the linear functional $g\otimes x^{**}:\mathrm{lin}(\Gamma(\mathbb{D}))\otimes X^*\to\mathbb{C}$ given by

$$(g\otimes x^{**})(\gamma_z\otimes x^*)=g'(z)x^{**}(x^*)$$

is bounded on $\mathrm{lin}(\Gamma(\mathbb{D}))\otimes_\alpha X^*$ with $\|g\otimes x^{**}\|\leq\rho_{\mathcal{B}}(g)\,\|x^{**}\|$.

Definition 2. *Let X be a complex Banach space. We define $\mathcal{R}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}$ on $\mathrm{lin}(\Gamma(\mathbb{D}))\otimes X^*$ as follows*

$$\mathcal{R}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\gamma)=\inf\left\{\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^{n}\left(|\beta_i|\,\frac{1}{(1-|z_i|^2)^\theta}\left|h'(z_i)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}}\right)^{\frac{1-\theta}{p}}\right.$$
$$\left.\times\sup_{x^{**}\in B_{X^{**}}}\left(\sum_{i=1}^{n}\left(\left|x^{**}(x_i^*)\right|^{1-\eta}\|x_i^*\|^\eta\right)^{\frac{q}{1-\eta}}\right)^{\frac{1-\eta}{q}}\right\},$$

where the infimum is taken over all representations of γ in the form $\gamma=\sum_{i=1}^n\beta_i\gamma_{z_i}\otimes x_i^*$ for all $(z_i)_{i=1}^n$ in $\mathbb{D}$, $(\beta_i)_{i=1}^n$ in $\mathbb{C}$ and $\left(x_i^*\right)_{i=1}^n$ in X^*.

Theorem 3. *Defined above $\mathcal{R}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}$ is a Bloch reasonable crossnorm on* $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X^*$.

Proof. Using the same techniques as those employed in [4, Theorem 5.2] and the inequality (4), we can prove that $\mathcal{R}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}$ is a norm on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X^*$.

(1) Given $z \in \mathbb{D}$ and $x^* \in X^*$, we get

$$\begin{aligned}
&\mathcal{R}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\gamma_z \otimes x^*) \\
&\quad \leq \sup_{h \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\left(\frac{1}{(1-|z|^2)^\theta} \left| h'(z) \right|^{1-\theta} \right)^{\frac{p}{1-\theta}} \right)^{\frac{1-\theta}{p}} \sup_{x^{**} \in B_{X^{**}}} \left(\left(|x^{**}(x^*)|^{1-\eta} \, \|x^*\|^\eta \right)^{\frac{q}{1-\eta}} \right)^{\frac{1-\eta}{q}} \\
&\quad \leq \left(\frac{1}{1-|z|^2} \right)^\theta \left(\frac{1}{1-|z|^2} \right)^{1-\theta} \|x^*\| = \frac{\|x^*\|}{1-|z|^2} = \|\gamma_z\| \, \|x^*\|.
\end{aligned}$$

(2) For any $g \in \widehat{\mathcal{B}}(\mathbb{D})$ and $x^{**} \in X^{**}$, an application of Hahn-Banach theorem and Hölder inequality yields

$$\begin{aligned}
\left| (g \otimes x^{**})(\gamma) \right| &= \left| \sum_{i=1}^n \beta_i (g \otimes x^{**})(\gamma_{z_i} \otimes x_i^*) \right| = \left| \sum_{i=1}^n \beta_i g'(z_i) \, x^{**}(x_i^*) \right| \\
&\leq \sum_{i=1}^n |\beta_i| \left| g'(z_i) \right| \left| x^{**}(x_i^*) \right| \leq \rho_{\mathcal{B}}(g) \, \|x^{**}\| \sum_{i=1}^n \frac{|\beta_i|}{1-|z_i|^2} \|x_i^*\| \\
&= \rho_{\mathcal{B}}(g) \, \|x^{**}\| \sum_{i=1}^n |\beta_i| \left| f'_{z_i}(z_i) \right| \left| x_i^{**}(x_i^*) \right| \\
&= \rho_{\mathcal{B}}(g) \, \|x^{**}\| \sum_{i=1}^n |\beta_i| \frac{1}{(1-|z_i|^2)^\theta} \left| f'_{z_i}(z_i) \right|^{1-\theta} \left| x_i^{**}(x_i^*) \right|^{1-\eta} \|x_i^*\|^\eta \\
&= \rho_{\mathcal{B}}(g) \, \|x^{**}\| \left(\sum_{i=1}^n \left(|\beta_i| \frac{1}{(1-|z_i|^2)^\theta} \left| f'_{z_i}(z_i) \right|^{1-\theta} \right)^{\frac{p}{1-\theta}} \right)^{\frac{1-\theta}{p}} \\
&\qquad \times \left(\sum_{i=1}^n \left(\left| x_i^{**}(x_i^*) \right|^{1-\theta} \|x_i^*\|^\theta \right)^{\frac{q}{1-\eta}} \right)^{\frac{1-\eta}{q}} \\
&\leq \rho_{\mathcal{B}}(g) \, \|x^{**}\| \sup_{h \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^n \left(|\beta_i| \frac{1}{(1-|z_i|^2)^\theta} \left| h'(z_i) \right|^{1-\theta} \right)^{\frac{p}{1-\theta}} \right)^{\frac{1-\theta}{p}} \\
&\qquad \times \sup_{x^{**} \in B_{X^{**}}} \left(\sum_{i=1}^n \left(\left| x^{**}(x_i^*) \right|^{1-\eta} \|x_i^*\|^\eta \right)^{\frac{q}{1-\eta}} \right)^{\frac{1-\eta}{q}},
\end{aligned}$$

where for each $i = 1, \ldots, n$ we have taken a functional $x_i^{**} \in B_{X^{**}}$ such that $\left| x_i^{**}(x_i^*) \right| = \|x_i^*\|$. Passing to the infimum over all the representations of γ, we obtain

$$\left| (g \otimes x^{**})(\gamma) \right| \leq \rho_{\mathcal{B}}(g) \, \|x^{**}\| \, \mathcal{R}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\gamma).$$

Hence $g \otimes x^{**} \in \left(\mathrm{lin}(\Gamma(\mathbb{D})) \otimes_{\mathcal{R}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}} X^* \right)^*$ and $\|g \otimes x^{**}\| \leq \rho_{\mathcal{B}}(g) \, \|x^{**}\|$. □

We are now ready to investigate the duality of the space of (p, θ, q, η)-nuclear Bloch maps from $\mathbb{D}$ into a complex Banach space X.

Theorem 4 (Duality). *The space* $\left(\mathcal{N}^{\widehat{\mathcal{B}}}_{p,\theta,q,\eta}(\mathbb{D},X);\|\cdot\|_{\mathcal{N}^{\mathcal{B}}_{p,\theta,q,\eta}}\right)$ *is isometrically isomorphic to* $\left(\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_{\mathcal{R}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}}X^*\right)^*$.

Proof. It is easy to see that the map

$$\Lambda\colon \left(\mathcal{N}^{\widehat{\mathcal{B}}}_{p,\theta,q,\eta}(\mathbb{D},X);\|\cdot\|_{\mathcal{N}^{\mathcal{B}}_{p,\theta,q,\eta}}\right)\to\left(\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_{\mathcal{R}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}}X^*\right)^*,$$

defined by

$$\Lambda(f)(\gamma_z\otimes x^*)=x^*\left(f'(z)\right),\qquad f\in\mathcal{N}^{\widehat{\mathcal{B}}}_{p,\theta,q,\eta}(\mathbb{D},X),\ z\in\mathbb{D},\ x^*\in X^*,$$

is linear and injective. Fix $f\in\mathcal{N}^{\widehat{\mathcal{B}}}_{p,\theta,q,\eta}(\mathbb{D},X)$. For $\gamma=\sum_{i=1}^n\lambda_i\gamma_{z_i}\otimes x_i^*\in\mathrm{lin}(\Gamma(\mathbb{D}))\otimes X^*$, an application of Theorem 1 gives

$$\begin{aligned}|\Lambda(f)(\gamma)| &\le \sum_{i=1}^n|\beta_i|\left|x_i^*\left(f'(z_i)\right)\right|\\ &\le \|f\|_{\mathcal{N}^{\mathcal{B}}_{p,\theta,q,\eta}}\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^n\left(|\beta_i|\frac{1}{(1-|z_i|^2)^\theta}\left|h'(z_i)\right|^{1-\theta}\right)^{\frac{p}{1-\theta}}\right)^{\frac{1-\theta}{p}}\\ &\quad\times\sup_{x^{**}\in B_{X^{**}}}\left(\sum_{i=1}^n\left(|x^{**}(x_i^*)|^{1-\eta}\|x_i^*\|^\eta\right)^{\frac{q}{1-\eta}}\right)^{\frac{1-\eta}{q}}.\end{aligned}$$

Taking the infimun over all the representation of γ, we get

$$|\Lambda(f)(\gamma)|\le\|f\|_{\mathcal{N}^{\mathcal{B}}_{p,\theta,q,\eta}}\mathcal{R}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}(\gamma),$$

and therefore $\|\Lambda(f)\|\le\|f\|_{\mathcal{N}^{\mathcal{B}}_{p,\theta,q,\eta}}$. In order to establish the reverse inequality and the surjectivity of Λ, let $\phi\in\left(\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_{\mathcal{R}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}}X^*\right)^*$. Define $F_\phi\colon\mathbb{D}\to X$ by

$$x^*\left(F_\phi(z)\right)=\phi\left(\gamma_z\otimes x^*\right),\qquad z\in\mathbb{D},\ x^*\in X^*.$$

A look at the proof of [5, Proposition 2.4] shows that $F_\phi\in\mathcal{H}(\mathbb{D},X)$ and $F_\phi=f'_\phi$ for a convenient map $f_\phi\in\widehat{\mathcal{B}}(\mathbb{D},X)$ with $\rho_{\mathcal{B}}(f_\phi)\le\|\phi\|$.

To prove that $f_\phi\in\mathcal{N}^{\widehat{\mathcal{B}}}_{p,\theta,q,\eta}(\mathbb{D},X)$, let $n\in\mathbb{N}$, $\beta_i\in\mathbb{C}$ and $z_i\in\mathbb{D}$ for $i=1,\ldots,n$. For each $i\in\{1,\ldots,n\}$, we can take a functional $x_i^*\in X^*$ with $\|x_i^*\|=1$ so that $x_i^*\left(f'_\phi(z_i)\right)=\left|f'_\phi(z_i)\right|$. Obviously, the function $T\colon\mathbb{C}^n\to\mathbb{C}$ defined by

$$T(t_1,\ldots,t_n)=\sum_{i=1}^n t_i\beta_i\left\|f'_\phi(z_i)\right\|,\qquad (t_1,\ldots,t_n)\in\mathbb{C}^n,$$

is in $(\mathbb{C}^n,\|\cdot\|_\infty)^*$ and $\|T\|=\sum_{i=1}^n|\beta_i|\left|f'_\phi(z_i)\right|$.

For any $(t_1, \ldots, t_n) \in \mathbb{C}^n$ with $\|(t_1, \ldots, t_n)\|_\infty \leq 1$, we get

$$\begin{aligned}
|T(t_1, \ldots, t_n)| &= \left|\phi\Big(\sum_{i=1}^n t_i\beta_i\gamma_{z_i} \otimes x_i^*\Big)\right| \\
&\leq \|\phi\|\, \mathcal{R}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}\Big(\sum_{i=1}^n \beta_i\gamma_{z_i} \otimes t_i x_i^*\Big) \\
&\leq \|\phi\| \sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^n \left(|\beta_i|\,\frac{1}{(1-|z_i|^2)^\theta}\,|h'(z_i)|^{1-\theta}\right)^{\frac{p}{1-\theta}}\right)^{\frac{1-\theta}{p}} \\
&\quad\times \sup_{x^{**}\in B_{X^{**}}} \left(\sum_{i=1}^n \left(|x^{**}(t_i x_i^*)|^{1-\eta}\,\|t_i x_i^*\|^\eta\right)^{\frac{q}{1-\eta}}\right)^{\frac{1-\eta}{q}} \\
&\leq \|\phi\| \sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^n \left(|\beta_i|\,\frac{1}{(1-|z_i|^2)^\theta}\,|h'(z_i)|^{1-\theta}\right)^{\frac{p}{1-\theta}}\right)^{\frac{1-\theta}{p}} \\
&\quad\times \sup_{x^{**}\in B_{X^{**}}} \left(\sum_{i=1}^n \left(|x^{**}(x_i^*)|^{1-\eta}\,\|x_i^*\|^\eta\right)^{\frac{q}{1-\eta}}\right)^{\frac{1-\eta}{q}},
\end{aligned}$$

and therefore

$$\begin{aligned}
\sum_{i=1}^n |\beta_i|\left|x_i^*\left(f'_\phi(z_i)\right)\right| \leq \|\phi\| &\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^n \left(|\beta_i|\,\frac{1}{(1-|z_i|^2)^\theta}\,|h'(z_i)|^{1-\theta}\right)^{\frac{p}{1-\theta}}\right)^{\frac{1-\theta}{p}} \\
\times &\sup_{x^{**}\in B_{X^{**}}} \left(\sum_{i=1}^n \left(|x^{**}(x_i^*)|^{1-\eta}\,\|x_i^*\|^\eta\right)^{\frac{q}{1-\eta}}\right)^{\frac{1-\eta}{q}}.
\end{aligned}$$

Therefore, according to Theorem 1, it follows that $f_\phi \in \mathcal{N}^{\widehat{\mathcal{B}}}_{p,\theta,q,\eta}(\mathbb{D}, X)$ and $\|f_\phi\|_{\mathcal{N}^{\mathcal{B}}_{p,\theta,q,\eta}} \leq \|\phi\|$.

Now, for any $\gamma = \sum_{i=1}^n \beta_i\gamma_{z_i} \otimes x_i^* \in \operatorname{lin}(\Gamma(\mathbb{D})) \otimes X^*$, we have

$$\Lambda(f_\phi)(\gamma) = \sum_{i=1}^n \beta_i x_i^*\left(f'_\phi(z_i)\right) = \sum_{i=1}^n \beta_i\phi(\gamma_{z_i} \otimes x_i^*) = \phi\Big(\sum_{i=1}^n \beta_i\gamma_{z_i} \otimes x_i^*\Big) = \phi(\gamma),$$

and so $\Lambda(f_\phi) = \phi$ on $\operatorname{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_{\mathcal{R}^{\widehat{\mathcal{B}}}_{(p,\theta,q,\eta)}} X^*$. Hence

$$\|f_\phi\|_{\mathcal{N}^{\mathcal{B}}_{p,\theta,q,\eta}} \leq \|\Lambda(f_\phi)\|.$$

□

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Хаміду Ю.С., Бугутая А., Беласел А. *(p, θ, q, η)-Ядерні відображення Блоха* // Карпатські матем. публ. — 2025. — Т.17, №2. — С. 386–405.

У цій статті використано нові результати теорії ідеалів відображень Блоха для введення та аналізу властивостей (p, θ, q, η)-ядерних відображень Блоха з відкритого одиничного диска $\mathbb{D}$ у комплексний банахів простір X, де $1 \leq p, q < \infty$ та $0 \leq \theta, \eta < 1$ задовольняють умову $(1-\theta)/p + (1-\eta)/q = 1$. Основну увагу приділено означенню цих відображень, встановленню їхніх властивостей як банахових просторів і дослідженню фундаментальних характеристик, таких як домінування Пітча, компактність Блоха та інваріантність Мебіуса. Наприкінці статті представлено відповідну крос-норму Блоха та проілюстровано ізометричний ізоморфізм між визначеним простором і його спряженим простором.

Ключові слова і фрази: оператор сумування, векторнозначне відображення Блоха, компактне відображення Блоха, домінування Пітча, факторизація Квапєня.